COMPRESSED SENSING FOR APERTURE SYNTHESIS IMAGING

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ABSTRACT

The theory of compressed sensing has a natural application in interferometric aperture synthesis. As in many real-world applications, however, the assumption of random sampling, which is elementary to many propositions of this theory, is not met. Instead, the induced sampling patterns exhibit a large degree of regularity. In this paper, we statistically quantify the effects of this kind of regularity for the problem of radio interferometry where astronomical images are sparsely sampled in the frequency domain. Based on the favorable results of our statistical evaluation, we present a practical method for interferometric image reconstruction that is evaluated on observational data from the Very Large Array (VLA) telescope.

Index Terms— radio astronomy, synthetic aperture imaging, image reconstruction, interferometry

1. INTRODUCTION

The theory of *compressed sensing* (CS) [1, 2, 3] provides an innovative way of thinking about sampling. The assumption of a band-limited signal – as in classical Shannon-Nyquist sampling – is replaced by the assumption that the signal vector s has a *sparse representation* x in some basis S, so that $s = Sx$. The measurement vector y is then obtained by measuring a number of linear combinations of the signal components. It can be written in terms of the *measurement matrix* M as $y = Ms = MSx$. In many practical applications, M represents a sampling in the Fourier domain. The sparsity matrix S may be a wavelet basis or simply the identity matrix.

The central theorem of compressed sensing [4] states that a sparse signal x can be reconstructed from measurements y by computing the vector with minimal ℓ_1 norm among all signals that are consistent with the measurements:

$$x = \operatorname*{argmin}_{z} ||z||_1 \text{ subject to } y = MSz. \qquad (1)$$

The theorem holds if at most c components of x are nonzero and any set of at most c columns of MS is approximately orthonormal. This constraint is well met for random matrices [5] or randomly sampled Fourier transform matrices [6]. The effect of random sampling is that it distributes the sampling error evenly among the components of x, resulting in a *point spread function* that consists of a sharp peak with low background noise.

In many practical applications such as interferometric aperture synthesis, however, truly random sample distributions cannot be obtained because the sampling is constrained by the design of the measurement system. Radio telescope arrays consist of several antennas that pick up electromagnetic waves from astronomical sources, and the correlation of the signals at each pair of antennas, termed *visibility*, corresponds to a single complex entry in the Fourier transform of the image of the source. The location of this entry in the transform domain is determined by the distance vector, or *baseline*, between both antennas, Fig. 1.

The measurement process can be formalized as follows: for a fixed frequency ν, one defines the intensity $I_\nu(d)$ of the radiation from the direction d. Denoting the baseline as b, the corresponding visibility $V_\nu(b)$ is given as an integral over the visible part of the sky,

$$V_\nu(b) = \int I_\nu(d) \, e^{-2\pi i \, d \cdot b} \, d\Omega. \qquad (2)$$

Under the assumption that the sources being imaged are confined to a small region of the sky, we can write $d \approx (i, j, 1)$ and $b = (u, v, w)$, so that Eq. (2) can be approximated by a two-dimensional Fourier transform multiplied by a constant phase,

$$V_\nu(u, v, w) \approx e^{-2\pi i \, w} \sum_{i,j} I_\nu(i, j) \, e^{-2\pi i \, (iu + jv)}. \qquad (3)$$

This equation is the basis for radio interferometric image reconstruction. It can be efficiently implemented using the Fast Fourier Transform.

The way in which the sampling pattern arises introduces a large amount of regularity: the positions of all $n(n-1)$ samples are uniquely defined by only n sensor positions, Fig. 1.

(a) Sample array configuration.

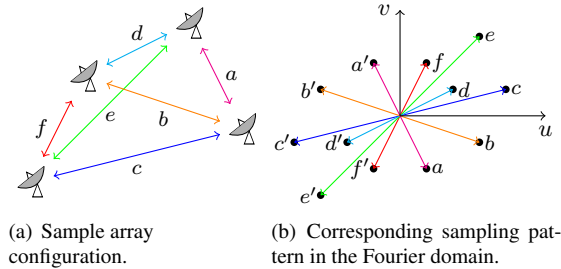(b) Corresponding sampling pattern in the Fourier domain.

