ECE137A, Notes Set 16: Second-Order Circuits

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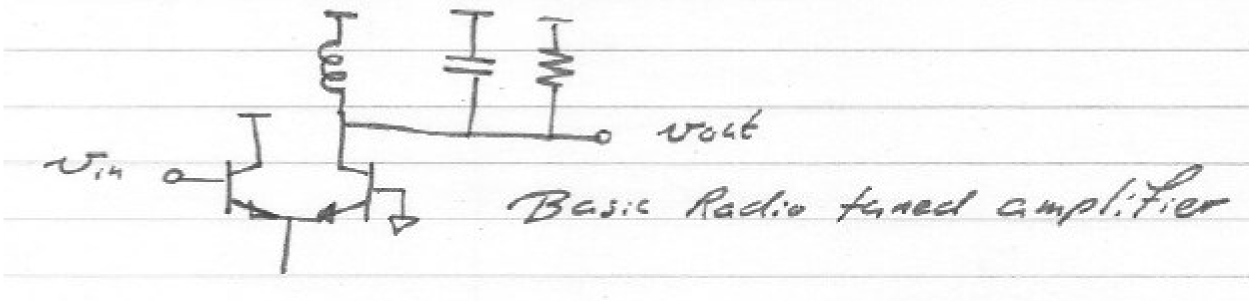
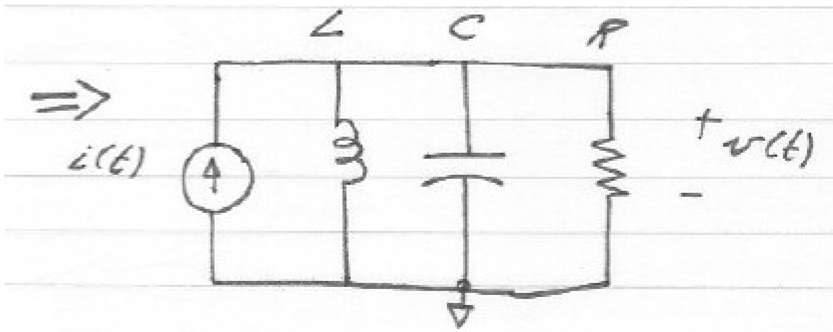
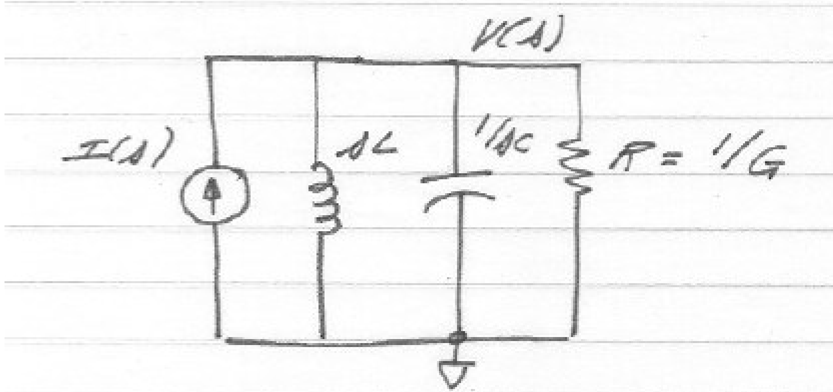
Parallel RLC Resonance







Parallel RLC Resonance: Analysis

$$V\left[sC + 1/sL + G\right] = I$$

$$V/I = \frac{1}{sC + 1/sL + G} = \frac{sL}{1 + sL/R + s^2LC}$$

put in standard ("canonical") form

$$\frac{V(s)}{I(s)} = Z(s) = R \cdot \frac{(2\zeta/\omega_n)s}{s^2/\omega_n^2 + (2\zeta/\omega_n)s + 1}$$

