Structural Properties of Cayley Digraphs with Applications to Mesh and Pruned Torus Interconnection Networks

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Abstract – Despite numerous interconnection schemes proposed for distributed multicomputing, systematic studies of classes of interprocessor networks, that offer speed-cost tradeoffs over a wide range, have been few and far in between. A notable exception is the study of Cayley graphs that model a wide array of symmetric networks of theoretical and practical interest. Properties established for all, or for certain subclasses of, Cayley graphs are extremely useful in view of their wide applicability. In this paper, we obtain a number of new relationships between Cayley (di)graphs and their subgraphs and coset graphs with respect to subgroups, focusing in particular on homomorphism between them and on relating their internode distances and diameters. We discuss applications of these results to well-known and useful interconnection networks such as hexagonal and honeycomb meshes as well as certain classes of pruned tori.

Keywords – Cayley digraph, Cellular network, Coset graph, Distributed system, Homomorphism, Interconnection network, Internode distance, Diameter, Parallel processing.

List of key notation

Unless explicitly specified, all graphs in this paper are undirected graphs.

$\bullet \le \bullet$	Subgroup relationship
$\bullet \lhd \bullet$	Normal subgroup relationship
$\bullet / \bullet$	Set of (right) cosets
$\bullet \times \bullet$	Graph or set cross-product
$(\bullet,\bullet)$	Edge
$\langle \bullet \rangle$	Group specified by its generators
$\bullet^{(i)}$	The symbol "•" repeated *i* times
$\bullet^{-1}$	Inverse of group element
$\rightarrow$	Mapping
$\cong$	Isomorphic to
Γ, Δ, Σ	Graphs or digraphs
ϕ	Homomorphism
1	Identity element of a group
Aut( )	Automorphism group
Cay( )	Cayley graph
Cos( )	Coset graph
dis( )	Distance function
D( )	Diameter of a graph
E( )	Edge set of a graph
G, *H*	Groups
K, *N*	Subgroups
S	Generator set, subset of *G*
V( )	Vertex set of a graph
wr	Wreath product
Z	Infinite cyclic group
Z_q	Cyclic group of order *q*
Z_d^q	Elem. abelian *d*-group of order d^q

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1. Introduction

Since the emergence of parallel processing in the 1960s, numerous networks have been proposed for connecting the processing nodes in distributed multicomputers, to the extent that a "sea of interconnection networks" is said to exist [10]. An implication of this terminology is that new networks, or designers trying to make sense of the wide array of options available to them, might drown in this sea. The study of one-of-a-kind networks, while useful in the sense of broadening the designers' repertoire, may be deemed counterproductive if various networks are not tied together by means of realistic comparative evaluations. It is for these reasons that classes of networks offering cost-performance tradeoffs within a wide range are extremely useful, because membership in the same class allows the application of theoretical results to make the task of performance evaluation both tractable and meaningful.

The fact that Cayley (di)graphs and coset graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation, is widely acknowledged [1], [2], [4], [6]. Many well-known interconnection networks are Cayley (di)graphs or coset graphs. For example, hypercube (binary q-cube), butterfly, and cube-connected cycles networks are Cayley graphs, while de Bruijn and shuffle-exchange networks are coset graphs [4], [13]. Other, lesser known, Cayley (di)graphs have been described in the technical literature and still others await discovery. Cayley graphs also play an important role in studies relating the three network parameters of size, node degree, and diameter. A recent review by Miller and Siran [8] reveals that roughly one-half of the largest known undirected graphs, with node degree up to 16 and diameter up to 10, are derived from Cayley graphs.

Much work on interconnection networks can be categorized as ad hoc design and evaluation. Typically, a new interconnection scheme is suggested and shown to be superior to some previously studied network(s) with respect to one or more performance or complexity attributes. Whereas Cayley (di)graphs have been used to explain and unify interconnection networks with many ensuing benefits [6], much work remains to be done. As suggested by Heydemann [4], general theorems are lacking for Cayley digraphs and more group theory has to be exploited to find properties of Cayley digraphs.

In this paper, we explore the relationships between Cayley (di)graphs and their subgraphs and coset graphs with respect to subgroups, focusing in particular on homomorphism between them and on relationships between their internode distances and diameters. We provide several applications of these results to well-known and useful interconnection networks such as hexagonal and honeycomb meshes as well as certain classes of pruned tori.

