

# The Thomas precession factor in spin–orbit interaction

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The origin of the Thomas factor 1/2 in the spin–orbit Hamiltonian can be understood by considering the case of a classical electron moving in crossed electric and magnetic fields chosen such that the electric Coulomb force is balanced by the magnetic Lorentz force. © 2004 American Association of

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## I. INTRODUCTION: THE PROBLEM

It is well known that the spin–orbit Hamiltonian differs by a factor 1/2—the Thomas precession factor—from what one might expect from a naive Lorentz transformation that assumes a uniform straight-line motion of the electron. The naive argument runs as follows.

When an electron moves with a velocity  $\mathbf{v}$  through space in the presence of an electric field  $\mathbf{E}$ , the electron will see, in its own frame of reference, a Lorentz-transformed magnetic field  $\mathbf{B}'$  given by the familiar expression

$$\mathbf{B}' = \frac{(\mathbf{E} \times \mathbf{v})/c^2}{\sqrt{1 - (v/c)^2}} \rightarrow_{v \ll c} \frac{(\mathbf{E} \times \mathbf{v})}{c^2}, \quad (1)$$

where  $c$  is the speed of light, and the limit  $v \ll c$  has been taken. It is this Lorentz-transformed magnetic field that is supposedly seen by the electron magnetic moment. This argument suggests that in the presence of electric fields, the magnetic field  $\mathbf{B}$  in the spin Hamiltonian should be replaced by  $\mathbf{B} + (\mathbf{E} \times \hat{\mathbf{v}})/c^2$ , where  $\hat{\mathbf{v}}$  is the velocity operator of the electron. This conclusion does not agree with observed atomic spectra.

Historically, this discrepancy provided a major puzzle,<sup>1</sup> until it was pointed out by Thomas<sup>2</sup> that this argument overlooks a second relativistic effect that is less widely known, but is of the same order of magnitude: An electric field with a component perpendicular to the electron velocity causes an additional acceleration of the electron perpendicular to its instantaneous velocity, leading to a curved electron trajectory. In essence, the electron moves in a rotating frame of reference, implying an additional precession of the electron, called the Thomas precession. A detailed treatment is given, for example, by Jackson,<sup>3</sup> where it is shown that this effect changes the interaction of the moving electron spin with the electric field, and that the net result is that the spin–orbit interaction is cut in half, as if the magnetic field seen by the electron has only one-half the value in Eq. (1),

$$\mathbf{B}' = \frac{1}{2} \frac{(\mathbf{E} \times \mathbf{v})}{c^2}. \quad (2)$$

It is this modified result that is in agreement with experiment.

A rigorous derivation<sup>3</sup> of this (classical) result requires knowledge of some aspects of relativistic kinematics, which, although not difficult, are unfamiliar to most students. In a course on relativistic quantum mechanics, the result (2) can be derived directly from the Dirac equation, without reference to classical relativistic kinematics. But in a nonrelativistic

QM course, the instructor is likely compelled to use an unsatisfactory “it can be shown that” argument, which is, in fact, the approach taken in most textbooks.

In my own 1994 textbook,<sup>4</sup> I tried to go beyond that approach by considering the case where the electron is forced to move along a straight line, by adding a magnetic field in the rest frame such that the magnetic Lorentz force would exactly balance the electric force. In this case, the Thomas factor of 1/2 was indeed obtained, but the argument—especially its extension to more general cases—lacked rigor. Moreover, because it was published outside the mainstream physics literature, it has remained largely unknown to potentially interested readers inside that mainstream. The purpose of the present note is to put the argument on a more rigorous basis, and to do so in a more readily accessible manner.

## II. CROSSED-FIELD TREATMENT

From our perspective, the central aspect of the standard Lorentz transform (1) is the following: The transform  $\mathbf{E}_{\text{rest}} \Rightarrow \mathbf{B}_{\text{electron}}$  must be linear in  $\mathbf{E}$ , and  $\mathbf{E}$  can occur only in the combination  $\mathbf{E} \times \mathbf{v}$ . This combination reflects the fact that only the component of  $\mathbf{E}$  perpendicular to the velocity  $\mathbf{v}$  can play a role, and that the resulting  $\mathbf{B}$  field must be perpendicular to both  $\mathbf{E}$  and  $\mathbf{v}$ .

In the absence of any specific arguments to the contrary, we would expect that this proportionality to  $\mathbf{E} \times \mathbf{v}$  carries over to the case of a curved trajectory. (For example, a component of  $\mathbf{E}$  parallel to  $\mathbf{v}$  would not contribute to a curved trajectory and to the accompanying rotation of the electron frame of reference.) If one accepts this argument, it follows that  $\mathbf{B}' \equiv \mathbf{B}_{\text{electron}}$  should be given by an expression of the general form

$$\mathbf{B}' = \alpha \frac{(\mathbf{E} \times \mathbf{v})}{c^2}, \quad (3)$$

with some scalar proportionality factor  $\alpha$ , which may still depend on the velocity, but which cannot depend on either  $\mathbf{E}$  or on any magnetic field  $\mathbf{B}$  in the rest frame. If there also is a magnetic field  $\mathbf{B}$  present in the rest frame, Eq. (3) should be generalized to

$$\mathbf{B}' = \alpha \frac{(\mathbf{E} \times \mathbf{v})}{c^2} + \beta \mathbf{B}, \quad (4)$$

where  $\beta$  is another constant subject to the same constraints as  $\alpha$ .

