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ECE 240A Final Exam Solutions

Time: 3 hours. **Instructions**: Do all four problems. You may use your text, hardcopy notes and a calculator.

(20 points) Problem 1: An *adaptive equalizer* is a device that attempts to invert a distorting channel impulse response in a digital communications sytem. Let the input to the channel be a known data sequence $d(l) = \pm 1$ for $l = 1,2,...k$. The channel output is

$$y(k) = \sum_{m=0}^{M} f_m d(k-m) + v(k)$$

Note that $y(k)$ consists of the desired data d(k), plus *intersymbol interference* (ISI) terms of the form $\sum_{m=1}^{M} f_m d(k-m)$. The noise v(k) is i.i.d. zero-mean Gaussian with variance σ_v^2. An equalizer is an FIR filter that operates on the sequence $y(k)$ to restore the data d(k). The equalizer attempts to remove the ISI by minimizing the following cost function.

$$J = \sum_{l=1}^{k} \left[d(l) - \sum_{n=0}^{N-1} w_n y(l-n) \right]^2$$

The problem is to determine the equalizer coefficients $\{w(0), w(1)..w(N-1)\}$ that minimize J.

(a) Write down the equivalent measurement matrix **H**, measurement vector **z**, and parameter vector θ for the equalizer problem in batch form using a vector-matrix equation. Thus the above J should be equivalent to $J = \| \mathbf{z} - \mathbf{H}\theta \|^2$. Assume that $y(l) = 0,\ l < 0$.

(b) The scalar RLS algorithm uses the measurement sequence $z(k) = \mathbf{h}(k)^T \theta + v(k)$ where θ is the parameter vector from part (a). Define the quantities $z(k), \mathbf{h}(k), \theta$ in terms of the channel output $y(k)$, data symbols $d(k)$ and equalizer coefficients w_n, so that RLS will recursively minimize J.

Answer:

(a)

$$\underbrace{\begin{bmatrix} d(k) \\ d(k-1) \\ \vdots \\ d(1) \end{bmatrix}}_{\mathbf{z}} = \underbrace{\begin{bmatrix} y(k) & y(k-1) & & y(k-N+1) \\ y(k-1) & & & y(k-N) \\ & & & \\ y(1) & y(0) & & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} w_0 \\ w_1 \\ \\ w_{N-1} \end{bmatrix}}_{\theta} + \mathbf{v}$$

(b) For RLS updates, the model is

$$z(k) = d(k) = \underbrace{[y(k)\ \ y(k-1)\ \ y(k-N+1)]}_{\mathbf{h}(k)^T} \underbrace{\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}}_{\theta} + v(k)$$

(20 points) Problem 2: Consider a MIMO two transmit/two receive antenna system where the received signal at time n is given by

$$\mathbf{r}(n) = \mathbf{H}\mathbf{b}(n) + \mathbf{n}(n)$$

where $\mathbf{n}(n)$ is i.i.d. zero-mean white Gaussian with covariance matrix $E\{\mathbf{n}(n)\mathbf{n}(k)^T\} = \sigma_n^2 \mathbf{I}\delta_{n,k}$. The transmitted data sequence $\mathbf{b}(n) \in R^2$ is approximated as i.i.d. white Gaussian, with $E\{\mathbf{b}(n)\} = \mathbf{0}$, $E\{\mathbf{b}(n)\mathbf{b}(n)^T\} = \mathbf{I}$. The MIMO antenna gain matrix is defined by

$$\mathbf{H} = \begin{bmatrix} 1 & 1/4 \\ 3/4 & 1 \end{bmatrix}$$

(a) Find the minimum mean-square error estimate $E\{\mathbf{b}(n) \mid \mathbf{r}(n)\}$ in as compact a form as possible using the above form of $\mathbf{H}$, with as many quantities expressed in numerical form as possible.

(b) Find the limiting form of $E\{\mathbf{b}(n) \mid \mathbf{r}(n)\}$ as $\sigma_n^2 \to 0$, again in as compact a form as possible solving for all quantities numerically where feasible.