Before proceeding further, we introduce some definitions and notations related to (di)graphs, Cayley (di)graphs in particular, and interconnection networks. For more definitions and mathematical results on graphs and groups we refer the reader to [3], for instance, and on interconnection networks to [7], [10]. Unless noted otherwise, all graphs in this paper are undirected graphs.

A digraph $\Gamma = (V, E)$ is defined by a set V of vertices and a set E of arcs or directed edges. The set E is a subset of elements (u, v) of $V \times V$. If the subset E is symmetric, that is, $(u, v) \in E$ implies $(v, u) \in E$, we identify two opposite arcs (u, v) and (v, u) by the undirected edge (u, v). Because we deal primarily with undirected graphs in this paper, no problem arises from using the same notation (u, v) for a directed arc from u to v or an undirected edge between u and v.

Let G be a (possibly infinite) group and S a subset of G. The subset S is said to be a generating set for G, and the elements of S are called generators of G, if every element of G can be expressed as a finite product of their powers. We also say that G is generated by S. The Cayley digraph of the group G and the subset S, denoted by $Cay(G, S)$, has vertices that are elements of G and arcs that are ordered pairs (g, gs) for $g \in G$, $s \in S$. If S is a generating set of G, we say that $Cay(G, S)$ is the Cayley digraph of G generated by S. When $1 \notin S$ (1 is the identity element of G) and $S = S^{-1}$, the graph $Cay(G, S)$ is a simple graph.

Assume that Γ and Σ are two digraphs. The mapping ϕ of $V(\Gamma)$ to $V(\Sigma)$ is a homomorphism from Γ to Σ if for any $(u, v) \in E(\Gamma)$ we have $(\phi(u), \phi(v)) \in E(\Sigma)$. In particular, if ϕ is a bijection such that both ϕ and the inverse of ϕ are homomorphisms, then ϕ is called an isomorphism of Γ to Σ. Let G be a (possibly infinite) group and S a subset of G. Assume that K is a subgroup of G (denoted as $K \le G$). Let ${}^{G}/_{K}$ denote the set of the right cosets of K in G. The (right) coset graph of G with respect to subgroup K and subset S, denoted by $Cos(G, K, S)$, is the digraph with vertex set ${}^{G}/_{K}$ such that there exists an arc (Kg, Kg') if and only if there exists $s \in S$ and $Kgs = Kg'$.

The following basic theorem, which can be easily proven, is helpful in establishing some of our subsequent results [14].

Theorem 1. For $g \in G$, $S \subseteq G$, and $K \le G$, the mapping $\phi: g \to Kg$ is a homomorphism from $Cay(G, S)$ to $Cos(G, K, S)$.

2. An Inequality for Diameter

For any digraph Ω, $D(\Omega)$ denotes the diameter of Ω, defined as the longest distance between any pair of vertices in Ω. The diameter of a network is important because it determines the worst-case communication latency. Additionally, in symmetric networks, the diameter is intimately related to the average internode distance, thus indirectly dictating the average communication performance as well for the class of networks that are of interest in this paper. Similar to Theorem 2 in [13], we have the following result.

Theorem 2. Assume that G is a finite group, $K \leq G$, $\Gamma = Cay(G, S)$, and $\Delta = Cos(G, K, S)$ for some generating set S of G, and let $D(\Gamma_K)$ denote the longest distance between vertices of K in Γ. Then, we have $D(\Gamma) \leq D(\Delta) + D(\Gamma_K)$.

Proof. Let u be any element of G. We consider the distance $dis(u, 1)$ between u and the identity element 1. Given that S is a generating set of G, we may assume that $Ku = Ks_1s_2\ldots s_t$ for $s_1, s_2, \ldots, s_t \in S$. Thus $u = ks_1s_2\ldots s_t$ for some $k \in K$. Since $D(\Delta)$ is the longest distance between pairs of vertices in Δ, we may assume that $t \leq D(\Delta)$. On the other hand, we have $dis(k, 1) \leq D(\Gamma_K)$ according to the definition of $D(\Gamma_K)$. Hence, we obtain $dis(u, 1) \leq t + dis(k, 1) \leq D(\Delta) + D(\Gamma_K)$. This leads to the desired conclusion $D(\Gamma) \leq D(\Delta) + D(\Gamma_K)$. ■

We can apply Theorem 2 to some well-known interconnection networks. Although many results on these interconnection networks are known, the unified treatment is still beneficial.

