Department of Electrical and Computer Engineering
Digital Speech Processing
Homework No. 6 Solutions

Problem 1

The complex cepstrum, $\hat{x}[n]$, of a sequence $x[n]$ is the inverse Fourier transform of the complex log spectrum

$$\hat{X}(e^{j\omega}) = \log|X(e^{j\omega})| + j\arg[X(e^{j\omega})]$$

Show that the cepstrum $c[n]$, defined as the inverse Fourier transform of the log magnitude, is the even part of $\hat{x}[n]$; i.e., show that

$$c[n] = \frac{\hat{x}[n] + \hat{x}[-n]}{2}$$

Solution

We begin with the basic definitions:

$$\hat{x}[n] \leftrightarrow \hat{X}(e^{j\omega}) = \log|X(e^{j\omega})| + j\arg[X(e^{j\omega})]$$
$$c[n] \leftrightarrow \log|X(e^{j\omega})|$$

and show that:

$$c[n] = \frac{\hat{x}[n] + \hat{x}[-n]}{2}$$

By definition we have

$$\begin{aligned} c[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|X(e^{j\omega})| e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|X(e^{j\omega})| [\cos(\omega n) + j\sin(\omega n)] d\omega \end{aligned}$$

Recall that for $x[n]$ real, $|X(e^{j\omega})|$ is an even function; therefore

$$\int_{-\pi}^{\pi} \log|X(e^{j\omega})| [j\sin(\omega n)] d\omega = 0$$

Leading to the result

$$c[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|X(e^{j\omega})| \cos(\omega n) d\omega$$

By inverse transforming $\hat{X}(e^{j\omega})$ we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log|X(e^{j\omega})| + j\arg\{X(e^{j\omega})\}] \cdot [\cos(\omega n) + j\sin(\omega n)]d\omega$$

For $x[n]$ real, $\arg\{X(e^{j\omega})\}$ is an odd function, therefore

$$\int_{-\pi}^{\pi} j\arg\{X(e^{j\omega})\}\cos(\omega n)d\omega = 0$$

Similarly, since $\log|X(e^{j\omega})|$ is an even function of ω and $\sin(\omega n)$ is an odd function of ω, therefore we have

$$\int_{-\pi}^{\pi} \log|X(e^{j\omega})|(j\sin(\omega n))d\omega = 0$$

Thus we get:

$$\begin{aligned}
\hat{x}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log|X(e^{j\omega})|][\cos(\omega n)]d\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \arg[X(e^{j\omega})][\sin(\omega n)]d\omega \\
\hat{x}[-n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log|X(e^{j\omega})|][\cos(\omega n)]d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} \arg[X(e^{j\omega})][\sin(\omega n)]d\omega \\
\frac{\hat{x}[n] + \hat{x}[-n]}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\log|X(e^{j\omega})|][\cos(\omega n)]d\omega = c[n]
\end{aligned}$$

Problem 2

A linear time-invariant system has the transfer function:

$$H(z) = 8\left[\frac{1 - 4z^{-1}}{1 - \frac{1}{6}z^{-1}}\right]$$

(a) Find the complex cepstral coefficients, $\hat{h}[n]$, for all n.
(b) Sketch $\hat{h}[n]$ versus n for the range $-10 \leq n \leq 10$.
(c) Solve for the (real) cepstrum coefficients, $c[n]$, for all n.

Solution

(a) We can write $H(z)$ in the normalized format:

$$H(z) = 8\left[\frac{1 - 4z^{-1}}{1 - \frac{1}{6}z^{-1}}\right] = 32z^{-1}\left[\frac{1 - z/4}{1 - z^{-1}/6}\right]$$

We now can form the log of $H(z)$ as:

$$\hat{H}(z) = \log|H(z)| = \log(32) + \log[-z^{-1}] + \log(1 - z/4) - \log(1 - z^{-1}/6)$$

where the log terms for the minus sign and the z^{-1} term have no effect and are omitted from subsequent computation. Recognizing the log series:

$$\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1$$

we get:

$$\hat{H}(z) = \log(32) + \sum_{n=-\infty}^{-1} \frac{(1/4)^{-n} z^n}{n} - \sum_{n=1}^{\infty} \frac{(1/6)^n z^{-n}}{n}$$

giving:

$$\hat{h}[n] = \begin{cases} \log(32) & n = 0 \\ \dfrac{(1/6)^n}{n} & n > 0 \\ \dfrac{(1/4)^{-n}}{n} & n < 0 \end{cases}$$

(b) Figure 1 (at the top) shows a plot of $\hat{h}[n]$. By setting $\hat{h}[n]$ to a value of zero, we can better see the behavior of $\hat{h}[n]$ for the range $-10 \leq n \leq n$ as shown in Figure 1 at the bottom.