Fig. 1. Hypothetical interferometric array configuration (a) and resulting sampling pattern in the frequency domain (b). Each vectorial distance between any pair of antennas corresponds to the location of one sample point. The array configuration and sampling pattern for the VLA telescope is shown in Figs. 2(a) and (c), respectively.

In addition, the resulting patterns are naturally biased towards low frequencies. In this paper, we statistically evaluate the effects of such non-random sampling on compressed sensing performance and present *SparseRI*, a practically usable method for compressed sensing reconstruction of radio interferometric measurements.

2. RELATED WORK

Many real-world applications rely on non-random sampling in the frequency domain, and compressed sensing has been successfully applied to some of these problems. One classical example is the reconstruction of a piecewise constant signal from simulated tomography measurements where the sampling pattern consists of radial lines. However, pathological cases exist where regular sampling of the Fourier domain makes reconstruction impossible [1]. For the case of magnetic resonance imaging (MRI), several types of non-random sampling have been compared [7]. While applications of compressed sensing to interferometry have already been proposed [8, 9], these works neither investigate the influence of non-random sampling, nor is the reconstruction algorithm validated on observational data.

The common image reconstruction method used in radio astronomy is the CLEAN algorithm [10], dating back to 1974. This algorithm is based on conventional least-squares energy minimization (all unknown Fourier samples are set to zero), followed by iterative deconvolution in the image domain. The deconvolution algorithm assumes that the image is sparse in the pixel domain; in the context of compressed sensing, this means that **S** is the identity matrix. Because CLEAN suffers from known instabilities [11], considerable manual work is often needed to obtain satisfactory results. Our approach, on the other hand, works well for a wide range of sources without the need for parameter tweaking or any other kind of user interaction.

New interferometric arrays like the *Square Kilometre Array* (SKA), the *Long Wavelength Array* (LWA) or the *Low Frequency Array* (LOFAR) feature a wide field of view that violates the assumption necessary to derive Eq. (3). To fully exploit the capabilities of these telescopes, automatic and more flexible algorithms are needed. With increasing parallel processing power of today's (graphics) hardware, direct evaluation of Eq. (2) comes within reach. Compressed sensing easily implements such generalizations.

3. IMPLEMENTATION AND EVALUATION

We propose the *SparseRI* algorithm, a compressed sensing method for interferometric image reconstruction that is based on the very general *SpaRSA* optimization framework [12]. In our implementation, the weight of the data term is successively increased until the consistency of the reconstructed image with the measurements reaches the observational noise level. In order to work with observational data, visibility information from interferometric arrays is resampled on a regular grid, and the measurement matrix is implemented as a Fast Fourier Transform followed by appropriate sampling. In addition to the pixel basis that is used in this paper, wavelet and total variation minimizers are also part of the proposed extension. These minimizers yield improved reconstruction results for many image types. The sparsity of natural images is imperfect (so that perfect reconstruction cannot be expected), but compressed sensing has been shown to be robust with respect to imperfect sparsity [5].

We investigate the applicability of compressed sensing to interferometric image reconstruction using the aforementioned optimization method. We statistically evaluate the influence of non-random sampling before we apply our method to actual interferometric data from the VLA telescope.

3.1. Numerical experiment

In order to evaluate the influence of random sampling on reconstruction quality, we compare sampling patterns that are generated from real or hypothetical sensor positions to randomly generated patterns. Fig. 2(a) shows the sensor positions for the VLA telescope; the corresponding sampling pattern is displayed in Fig. 2(c). From the actual sampling pattern, a randomized pattern, Fig. 2(e), is generated that inherits the large-scale frequency distribution, the number of samples and the point symmetry of the actual pattern, but does not exhibit the regularity discussed in Sect. 1. Finally, another point symmetric pattern with uniform random sample distribution, Fig. 2(g), is generated for comparison.