Example1. Diameter of hypercube network. We know that the hypercube $Q_q = Cay(Z_2^q, S)$, where $S = \{0^{(i-1)}10^{(q-i)} \mid i = 1, \ldots, q\}$. Let $K = Z_2$. Then we have $\Delta = Cos(Z_2^q, Z_2, S)$, leading to $D(Q_q) \leq D(\Delta) + 1$ by Theorem 2. But $\Delta \cong Q_{q-1}$ and thus $D(\Delta) \leq q - 1$ by induction. Therefore we obtain $D(Q_q) \leq q$. Since $dis(0^{(q)}, 1^{(q)}) = q$, we have $D(Q_q) = q$. ■

Example 2. Relating the butterfly network BF_q to the de Bruijn network DB_2^q. Let $N = Z_2^q$ and $K = Z_q$. Then, $G = Z_2 \; wr \; Z_q$ is a semidirect product of N by K. Assuming $S = \{0^{(q)}1, 0^{(q-1)}11\}$, from [13] we have $\Gamma = Cay(G, S) = BF_q$ and $\Delta = Cos(G, K, S) = DB_2^q$. Given that $D(DB_2^q) = q$, we obtain the inequality $D(BF_q) \leq q + \lfloor q/2 \rfloor$ by Theorem 2. In fact, it is readily verified that $D(BF_q) = q + \lfloor q/2 \rfloor$. ■

3. Hexagonal Torus Networks

Let $G = Z \times Z$, where Z is the infinite cyclic group of integers, and consider $\Gamma = Cay(G, S)$ with $S = \{(\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)\}$. It is evident that Γ is isomorphic to the hexagonal mesh network [9], [12]. Figure 1 depicts a small part of an infinite hexagonal mesh in which the six neighbors of the center node (0, 0) are shown. A finite hexagonal mesh is obtained by simply using the same connectivity rules for a finite subset of the nodes located within a regular boundary (often a rectangle or hexagon). In the latter case, wraparound links are sometimes provided to keep the node degree uniformly equal to 6. In this paper, we deal mainly with hexagonal torus networks.

Let $H = Z_l \times Z_k$, where Z_l and Z_k are cyclic groups of orders l and k respectively (l and k are both positive integers). Assume that S is defined as in the preceding paragraph. Then $\Delta = Cay(H, S)$ is the hexagonal torus of order lk. Let $K = \langle l \rangle \times \langle k \rangle$. Then $\Delta \cong \mathrm{Cos}(Z \times Z, K, S)$ and so the hexagonal torus is a homomorphic image of the infinite hexagonal mesh according to Theorem 1. Using the results on infinite hexagonal meshes, we may deal with problems on hexagonal tori which are generally more difficult. Let Δ be defined as above. Then we have the following result.