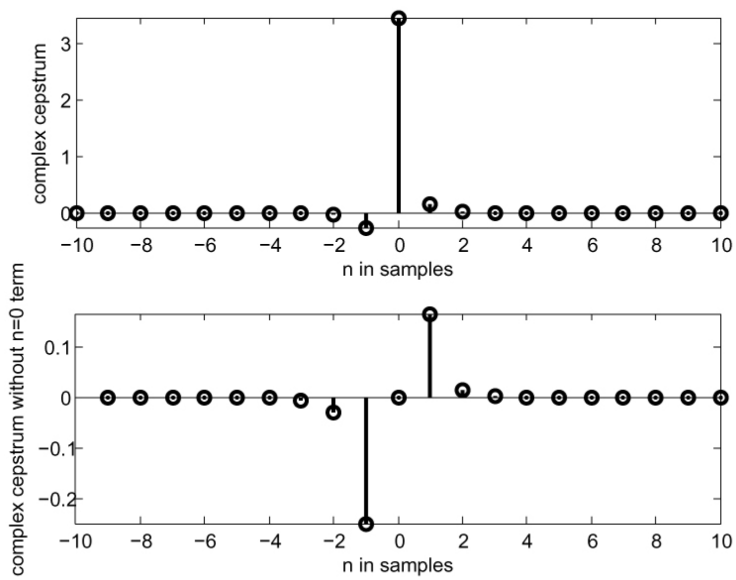


Figure 1: Plot of complex cepstral sequence; top panel with all cepstral values, bottom panel without cepstral value at n=0.

(c) We can solve for $c[n]$ as:

$$\begin{aligned} c[n] &= \frac{\hat{h}[n] + \hat{h}[-n]}{2} \\ &= \begin{cases} \log(32) & n = 0 \\ \dfrac{(1/6)^n}{2n} - \dfrac{(1/4)^n}{2n} & n > 0 \\ -\dfrac{(1/6)^{-n}}{2n} + \dfrac{(1/4)^{-n}}{2n} & n < 0 \end{cases} \end{aligned}$$

Problem 3

Consider a *finite length* minimum phase sequence $x[n]$ with complex cepstrum $\hat{x}[n]$, and a sequence

$$y[n] = \alpha^n x[n]$$

with complex cepstrum $\hat{y}[n]$.

(a) If $0 < \alpha < 1$, how will $\hat{y}[n]$ be related to $\hat{x}[n]$?
(b) How should α be chosen so that $y[n]$ would no longer be minimum phase?
(c) How should α be chosen so that $y[n]$ is maximum phase?

Solution

(a) $0 < \alpha < 1$; assume length N for $x[n]$, then we get

$$\begin{aligned} Y(z) &= \sum_{n=0}^{N-1} \alpha^n x[n] z^{-n} = \sum_{n=0}^{N-1} x[n](\alpha z^{-1})^n \\ &= X(z/\alpha) \end{aligned}$$

If the zeros of $X(z)$ occur at $z_0, z_1, \ldots, z_{N-1}$, then the zeros of $Y(z)$ occur at $\alpha z_0, \alpha z_1, \ldots, \alpha z_{N-1}$; therefore $\hat{y}[n] = \alpha^n \hat{x}[n]$.