Answer:

(a)

$$E\{\mathbf{b} \mid \mathbf{r}\} = E\{\mathbf{b}\} + E\{(\mathbf{b} - E\{\mathbf{b}\})(\mathbf{r} - E\{\mathbf{r}\})^T\} E\{(\mathbf{r} - E\{\mathbf{r}\})(\mathbf{r} - E\{\mathbf{r}\})^T\}^{-1} (\mathbf{r} - E\{\mathbf{r}\})$$

$$= \mathbf{0} + \mathbf{H}^T \left[\mathbf{H}\mathbf{H}^T \ \ + \sigma_n^2 \mathbf{I}\right]^{-1} \mathbf{r} = \left[\mathbf{H}^T\mathbf{H} + \sigma_n^2 \mathbf{I}\right]^{-1} \mathbf{H}^T \mathbf{r}$$

The matrix inverse using the second form is

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 1 & 3/4 \\ 1/4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/4 \\ 3/4 & 1 \end{bmatrix} = \begin{bmatrix} 25/16 & 1 \\ 1 & 17/16 \end{bmatrix}$$

$$\Rightarrow \left[\mathbf{H}^T\mathbf{H} + \sigma_n^2\mathbf{I}\right] = \begin{bmatrix} 25/16 + \sigma_n{}^2 & 1 \\ 1 & 17/16 + \sigma_n^2 \end{bmatrix}$$

$$\Rightarrow \left[\mathbf{H}^T\mathbf{H} + \sigma_n^2\mathbf{I}\right]^{-1} = \frac{1}{(25/16 + \sigma_n^2)(17/16 + \sigma_n^2) - 1}\begin{bmatrix} 17/16 + \sigma_n{}^2 & -1 \\ -1 & 25/16 + \sigma_n^2 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{b}}_{MS} = \frac{1}{(25/16 + \sigma_n^2)(17/16 + \sigma_n^2) - 1}\begin{bmatrix} 17/16 + \sigma_n{}^2 & -1 \\ -1 & 25/16 + \sigma_n^2 \end{bmatrix}\begin{bmatrix} 1 & 3/4 \\ 1/4 & 1 \end{bmatrix}\mathbf{r}$$

(b) As $\sigma_n^2 \to 0$,

$$\hat{b}_{MS} \to \left[\mathbf{H}^T\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{r} = \frac{1}{.66}\begin{bmatrix} 17/16 & -1 \\ -1 & 25/16 \end{bmatrix}\begin{bmatrix} 1 & 3/4 \\ 1/4 & 1 \end{bmatrix}\mathbf{r}$$

$(\textit{easier},\ \textit{note}\ \mathbf{H}\ \textit{is invertible})$

$$= \mathbf{H}^{-1}\mathbf{r} = \frac{1}{13/16}\begin{bmatrix} 1 & -1/4 \\ -3/4 & 1 \end{bmatrix}\mathbf{r} = \begin{bmatrix} 1.23 & -.3077 \\ -.9231 & 1.23 \end{bmatrix}\mathbf{r}$$

(20 points) Problem 3: A scalar measurement sequence is given by

$$z(k) = \alpha^k \exp(-\theta^2) + v(k)$$

where θ is a scalar parameter confined to $\theta \in [0, \infty)$ and $v(k)$ is a zero-mean independent white Gaussian noise sequence, with $E\{v(k)^2\} = \sigma_v{}^2$.

(a) Find the maximum-likelihood estimate in as compact a form as possible for θ using measurements $\{z(0), z(1)...z(k)\}$.

(b) Find the Cramer-Rao bound for estimating θ using $\{z(0), z(1)...z(k)\}$ in as compact a form as possible.