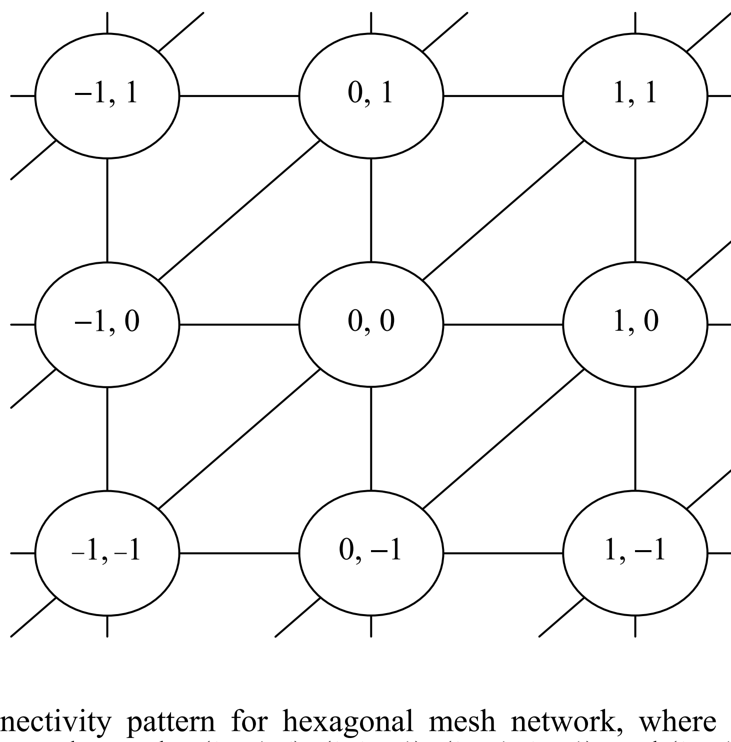

Fig. 1. Connectivity pattern for hexagonal mesh network, where node (i, j) is connected to nodes $(i \pm 1, j)$, $(i, j \pm 1)$, $(i + 1, j + 1)$, and $(i - 1, j - 1)$.

Proposition 1. For the hexagonal torus Δ of order lk with integers a and b, $0 \le a < l$, $0 \le b < k$, we have $dis((0, 0), (a, b)) = min(max(a, b), max(l - a, k - b), l - a + b, k + a - b)$.

Proof. For $(a, b) \in H = Z_l \times Z_k$, we have $(a, b) = (a - l, b - k) = (a - l, b) = (a, b - k)$. According to Proposition 2 in [14], we know that in the hexagonal mesh Γ, $dis((0, 0), (u, v))$ is given by $max(|u|, |v|)$ if u and v have the same sign and by $|u| + |v|$ otherwise. Hence we have in Γ, $dis((0, 0), (a, b)) = max(a, b)$, $dis((0, 0), (a - l, b - k)) = max(l - a, k - b)$, $dis((0, 0), (a - l, b)) = l - a + b$, and $dis((0, 0), (a, b - k)) = k + a - b$. Thus in the hexagonal torus Δ, we have $dis((0, 0), (a, b)) = min(max(a, b), max(l - a, k - b), l - a + b, k + a - b)$. ■

Remark 1. We have been unable to obtain a formula for the diameter of the hexagonal torus Δ as a function of l and k. This constitutes a seemingly difficult open problem. ■

4. Structure of Pruned 3D Tori

Let G be a (possibly infinite) group and S a subset of G and consider the problem of constructing a group G'' and its generating set S'' such that $G'' = G$ as sets and $S'' \subseteq S$, and a homomorphism $\phi: \Gamma'' \to \Gamma$, where $\Gamma = Cay(G, S)$ and $\Gamma'' = Cay(G'', S'')$. It is shown in [14] that a number of pruning schemes, including the one studied in [11], are equivalent to the construction above. Pruning of interconnection networks constitutes a way of obtaining variants with lower implementation cost, and greater scalability [5]. If pruning is done with care, and in a systematic fashion, many of the desirable properties of the original (unpruned) network, including (node, edge) symmetry and regularity, can be maintained while reducing both the node degree and wiring density. We give new proofs of the construction above in the following examples.

Example 3. Pruned three-dimensional toroidal network T_1 of [5]. Let $G = (\langle a\rangle \langle b\rangle)\langle c\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^k = b^k = c^k = 1$, $ab = ba$, $c^{-1}ac = b^{-1}$, $c^{-1}bc = a^{-1}$. Here, k is even. Thus the group $\langle a\rangle \langle b\rangle = \langle a, b\rangle$ is a direct product of $\langle a\rangle$ and $\langle b\rangle$, and G is a semidirect product of $\langle a, b\rangle$ by $\langle c\rangle$. Let $S = \{a, a^{-1}, c, c^{-1}\}$ and $\Delta_1 = Cay(G, S)$. We now prove that Δ_1 is isomorphic to the pruned three-dimensional toroidal network T_1 in [5], as shown in Fig. 2. In fact, let $a_1 = (1, 0, 0)^T$, $b_1 = (0, 1, 0)^T$, $c_1 = (0, 0, 1)^T$. It is easily shown that a_1, b_1, and c_1 satisfy the same relations as those of Δ_1; namely, $a_1^k = b_1^k = c_1^k = 1$, $a_1b_1 = b_1a_1$, $c_1^{-1}a_1c_1 = b_1^{-1}$, $c_1^{-1}b_1c_1 = a_1^{-1}$. Hence the mapping $a \to a_1$, $b \to b_1$, $c \to c_1$ is an isomorphism of Δ_1 to T_1. ■