(b) Let $\max_i |z_i|$ denote the magnitude of the maximum magnitude zero of $X(z)$. $y[n]$ is not minimum phase if any zero of $Y(z)$ lies on or outside the unit circle. Therefore choose α so that

$$|\alpha| \max_i |z_i| \geq 1 \Rightarrow |\alpha| \geq \frac{1}{\max\limits_i |z_i|}$$

(c) Let $\min_i |z_i|$ denote the magnitude of the minimum magnitude zero of $X(z)$. $y[n]$ is maximum phase if all the zeros of $Y(z)$ lie outside the unit circle. Therefore choose α so that

$$|\alpha| \min_i |z_i| > 1 \Rightarrow |\alpha| > \frac{1}{\min\limits_i |z_i|}$$

Problem 4

Given the signal, $x[n]$, with complex cepstrum, $\hat{x}[n]$, it was shown in class that their z-transforms are related by the relation:

$$\hat{X}(z) = \log X(z)$$

The complex cepstrum of a causal, minimum phase signal can be generated with the use of DFTs. However, there also exists a recursive relationship for determining $\hat{x}[n]$ directly from $x[n]$. The purpose of this problem is to derive this recursive relationship between $\hat{x}[n]$ and $x[n]$. We do this as follows:

(a) by differentialing $\hat{X}(z)$ and $X(z)$, show that $x[n]$ and $\hat{x}[n]$ are related by:

$$n\hat{x}[n] * x[n] = nx[n]$$

(b) assuming that $x[n]$ is minimum phase, use this expression to derive the following recursive formula for generating $\hat{x}[n]$

$$\hat{x}[n] = \begin{cases} 0 & n < 0 \\ \log x[0] & n = 0 \\ \dfrac{x[n]}{x[0]} - \dfrac{1}{nx[0]} \displaystyle\sum_{k=0}^{n-1} k\hat{x}[k]x[n-k] & n > 0 \end{cases}$$

(Hint: use the initial value theorem for $n = 0$)

$$x[0] = \lim_{z \to \infty} X(z)$$

Solution

(a)

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ \frac{d}{dz}X(z) &= \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right] \\ &= \sum_{n=-\infty}^{\infty} - nx[n]z^{-n-1} \\ -z\frac{d}{dz}X(z) &= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} \end{aligned}$$

$$\Rightarrow \quad nx[n] \stackrel{ZT}{\longleftrightarrow} -z\frac{d}{dz}X(z)$$

Since

$$\begin{aligned}
\hat{X}(z) &= \log X(z) \\
\frac{d}{dz}\hat{X}(z) &= \frac{d}{dz}\log X(z) = \frac{1}{X(z)}\frac{d}{dz}X(z) \\
X(z)\frac{d}{dz}\hat{X}(z) &= \frac{d}{dz}X(z)
\end{aligned}$$

Multiplying both sides by $-z$ and inverse transforming gives:

$$\left[-z\frac{d}{dz}\hat{X}(z)\right]X(z) = -z\frac{d}{dz}X(z)$$

$$n\hat{x}[n] * x[n] = nx[n]$$

(b) If $x[n]$ is minimum phase, then $X(z)$ has no poles or zeros outside the unit circle. Thus both $x[n]$ and $\hat{x}[n]$ are causal, i.e.,
Step 1:

$$x[n] = 0 \quad n < 0$$

$$\hat{x}[n] = 0 \quad n < 0$$

Using the initial value theorem we get:
Step 2:

$$\begin{aligned}
\hat{x}[0] &= \lim_{z\to\infty}\hat{X}(z) = \lim_{z\to\infty}[\log X(z)] \\
&= \lim_{z\to\infty}\left[\log(x[0] + x[1]z^{-1} + \ldots)\right] = \log x[0]
\end{aligned}$$

Finally, using the expression from part (a) we get:

$$\begin{aligned}
nx[n] &= n\hat{x}[n] * x[n] = \sum_{k=-\infty}^{\infty} k\hat{x}[k]x[n-k] \\
x[n] &= \frac{1}{n}\sum_{k=-\infty}^{\infty} k\hat{x}[k]x[n-k]
\end{aligned}$$

Since $x[n] = 0$, for $n < 0$, and $\hat{x}[n] = 0$, for $n < 0$, we can change the limits of summation, giving:

$$x[n] = \frac{1}{n}\sum_{k=0}^{n} k\hat{x}[k]x[n-k]$$

Isolating the $k = n$ term gives:

$$x[n] = \frac{1}{n}\sum_{k=0}^{n-1} k\hat{x}[k]x[n-k] + \hat{x}[n]x[0]$$

Step 3:

$$\hat{x}[n] = \frac{x[n]}{x[0]} - \frac{1}{nx[0]} \sum_{k=0}^{n-1} k\hat{x}[k]x[n-k]$$

