

ECE 140 Final Exam Solutions

Time: 3 Hours. **Instructions:** Do all four problems. You may use your text, notes and a calculator.

(20 points) 1. A two-dimensional density function is given by

$$f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} c_1 \cos(x+y)u(x)u(y), & 0 < x+y < \pi/2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the normalization constant c_1 such that $f_{\tilde{x},\tilde{y}}(x,y)$ is a valid probability density function.

(b) Find the marginal densities $f_{\tilde{x}}(x), f_{\tilde{y}}(y)$. Are x and y independent r.v.s? Why or why not?

Answer

(a)

$$\begin{aligned} & \int_0^{\pi/2} dx \int_0^{\pi/2-x} dy \cos(x+y) \\ &= \int_0^{\pi/2} dx [\sin(x+\pi/2-x) - \sin(x)] \\ &= \int_0^{\pi/2} dx [1 - \sin(x)] = \frac{\pi}{2} - [-\cos(\pi/2) + \cos(0)] \\ &= \pi/2 - 1 \\ \Rightarrow c_1 &= \frac{1}{\pi/2 - 1} = 1.7519 \end{aligned}$$

(b)

$$\begin{aligned} f_{\tilde{x}}(x) &= \int f_{\tilde{x}\tilde{y}}(x,y) dy = c_1 \int_0^{\pi/2-x} dy \cos(x+y) \\ &= c_1 \sin(x+y) \Big|_{y=0}^{\pi/2-x} \\ &= c_1(1 - \sin(x)), \quad 0 < x < \pi/2 \end{aligned}$$

$$\begin{aligned} f_{\tilde{y}}(y) &= c_1 \int_0^{\pi/2-x} dx \cos(x+y) \\ &= c_1(1 - \sin(y)), \quad 0 < y < \pi/2 \end{aligned}$$

x and y are clearly not independent, since $f(x, y) \neq f(x)f(y)$.

(20 points) 2. Let a random process be defined by

$$\tilde{x}(t) = \tilde{y} \exp(-t)u(t) + \tilde{w} \exp(-2t)u(t)$$

The random variables y and w are independent Gaussian, both with zero mean and unit variance.

(a) Find the mean function and autocorrelation function of x(t), $E\{\tilde{x}(t)\}$, $E\{\tilde{x}(t_1)\tilde{x}(t_2)\}$. Is x(t) wide-sense stationary? Explain.

(b) Find the average magnitude-squared short-time Fourier Transform of x(t), $E\{|\tilde{X}_T(\omega)|^2\}$. Recall this is

$$E\{|\tilde{X}_T(\omega)|^2\} = E\left\{\left|\int_{-T}^T \tilde{x}(t)e^{-j\omega t} dt\right|^2\right\}$$

Express your result in closed-form in terms of purely real-valued quantities. Using your result, what is the power spectral density $S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E\{|\tilde{X}_T(\omega)|^2\}$ of x(t)?

Answer

(a)

$$E\{\tilde{x}(t)\} = 0, \text{ linearity } E\{\tilde{y}\} = E\{\tilde{w}\} = 0$$

$$E\{\tilde{x}(t_1)\tilde{x}(t_2)\} = E\{(\tilde{y} \exp(-t_1)u(t_1) + \tilde{w} \exp(-2t_1)u(t_1))(\tilde{y} \exp(-t_2)u(t_2) + \tilde{w} \exp(-2t_2)u(t_2))\}$$

$$= E\{\tilde{y}^2\} \exp(-(t_1 + t_2))u(t_1)u(t_2) + E\{\tilde{w}^2\} \exp(-2(t_1 + t_2))u(t_1)u(t_2)$$

$$(\text{Note } E\{\tilde{x}\tilde{y}\} = 0)$$

$$E\{\tilde{w}^2\} = E\{\tilde{y}^2\} = 1$$

$$\Rightarrow E\{\tilde{x}(t_1)\tilde{x}(t_2)\} = \exp(-(t_1 + t_2))u(t_1)u(t_2) + \exp(-2(t_1 + t_2))u(t_1)u(t_2)$$

x(t) is not w.s.s., since $R_{xx}(t_1, t_2) \neq R_{xx}(t_1 - t_2)$.