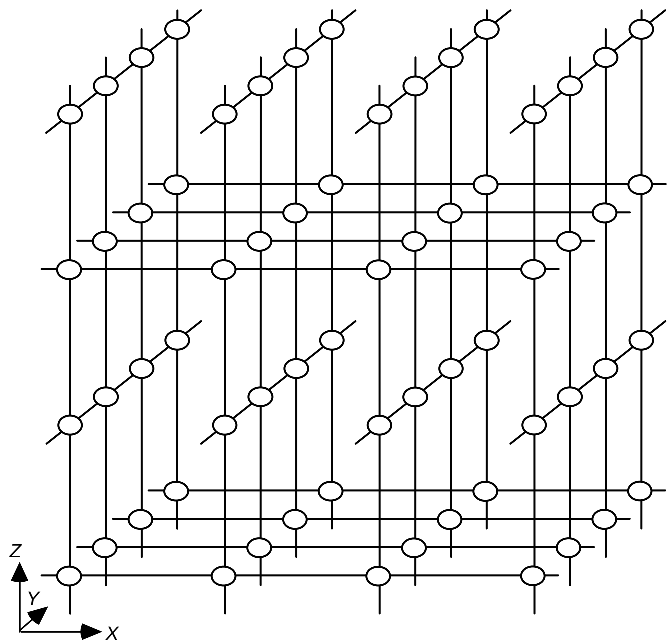

Fig. 2. Pruned 3D torus network T_1 of Reference [5]. To avoid clutter, wraparound links along X, Y, and Z directions are not drawn fully.

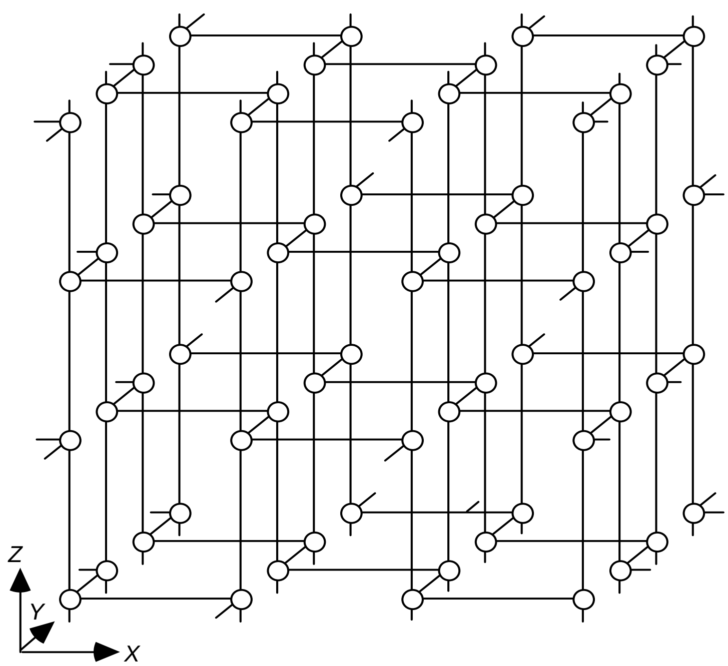

Fig. 3. Pruned 3D torus network T_2 of Reference [5]. To avoid clutter, wraparound links along X, Y, and Z directions are not drawn fully.

Example 4. Pruned three-dimensional toroidal network T_2 of [5], depicted in Fig. 3. We obtain the results for the network T_2 in a manner similar to those for T_1 of Example 3. Let $G = \langle a, b\rangle \langle c\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^{2k} = b^{2k} = c^k = 1$, $a^2 = b^2$, $(ab)^{k/2} = (ba)^{k/2} = 1$, $c^{-1}ac = b$, $c^{-1}bc = a$. Here k is even and $\langle a, b\rangle = \langle ab\rangle \langle a\rangle$ is a complex group. Let $S = \{a, a^{-1}, c, c^{-1}\}$ and $\Delta_2 = Cay(G, S)$. Then, the mapping $a \to (1, 0, 0)^T$, $b \to (0, -1, 0)^T$, $c \to (0, 0, 1)^T$ is an isomorphism of Δ_2 to T_2. ■