Problem 5

Write a MATLAB program to compute the complex and real cepstrums of the following signals:

1. $x_1[n] = \delta[n] - 0.85\delta[n-99]$
2. $x_2[n] = \sin\left[\frac{2\pi n}{100}\right] \quad 0 \le n \le 99$
3. $x_3[n] = 0.95^n \quad 0 \le n \le 99$
4. a section of voiced speech
5. a section of unvoiced speech

For each of the first 3 signals, plot the waveform, the complex cepstrum, and the real cepstrum. For the speech signals (numbers 4 and 5) plot the signal, the log magnitude spectrum, the real cepstrum, and the lowpass liftered log magnitude spectrum. Use the file test_16k.wav to test your program. Specify the unvoiced section as starting at sample 3400 in the file, and of duration 400 samples. Specify the voiced section as starting at sample 13000 in the file, and of duration 400 samples. Use a Hamming window before computing the cepstrums of the speech files.

Solution

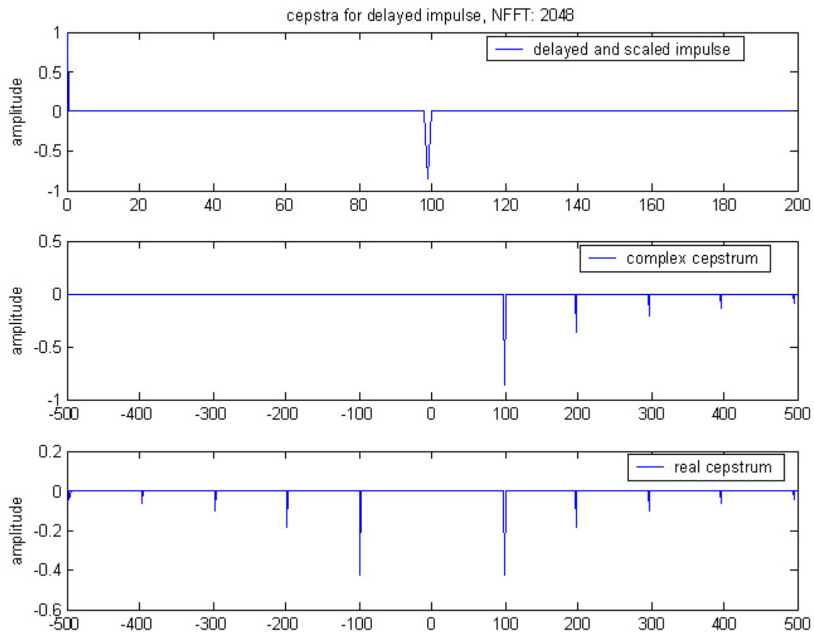
The following MATLAB code computes the complex and real cepstrums of the signals as well as the speech waveforms.

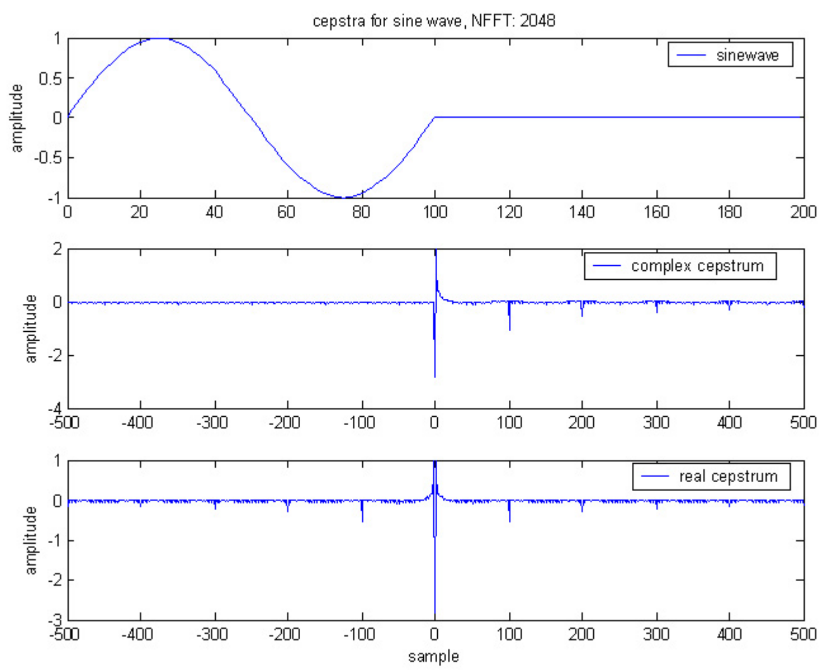
```
%  test_cepstrum_3.m
%  cepstrum.m

    NFFT=input('FFT size for cepstral calculations:');
%
%  generate impulse plus delayed impulse
    n=0:NFFT-1;
    x1=zeros(1,NFFT);
    x1(1)=1;
    x1(100)=-.85;

%  generate sine wave pulse with period 100 Hz
```