Answer:
(a)

$$\hat{\theta}_{ML} = \arg \min_{\theta} \sum_{l=1}^{k} \left[z(l) - \alpha^l \exp(-\theta^2) \right]^2$$

$$\frac{d}{d\theta} \sum_{l=1}^{k} \left[z(l) - \alpha^l \exp(-\theta^2) \right]^2 = \sum_{l=1}^{k} \left(2\alpha^l \theta \right) \exp(-\theta^2) \left[z(l) - \alpha^l \exp(-\theta^2) \right]$$

$$\Rightarrow \sum_{l=0}^{k} z(l)\alpha^l = \sum_{l=0}^{k} \alpha^{2l} \exp(-\theta^2)$$

$$\exp(-\theta^2) = \frac{(1-\alpha^2)\sum_{l=0}^{k} \alpha^l z(l)}{\left(1-\alpha^{2(k+1)}\right)}$$

$$\Rightarrow \hat{\theta}_{ML} = \sqrt{\log\left(\frac{1-\alpha^{2(k+1)}}{(1-\alpha^2)\sum_{l=0}^{k} \alpha^l z(l)} \right)}$$

$$\hat{\theta}_{ML} \to \infty \ \textit{if} \ \sum_{l=0}^{k} \alpha^l z(l) < 0$$

(b)

$$J^{-1} = \frac{\sigma_v^2}{\sum_{l=0}^{k} \left(\frac{d}{d\theta} \left(\alpha^l \exp(-\theta^2) \right) \right)^2} = \frac{\sigma_v^2}{\sum_{l=0}^{k} (2\theta)^2 \exp(-2\theta^2) \alpha^{2l}}$$

$$= \frac{(1-\alpha^2)\sigma_v^2 \left(\exp(2\theta^2)/(4\theta^2) \right)}{(1-\alpha^{2(k+1)})}$$

(20 points) Problem 4: A scalar state variable propagates as

$$x(k+1) = .75x(k) + w(k)$$

where $w(k)$ is i.i.d. Gaussian zero mean with $E\{w(k)^2\} = 1$. The measurement sequence is given by

$$\mathbf{z}(k) = \begin{bmatrix} 1 \\ .5 \\ .25 \end{bmatrix} x(k) + \mathbf{v}(k)$$

where $\mathbf{v}(k)$ is i.i.d. Gaussian mean zero and covariance matrix $E\{\mathbf{v}(k)\mathbf{v}(k)^T\} = \mathbf{I}$.

(a) Write down the Kalman filter equations in a form that does not require matrix inversion for estimating $x(k)$. Give numerical values for quantities where possible and express the equations in as compact a form as possible.

(b) Find the steady-state prediction covariance $p(k+1|k)$ using your answer to (a).

Answer

(a)

$$p(k|k)^{-1} = p(k|k-1)^{-1} + \mathbf{H}(k)^T \mathbf{R}^{-1}\mathbf{H}(k)$$

$$\Rightarrow \frac{1}{p(k|k)} = \frac{1}{p(k|k-1)} + [1\ .5\ .25]\mathbf{I}\begin{bmatrix}1\\ .5\\ .25\end{bmatrix} = \frac{1}{p(k|k-1)} + 1.3125$$

$$\hat{x}(k|k) = \hat{x}(k|k-1) + p(k|k)\mathbf{H}(k)^T \mathbf{R}^{-1}[\mathbf{z}(k) - \mathbf{H}(k)\hat{x}(k|k-1)]$$

$$= \hat{x}(k|k-1) + p(k|k)[1\ .5\ \ .25]\left[\mathbf{z}(k) - \begin{bmatrix}1\\ .5\\ .25\end{bmatrix}\hat{x}(k|k-1)\right]$$

$$\hat{x}(k+1|k) = .75\hat{x}(k|k)$$

$$p(k+1|k) = .75^2 p(k|k) + \sigma_w^2 = .5625p(k|k) + 1$$

(b)

$$p(k+1|k) = .5625p(k|k) + 1 = .5625\frac{1}{\frac{1}{p(k|k-1)} + 1.3125} + 1$$

$$\Rightarrow \bar{p} = .5625\frac{\bar{p}}{1.3125\bar{p} + 1} + 1$$

$$\Rightarrow 1.3125\bar{p}^2 + \bar{p} - .5625\bar{p} - 1.3125\bar{p} - 1 = 0$$

$$1.3125\bar{p}^2 - .875\bar{p} - 1 = 0$$

$$\bar{p} = \frac{.875 \pm \sqrt{.7656 + 5.25}}{2.625} = 1.2677\ (pos.\ root)$$