We simulate measurements on a radio image of the galaxy 3C31, Fig. 2(b), based on these patterns and reconstruct the image using our method, Figs. 2(d,f,h).The reconstruction results from actual, Fig. 2(d), and randomized, Fig. 2(f), patterns both provide much better visual reconstruction quality
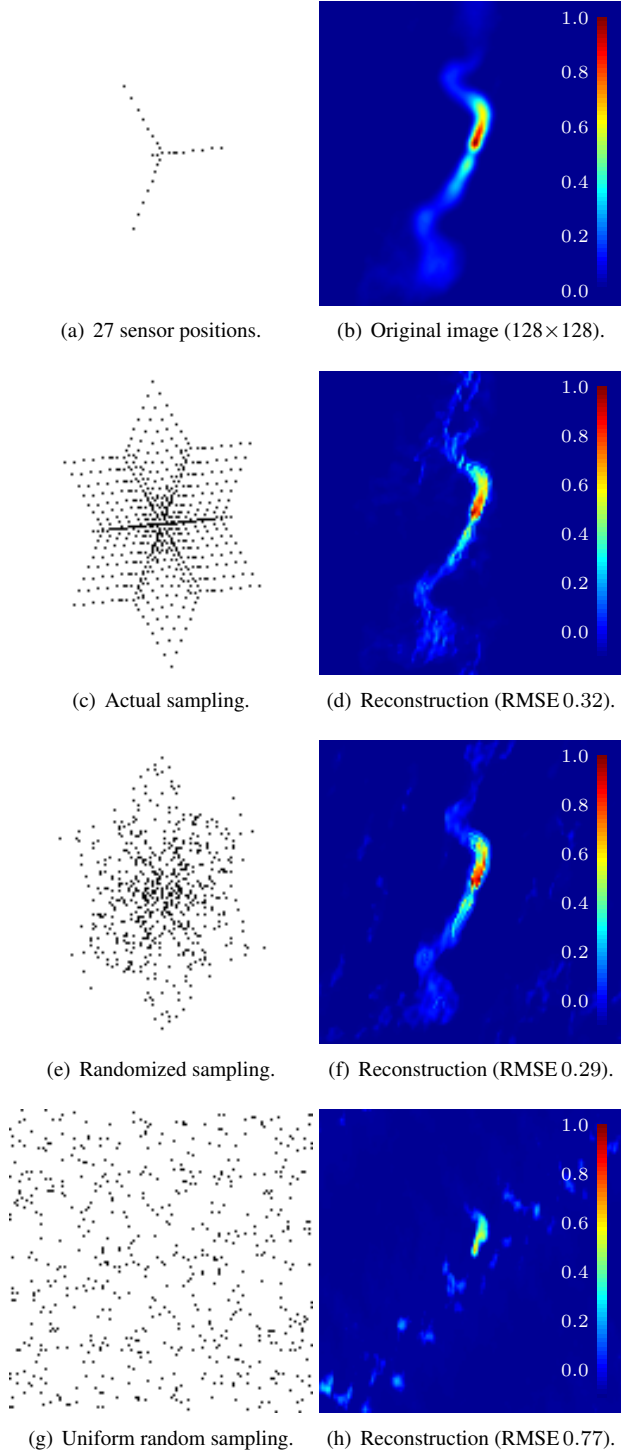
(a) 27 sensor positions.

(b) Original image (128×128).

(c) Actual sampling.

(d) Reconstruction (RMSE 0.32).

(e) Randomized sampling.

(f) Reconstruction (RMSE 0.29).

(g) Uniform random sampling.

(h) Reconstruction (RMSE 0.77).

Fig. 2. Sample reconstruction results, using SparseRI with the pixel basis, for the radio galaxy 3C31 (b) from simulated measurements. The 'D' configuration of the VLA telescope (a) was used to generate an actual sampling pattern (c), a randomized sampling pattern with the same large-scale distribution and number of samples (e), and a uniform random sampling pattern (g). (e) and (g) cannot be realized by any physical sensor configuration.
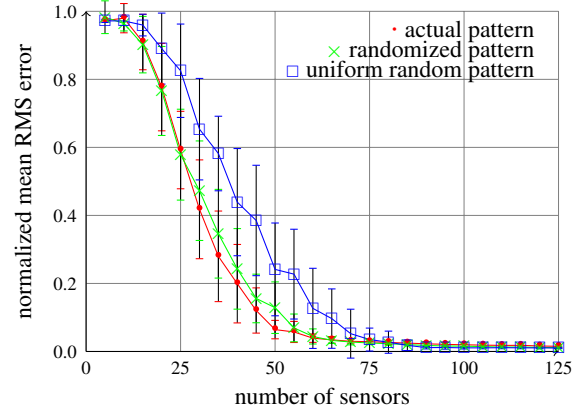