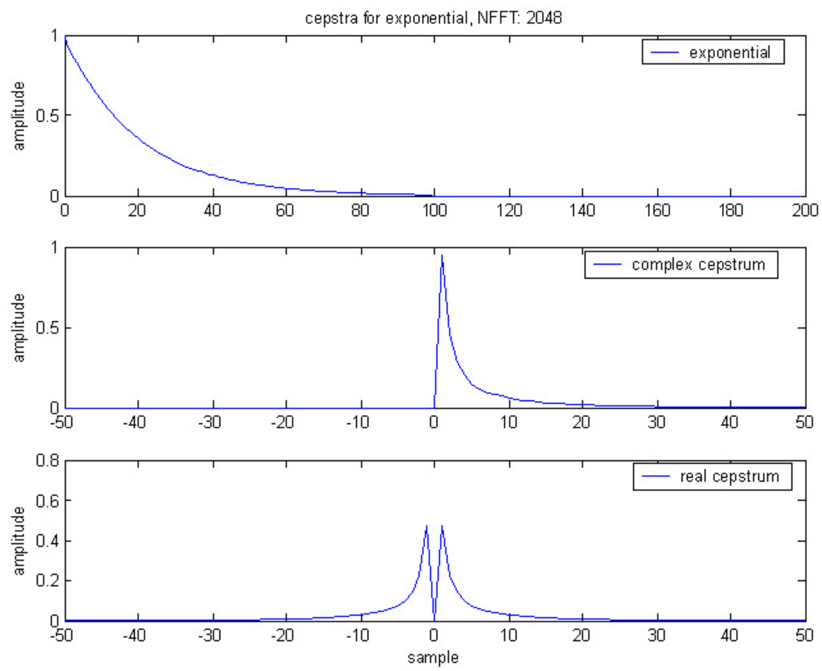
```
    n1=0:99;
    x2=sin(2*pi*n1/10);
    x2=[x2 zeros(1,NFFT-100)];

%  generate exponential
    x3=.95.^n1;
    x3=[x3 zeros(1,NFFT-100)];

%  compute complex and real cepstrum for all three signals
%
    xhat1=cceps(x1,NFFT);
    rhat1=rceps(x1);
    xhat2=cceps(x2,NFFT);
    rhat2=rceps(x2);
    xhat3=cceps(x3,NFFT);
    rhat3=rceps(x3);

% read in speech file
% [xin,fs,mode,format]=loadwav('ah_truncated.wav');xin)
    filename=input('enter speech filename:','s');
    [xin,fs,mode,format]=loadwav(filename);
% choose section to plot
    m1=input('starting speech sample in file for unvoiced section:');
    m2=input('starting speech sample in file for voiced section:');
    stitle=sprintf('file: %s, unvoiced/voiced starting sample: %d %d',filename,m1,m2);

% choose analysis parameters
    nwin=400;
    x4=(xin(m1:m1+nwin-1).*hamming(nwin))';
    x4=[x4 zeros(1,NFFT-nwin)];
    X4m=log(abs(fft(x4,NFFT)));
    xh4=fft(X4m,NFFT)/NFFT;
    lifter=zeros(1,NFFT);
    lifter(1:50)=1;
    lifter(NFFT-48:NFFT)=1;
    xh4p=xh4.*lifter;
    X4mm=ifft(xh4p,NFFT)*NFFT;

    x5=(xin(m2:m2+nwin-1).*hamming(nwin))';
    x5=[x5 zeros(1,NFFT-nwin)];
    X5m=log(abs(fft(x5,NFFT)));
    xh5=fft(X5m,NFFT)/NFFT;
    lifter=zeros(1,NFFT);
    lifter(1:50)=1;
    lifter(NFFT-48:NFFT)=1;
    xh5p=xh5.*lifter;
```