To determine the constants  $\alpha$  and  $\beta$ , we consider not simply the Lorentz transformation of a pure electric field  $\mathbf{E}$ , but of a specific combination of electric and magnetic fields such that

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}, \quad (5a)$$

where  $\mathbf{v}$  is the velocity of the electron, and  $\mathbf{B}$  is chosen perpendicular to  $\mathbf{v}$ . For this combination, the electric Coulomb force and the magnetic Lorentz force on the electron cancel, and the electron will move along a straight line. But in this case the simple Lorentz transformation for uniform straight-line motion is rigorously applicable, without having to worry about a rotating frame of reference.

Without loss of generality, we may choose a Cartesian coordinate system such that the velocity is in the  $x$  direction, the magnetic field is in the  $z$  direction; and the electric field is in the  $y$  direction,

$$E_y = v_x B_z. \quad (5b)$$

If this combination of the two fields is Lorentz-transformed into the uniformly moving frame of the electron, the magnetic field  $B'_z$  in that frame is

$$B'_z = \frac{B_z - E_y v_x / c^2}{\sqrt{1 - (v_x/c)^2}}. \quad (6)$$

Inasmuch as we are interested only in the limit  $v \ll c$ , we may expand the right-hand side of Eq. (6) in powers of  $v_x$ . The result may be written as

$$B'_z = B_z + \frac{B_z v_x^2}{c^2} \left[ \frac{1}{2} + \frac{3}{8} \left( \frac{v_x}{c} \right)^2 \right] - \frac{E_y v_x}{c^2} \left[ 1 + \frac{1}{2} \left( \frac{v_x}{c} \right)^2 \right] + \dots, \quad (7)$$

where the dots represent omitted terms of order higher than  $v_x^4$ . But under the condition (5b), we have  $B_z v_x^2 = E_y v_x$ , and Eq. (7) may be simplified by combining terms to read

$$B'_z = B_z - \frac{E_y v_x}{c^2} \left[ \frac{1}{2} + \frac{1}{8} \left( \frac{v_x}{c} \right)^2 \right] + \dots. \quad (8)$$

In the limit  $v \ll c$ , we obtain

$$B'_z = B_z - \frac{1}{2} \frac{E_y v_x}{c^2}. \quad (9a)$$

In three-dimensional vector form,

$$\mathbf{B}' = \mathbf{B} + \frac{1}{2} \frac{\mathbf{E} \times \mathbf{v}}{c^2}, \quad (9b)$$

which is of the required form (4), with  $\alpha = 1/2$  and  $\beta = 1$ .

Equation (9) is the magnetic field that is seen by the intrinsic magnetic moment of the electron. It determines the potential energy of that moment in the presence of both electric and magnetic fields for our specific combination of crossed fields, up to terms linear in  $\mathbf{E} \times \mathbf{v}$ . The leading  $\mathbf{B}$  term already is included in the nonrelativistic spin Hamiltonian; the remainder is the first-order correction due to the spin-orbit interaction. Note that it contains the Thomas factor  $1/2$ , in agreement with Eq. (2).

The rest is straightforward. Our argument suggests that the magnetic field  $\mathbf{B}$  in the spin Hamiltonian should be replaced by an operator that is equivalent to Eq. (9b).

Following standard textbook arguments, we obtain the familiar spin-orbit contribution to the Hamiltonian,

$$\hat{H}_{\text{SO}} = \mu_e \hat{\sigma} \cdot \hat{\mathbf{B}}' = \frac{\mu_e}{2c^2} \hat{\sigma} \cdot (\mathbf{E} \times \hat{\mathbf{v}}), \quad (10)$$

where  $\mu_e$  is the intrinsic magnetic moment of the electron, and  $\hat{\mathbf{v}}$  is now the *operator* for the velocity.

### III. DISCUSSION

The central assumption in our derivation was that the proportionality of  $\mathbf{B}'$  to  $\mathbf{E} \times \mathbf{v}$  carries over to a rotating frame of reference. This proportionality is, of course, an exact result. But in a paper explicitly dedicated to the classroom didactics of the Thomas factor without invoking less well-known aspects of relativistic kinematics, it is probably best to treat it as an eminently plausible assumption.

I close with a comment on our retention of the  $B_z v_x^2$  term in Eq. (7), albeit converted into a  $E_y v_x$  term. This term replaces rotating-frame corrections in the case of a pure electric field. Neglecting it would be exactly equivalent to neglecting the effects of a rotating frame of reference for a more general choice of fields.

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<sup>1</sup>See, for example, M. Jammer, *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966), p. 152.

<sup>2</sup>L. H. Thomas, "The motion of the spinning electron," *Nature* (London) **117**, 514 (1926).

<sup>3</sup>J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Sec. 11.5.

<sup>4</sup>H. Kroemer, *Quantum Mechanics* (Prentice-Hall, Englewood Cliffs, 1994), Sec. 12.4.1.