5. Honeycomb Torus Networks

The authors of [11] studied the honeycomb torus network as a pruned 2D torus. They also proved that the honeycomb torus network is a Cayley graph, without explicating its associated group. We filled this gap in [14], while also showing why the side-length parameter k in [11] must be even. Let $G = (\langle c\rangle \langle b\rangle) \langle a\rangle$ be the group generated by the elements a, b, c, satisfying the relations $a^k = b^2 = c^{l/2} = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Here, k and l are even integers. Thus the group $\langle c\rangle \langle b\rangle = \langle c, b\rangle$ is a semidirect product of $\langle c\rangle$ by $\langle b\rangle$, and G is a semidirect product of $\langle c, b\rangle$ by $\langle a\rangle$. Let $S = \{a, a^{-1}, b\}$ and $\Delta = Cay(G, S)$. We have shown in [14] that Δ is isomorphic to the honeycomb torus network in [11].

Proposition 2. In [14], we introduced the infinite honeycomb network as a Cayley graph of a different infinite group. Let $G = (\langle c\rangle \langle b\rangle) \langle a\rangle$, where $\langle c\rangle$ and $\langle a\rangle$ are infinite cyclic groups, and c, b, a satisfy the relations $b^2 = 1$, $bcb = c^{-1}$, $aba^{-1} = c^{-1}b$, $aca^{-1} = c^{-1}$. Let $S = \{a, a^{-1}, b\}$ and $\Delta_\infty = Cay(G, S)$. Then Δ_∞ is isomorphic to the infinite honeycomb network (see Fig. 4). ■

Now let $N = \langle a^k\rangle \langle c^{l/2}\rangle$, where k and l are even integers. We can easily verify that $N \lhd G$. Construct the quotient group $G' = {}^G/_N$ and let $S' = \{Na, Na^{-1}, Nb\}$; the graph $Cay(G', S')$ is isomorphic to the honeycomb torus network. Thus the honeycomb torus is a homomorphic image of the infinite honeycomb network by Theorem 1.

For the infinite honeycomb network Δ_∞ any element of G can be expressed as the product $c^j b^l a^i$, where l is 0 or 1 and j and i are integers. We obtained in [14] the distance formula between vertices 1 (the identity of G) and $c^j b^l a^i$, as stated in the following theorem.

Theorem 3. In the infinite honeycomb network Δ_∞, when $|i| \le |2j + l|$, we have $dis(1, c^j b^l a^i) = |4j + l + ½[(-1)^{i+l} - (-1)^l]|$; otherwise, $dis(1, c^j b^l a^i) = |i| + |2j + l|$. ■