(b)

$$\begin{aligned}
\tilde{X}_T(\omega) &= \int_{-T}^T [\tilde{x}e^{-t}u(t) + \tilde{y}e^{-2t}u(t)]e^{-j\omega t} dt \\
&= \tilde{x} \int_0^T e^{-t(1+j\omega)} dt + \tilde{y} \int_0^T e^{-t(2+j\omega)} dt \\
&= \tilde{x} \frac{1 - e^{-T(1+j\omega)}}{1 + j\omega} + \tilde{y} \frac{1 - e^{-T(2+j\omega)}}{(2 + j\omega)} \\
E\{|\tilde{X}_T(\omega)|^2\} &= E\left\{\left|\tilde{x} \frac{1 - e^{-T(1+j\omega)}}{1 + j\omega} + \tilde{y} \frac{1 - e^{-T(2+j\omega)}}{(2 + j\omega)}\right|^2\right\} \\
&= \left|\frac{1 - e^{-T(1+j\omega)}}{1 + j\omega}\right|^2 + \left|\frac{1 - e^{-T(2+j\omega)}}{2 + j\omega}\right|^2 \\
\left|\frac{1 - e^{-T(1+j\omega)}}{1 + j\omega}\right|^2 &= \frac{1 - 2e^{-T} \cos(T\omega) + e^{-2T}}{1 + \omega^2} \\
\Rightarrow \lim_{T \rightarrow \infty} E\{|\tilde{X}_T(\omega)|^2\} &= \frac{1}{1 + \omega^2} + \frac{1}{4 + \omega^2} \\
\Rightarrow S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} E\{|\tilde{X}_T(\omega)|^2\} &= 0
\end{aligned}$$

(20 points) 3. A bandpass power spectral density corresponding to a w.s.s. process $n(t)$ is defined by

$$S_m(\omega) = \begin{cases} \exp(-|\omega - 100|), & |\omega - 100| < 10 \\ \exp(-|\omega + 100|), & |\omega + 100| < 10 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the equivalent low-pass power spectral density.

(b) Find the autocorrelation function of the low-pass equivalent process $z(t)$.

Answer

(a) Note that $S_{xy}(\omega) = 0$ from the symmetry of the spectral density about $\omega_0 = 100$.

$$S_m(\omega) = \frac{1}{2} S_{xx}(\omega - \omega_0) + \frac{1}{2} S_{xx}(\omega + \omega_0)$$

$$\Rightarrow S_{xx}(\omega) = 2 \exp(-|\omega|), \quad |\omega| < 10$$

(b)

$$R_{zz}(\tau) = 2R_{xx}(\tau) + 2jR_{xy}(\tau), \quad R_{xy}(\tau) = 0$$

$$\begin{aligned} R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-10}^{10} \exp(-|\omega|\tau) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_0^{10} e^{\omega(j\tau-1)} d\omega + \frac{1}{2\pi} \int_{-10}^0 e^{j\omega(j\tau+1)} d\omega \\ &= \frac{1}{2\pi} \left[\frac{1}{j\tau-1} e^{10(j\tau-1)} - \frac{1}{j\tau-1} \right] + \frac{1}{2\pi} \left[\frac{1}{j\tau+1} - \frac{1}{j\tau+1} e^{-10(j\tau+1)} \right] \end{aligned}$$

(20 points) 4. The difference equation characterizing a discrete-time linear system driven by white noise $x(n)$ is

$$\tilde{y}(n) = .5\tilde{y}(n-1) - .04\tilde{y}(n-2) + \tilde{x}(n)$$

where $E\{\tilde{x}(n)\} = 0$, $E\{\tilde{x}(n)\tilde{x}(m)\} = \delta_{n,m}$. Assume the system is in steady-state.

(a) Find the power spectral density of the output $y(n)$ in the Z-domain, $S_{yy}(z)$.

(b) Find the autocorrelation function $R_{yy}(k)$.

Answer

(a)

$$Y(z) = .5z^{-1}Y(z) - .04z^{-2}Y(z) + X(z)$$

$$H(z) = \frac{1}{1 - .5z^{-1} + .04z^{-2}} = \frac{1}{(1 - .1z^{-1})(1 - .4z^{-1})}$$

$$S_{yy}(z) = \frac{1}{(1 - .1z^{-1})(1 - .4z^{-1})} \frac{1}{(1 - .1z)(1 - .4z)}$$

(b)

$$\begin{aligned}
S_{yy}(z) &= \frac{1}{(1-.1z^{-1})(1-.4z^{-1})} \frac{1}{(1-.1z)(1-.4z)} \\
&= \frac{A}{(1-.1z^{-1})} + \frac{B}{(1-.4z^{-1})} + \frac{C}{(1-.1z)} + \frac{D}{(1-.4z)} \\
A &= \frac{1}{(1-.4/.1)} \frac{1}{(1-.1(.1))(1-.4(.1))} = -.3507 \\
B &= \frac{1}{(1-.1/.4)} \frac{1}{(1-.1(.4))(1-.4^2)} = 1.6534 \\
R_{yy}(k) &= 1.6534(.4)^k - .3507(.1)^k, \quad k \geq 0 \\
R_{yy}(k) &= 1.6534(.4)^{|k|} - .3507(.1)^{|k|}
\end{aligned}$$