Fig. 3. Mean normalized RMS error of reconstruction results for 3C31 from simulated measurements. For each number of sensors, 28 sampling patterns were generated from random sensor positions (red). From each sampling, random patterns with the same large-scale distribution and number of sampling points were created (green). For comparison, patterns with uniform random sample distribution were used (blue). The error bars indicate one standard deviation.

and lower root mean squared error (RMSE) than the reconstruction from a uniformly, randomly sampled Fourier domain, Fig. 2(h). This observation implies that the emphasis on low frequencies induced by actual sampling is more important to reconstruction quality than randomness of the sampling. This is due to the higher energy content of low frequencies in natural images and overcompensates for the fact that some high frequency regions are not sampled at all by actual interferometric array patterns.

3.2. Statistical evaluation

Based on this initial result, we further investigate the influence of non-random and non-uniform sampling for a wide range of parameters. For different numbers of sensors, sensor positions are chosen at random, and the resulting sampling pattern is computed. Subsequently, randomized and uniform random patterns are generated as in Sect. 3.1. For each number of sensors, 28 such configurations are generated, measurements of the source image, Fig. 2(b), are simulated, and the image is reconstructed. The resulting RMSE values are shown in Fig. 3.

The result of the statistical evaluation largely supports our hypothesis. Over the entire parameter range, randomized sampling differs only minimally from actual sampling, while uniform random sampling requires more samples to attain comparable RMS error. As the number of sensors approaches the square root of the number of pixels, all kinds of sampling contain enough information to reconstruct the image almost perfectly, in accordance with CS theory.
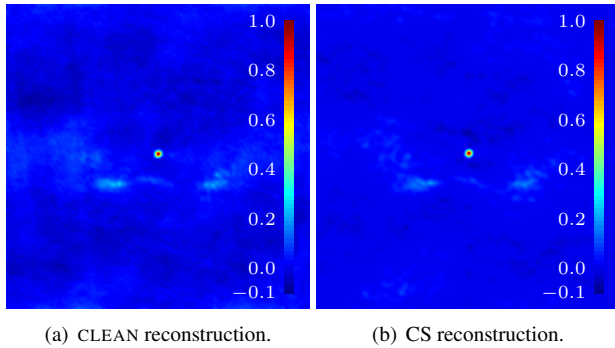
(a) CLEAN reconstruction.   (b) CS reconstruction.

Fig. 4. Reconstruction results for the radio source 3C 338 from VLA observations, using CLEAN (a) and compressed sensing (b). CLEAN suffers from striping artifacts and an uneven background. These artifacts are considerably reduced in the compressed sensing reconstruction.

3.3. Application to real data

As we have seen that the kind of non-random sampling that occurs in radio interferometric imaging does not impair the quality of image reconstruction, our method can be expected to reach the full performance of compressed sensing when applied to such measurements. In order to exemplify its practical applicability, we present reconstruction results for the radio source 3C 338 from VLA observations, Fig. 4(b).

When compared to the traditional CLEAN reconstruction result, Fig. 4(a), CS features several advantages. The background of the compressed sensing reconstruction appears more even and exhibits less regions of (unphysical) negative pixel intensities. In addition, the low amount of information contained in the data does not result in conspicuous striping artifacts. The reconstruction error is distributed more evenly across the image. The runtime of both algorithms is about two seconds on a conventional PC, but the CS algorithm leaves room for optimization and parallelization.

4. CONCLUSION

We have evaluated the use of compressed sensing for aperture synthesis imaging based on our proposed algorithm *SparseRI*. Our statistical evaluation of the influence of different sampling patterns on reconstruction quality shows that the kind of non-random sampling that occurs in interferometric measurements does not introduce deteriorating effects. Instead, the natural emphasis of low frequencies in realistic sampling increases reconstruction quality for natural images when compared to uniform random patterns. We have demonstrated the practical usability of our reconstruction method by applying it to radio interferometric observations from the VLA telescope. The comparison to traditional reconstruction algorithms shows a noticeable increase in visual image quality.

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