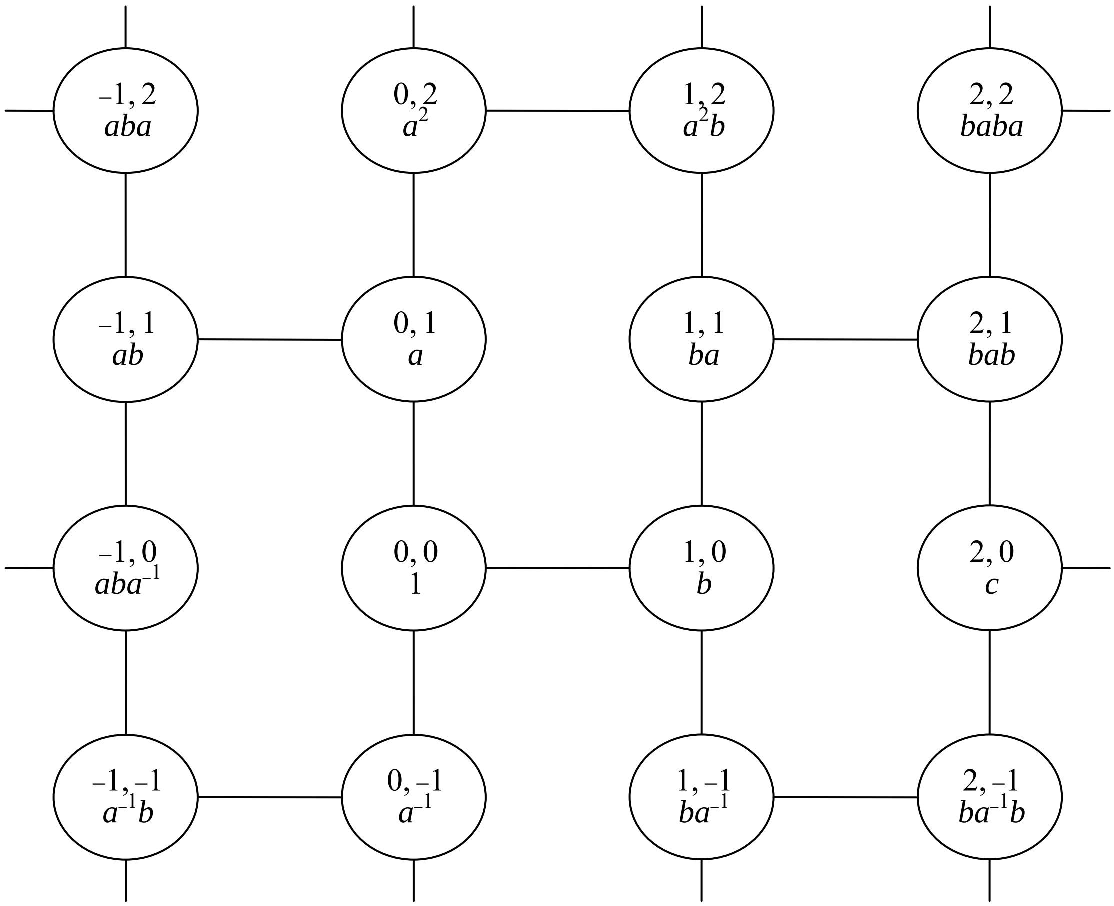

Fig. 4. Connectivity pattern for the honeycomb mesh network. Each node is labeled in two ways corresponding to its integer coordinates on the grid (upper label) and the notation in Proposition 2 (lower label), with the associations being (0, 1) = *a*, (1, 0) = *b*, (2, 0) = *c*.

In [14] we proved that Theorem 3 has the following corollary.

Corollary 1. In the infinite honeycomb network Δ_∞, the distance between nodes (x, y) and (u, v) is given as follows: if $|v - y| \leq |u - x|$, then $dis((x, y), (u, v))$ equals $|2(u - x) + ½ [(-1)^{u+v} - 1]|$ when $x + y \equiv 0 \bmod 2$, and $|2(x - u) + ½ [(-1)^{u+v+1} - 1]|$ otherwise. In the remaining case corresponding to $|v - y| \geq |u - x|$, we have $dis((x, y), (u, v)) = |u - x| + |v - y|$. ■

Using Theorem 3 and Corollary 1, we obtain a result on the diameter of the honeycomb torus network Δ, which generalizes theorem 3 in [11] which states that $D(\Delta) = l$ when $l = k$.

Theorem 4. For the honeycomb torus network Δ, we have $D(\Delta) = \max(l, (l + k)/2)$.

Proof. We consider the two cases of $l > k$ and $l \le k$ separately.

(Case 1) $l > k$: From Theorem 3, we have $dis(1, c^{l/4}) = l$ when $l/2$ is even and $dis(1, c^{(l-2)/4}ba) = l$ if $l/2$ is odd.

(Case 2) $l \le k$: From Theorem 3, we have $dis(1, c^{l/4}a^{k/2}) = (l + k)/2$ when $l/2$ is even and $dis(1, c^{(l-2)/4}ba^{k/2}) = (l + k)/2$ if $l/2$ is odd.

By Corollary 1, in the infinite honeycomb network, $dis(1, (j, i))$ equals $|2j + ½ [(-1)^{i+j} - 1]|$ if $|i| \le |j|$, and $|i| + |j|$ otherwise. Hence, we easily verify that $D(\Delta) \le \max(l, (l + k)/2)$ in the honeycomb torus network Δ. The desired conclusion of the theorem follows from cases 1 and 2 above. ■

6. Properties of Pruned Tori

As an application of our construction, we consider the pruned three-dimensional toroidal network T_1 in Example 3 further in this section. We shall derive a formula of the distance between the identity element 1 and the vertex $a^ib^jc^l$, where $0 \le i, j, l < k$. Theorem 3 in [5] pertaining to T_1 is a direct corollary of this formula.