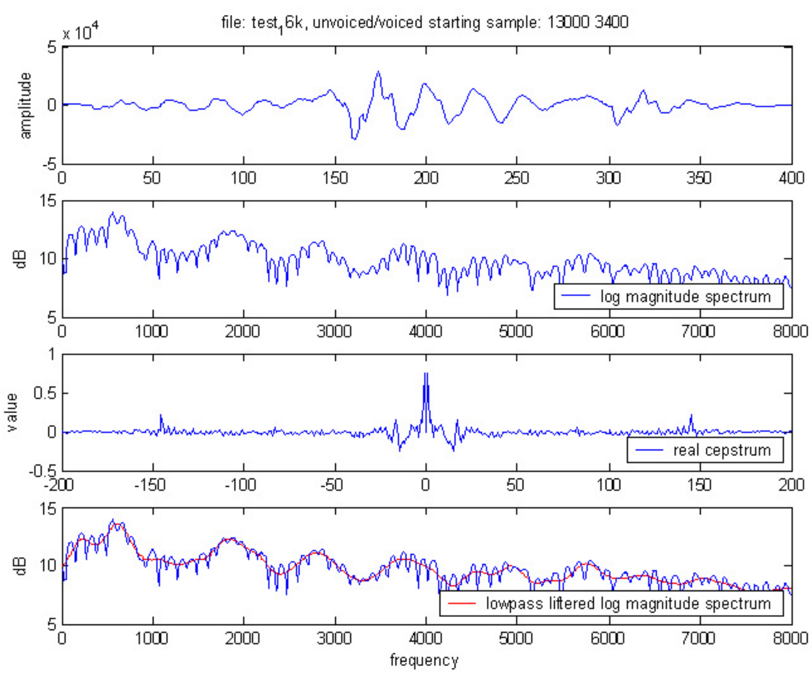
```
    X5mm=ifft(xh5p,NFFT)*NFFT;

% plot results for speech section--zero out first real cepstrum coefficient for plotting
%
    colordef white;
    ptitle1=sprintf('cepstra for delayed impulse, NFFT: %d',NFFT);
    ptitle2=sprintf('cepstra for sine wave, NFFT: %d',NFFT);
    ptitle3=sprintf('cepstra for exponential, NFFT: %d',NFFT);

% plot signal and cepstra for 3 basic signals and for speech section
    figure;
    subplot(3,1,1),plot(n,x1),title(ptitle1),legend('delayed and scaled impulse',0),ylabel(
    subplot(3,1,2),plot(n,xhat1),legend('complex cepstrum',0),ylabel('amplitude');
    subplot(3,1,3),plot(n,rhat1),legend('real cepstrum',0),ylabel('amplitude');

    figure;
    subplot(3,1,1),plot(n,x2),title(ptitle2),legend('sinewave',0),ylabel('amplitude');
    subplot(3,1,2),plot(n,xhat2),legend('complex cepstrum',0),ylabel('amplitude');
    subplot(3,1,3),plot(n,rhat2),legend('real cepstrum',0),xlabel('sample'),ylabel('amplitu

    figure;
    subplot(3,1,1),plot(n,x3),title(ptitle3),legend('exponential',0),ylabel('amplitude');
    subplot(3,1,2),plot(n,xhat3),legend('complex cepstrum',0),ylabel('amplitude');
    subplot(3,1,3),plot(n,rhat3),legend('real cepstrum',0),xlabel('sample'),ylabel('amplitu

% plot cepstra for voiced and unvoiced speech
    figure;
    xfreq=(0:NFFT/2)*(fs/NFFT);
    subplot(4,1,1),plot(x4(1:nwin)),title(stitle),ylabel('amplitude');
    subplot(4,1,2),plot(xfreq,X4m(1:NFFT/2+1)),ylabel('dB'),legend('log magnitude spectrum'
    subplot(4,1,3),plot(1:NFFT/2,xh4(2:NFFT/2+1)),ylabel('value'),legend('real cepstrum',4)
    subplot(4,1,4),plot(xfreq,X4mm(1:NFFT/2+1),'r',xfreq,X4m(1:NFFT/2+1),'b');
    xlabel('frequency'),ylabel('dB'),legend('lowpass liftered log magnitude spectrum',4);

% plot cepstra for voiced and unvoiced speech
    figure;
    subplot(4,1,1),plot(x5(1:nwin)),title(stitle),ylabel('amplitude');
    subplot(4,1,2),plot(xfreq,X5m(1:NFFT/2+1)),ylabel('dB'),legend('log magnitude spectrum'
    subplot(4,1,3),plot(1:NFFT/2,xh5(2:NFFT/2+1)),ylabel('value'),legend('real cepstrum',1)
    subplot(4,1,4),plot(xfreq,X5mm(1:NFFT/2+1),'r',xfreq,X5m(1:NFFT/2+1),'b'),
    xlabel('frequency'),ylabel('dB'),legend('lowpass liftered log magnitude spectrum',1);
```