Theorem 5. For T_1 we have $dis(1, a^ib^jc^l) = min(i, k - i) + min(j, k - j) + min(l, k - l)$ if $l > 0$. When $l = 0$, we have $dis(1, a^ib^j) = min(i, k - i) + min(j, k - j) + 2$ if $j > 0$ and $min(i, k - i)$ otherwise.

Proof. It is evident that $dis(1, a^i) = min(i, k - i)$, since $a^i = (a^{-1})^{k-i}$. Let $j > 0$. Because $a^ib^j = a^ica^{-j}c^{-1}$, we have $dis(1, a^ib^j) = min(i, k - i) + min(j, k - j) + 2$. Now, assuming $l > 0$, we have:

(1) $a^ib^jc^l = a^ica^{-j}c^{l-1} = a^ic^{-1}a^{-j}c^{l+1}$ for $0 \le i, j < k$

Continuing with the assumption $l > 0$, we have:

(2) $min(min(l - 1, k - l + 1), min(l + 1, k - l - 1)) = min(l - 1, k - l - 1) = min(l, k - l) - 1$

Thus, considering both (1) and (2), we obtain $dis(1, a^ib^jc^l) = min(i, k - i) + min(j, k - j) + min(min(l - 1, k - l + 1), min(l + 1, k - l - 1)) + 1 = min(i, k - i) + min(j, k - j) + min(l, k - l)$. ■

Corollary 2. For the network T_1 with $k \geq 4$, we have $D(T_1) = 3k/2$. ■

Finally, we show that Theorem 2 in [5] does not hold in general. The following example shows that the pruned three-dimensional toroidal network T_1 of Example 3 is not edge-symmetric in general.

Example 5. Consider the case of $k = 4$ and let $A = Aut(T_1)$ be the automorphism group of T_1. We show that there is no $\sigma \in$ A such that $\sigma(1) = 1$ and $\sigma(c) = a$. Similarly, we show that there is no $\tau \in$ A such that $\tau(1) = a$ and $\tau(c) = 1$. Hence T_1 is not edge-symmetric for $k = 4$. In fact, we show that the assumption $\sigma \in A$, such that $\sigma(1) = 1$ and $\sigma(c) = a$ leads to a contradiction. Since the edge $(1, c)$ is in the cycle $C = \{1, c, c^2, c^3\}$ and the edge $(1, a)$ is only in the cycle $A' = \{1, a, a^2, a^3\}$, C is mapped to A' by σ. Hence, $\sigma(c^2) = a^2$ and $\sigma(c^3) = a^3$. Now consider the cycle $B = \{a, ac, ac^2, ac^3\}$. Since $\sigma(1) = 1$ and $(1, a)$ is an edge, $(1, \sigma(a))$ is also an edge. This implies that $\sigma(a)$ equals c or c^{-1}. Let $\sigma(a) = c$; the case of $\sigma(a) = c^{-1}$ is similar. Since (a, ac) is an edge, $(c, \sigma(ac))$ is also an edge. Because the cycle B cannot be mapped to the cycle C by σ, we have $\sigma(ac) = ca$ or ca^{-1}. Given that (c^2, c^2a) is an edge and $\sigma(c^2) = a^2$, $(a^2, \sigma(c^2a))$ is also an edge. Therefore, $\sigma(c^2a)$ equals a^2c or a^2c^{-1}. Since $c^2a = ac^2$, $\{a, ac, c^2a\}$ is in the cycle B. But, $\{\sigma(a), \sigma(ac), \sigma(c^2a)\}$ is not in any cycle of order four. This is a contradiction. ■

7. Conclusion

In this paper, we have derived new relationships between Cayley (di)graphs and their subgraphs and coset graphs with respect to subgroups, focusing in particular on homomorphism between them and on relating their internode distances and diameters. We have also demonstrated the applications of these results to well-known and useful interconnection networks, including hexagonal and honeycomb tori and related networks. Because of the generality of these theorems, we expect that they will find many more applications.

We are currently investigating the applications of our method to the problems related to routing and average internode distance in certain subgraphs of honeycomb networks. We also aim to extend our results to other classes of networks as well as to other topological properties of networks. Such improvements and extensions, along with potential applications in the following areas will be reported in future:

- Load balancing and congestion control

- Scheduling and resource allocation
- Fault tolerance and graceful degradation

These constitute important practical problems in the design, evaluation, and efficient operation of parallel and distributed computer systems.

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