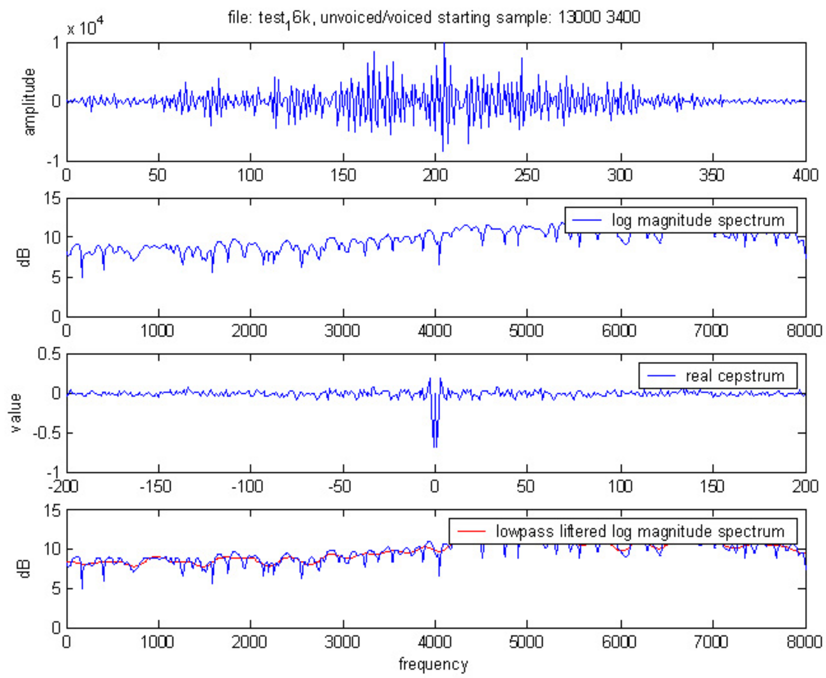
The following plots result (for the speech cepstra, the zeroth value was set to zero for plotting so as not to distort the scale):


cepstra for delayed impulse, NFFT: 2048
delayed and scaled impulse
complex cepstrum
real cepstrum
amplitude


cepstra for sine wave, NFFT: 2048
sinewave
complex cepstrum
real cepstrum
amplitude
sample


cepstra for exponential, NFFT: 2048
exponential
amplitude
complex cepstrum
amplitude
real cepstrum
amplitude
sample


file: test1 6k, unvoiced/voiced starting sample: 13000 3400
x 10^4
amplitude
dB
log magnitude spectrum
value
real cepstrum
dB
lowpass liftered log magnitude spectrum
frequency



file: test16k, unvoiced/voiced starting sample: 13000 3400
x 10^4
amplitude
log magnitude spectrum
dB
real cepstrum
value
lowpass filtered log magnitude spectrum
frequency