ω_n = the natural resonant frequency

$\alpha = \zeta\omega_n$ = the exponential damping coefficient

ζ = the damping factor

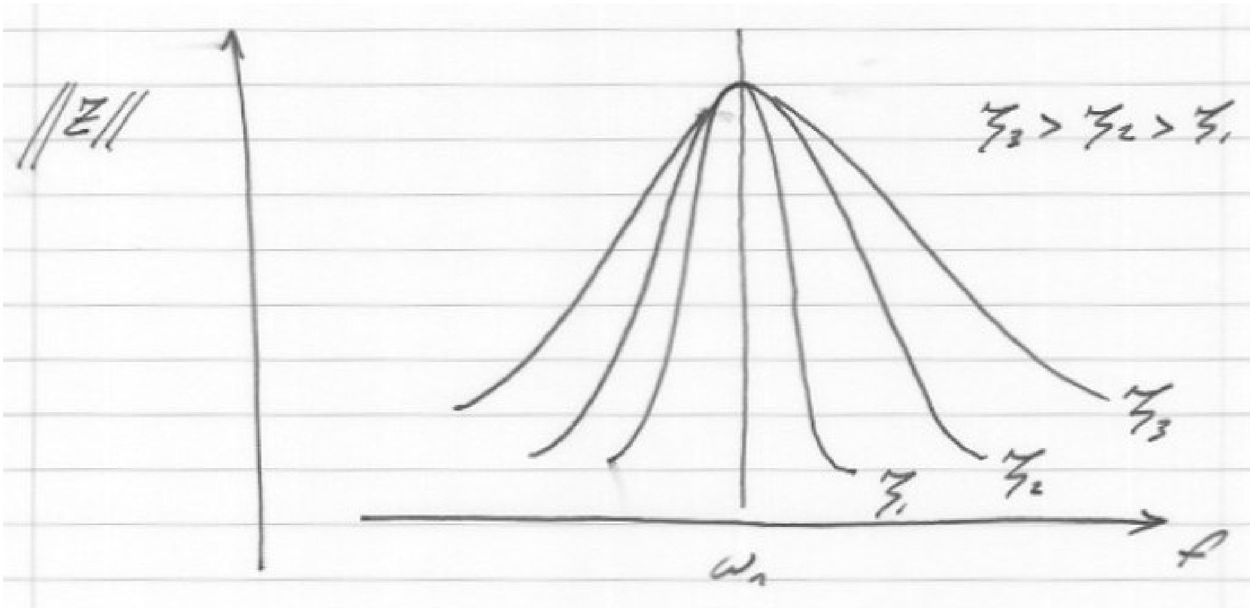
Equating terms for <u>this</u> <u>specific</u> problem <u>only</u>

$$\omega_n = 1/\sqrt{LC}, \quad \alpha = 1/2RC \qquad \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Parallel RLC Resonance: Frequency Reponse

response to signal $v_0 e^{j\omega_0 t}$ → $s = j\omega + 0$

$$\frac{V(j\omega)}{I(j\omega)} = Z(j\omega) = R\,\frac{j\omega(2\zeta/\omega_n)}{1 + j\omega(2\zeta/\omega_n) - \omega^2/\omega_n^2}$$

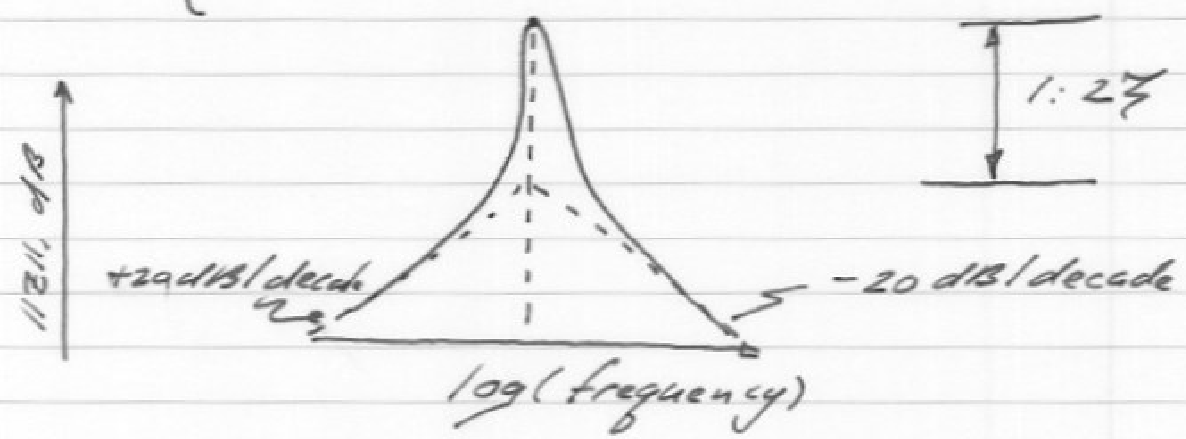


The larger ζ is, the wider the resonant peak in frequency.

Bode Plot of Resonance

$$Z(j\omega) = R\,\frac{j\omega(2\zeta/\omega_n)}{1 + j\omega(2\zeta/\omega_n) - \omega^2/\omega_n^2}$$

$$= \begin{cases} R\cdot(j\omega/\omega_n)\,2\zeta & \text{for } \omega \ll \omega_n \\ R\cdot 1 & \text{for } \omega = \omega_n \\ R\cdot(\omega_n/j\omega)\,2\zeta & \text{for } \omega \gg \omega_n \end{cases}$$



- Consider what will happen to the curve as we vary ζ

- Note asymptotes of ± 20 dB/decade and the $2\zeta:1$ resonant peak.

Poles, Zeros

$$V(s)/I(s) = Z(s) = R\,\frac{s\,(2\zeta/\omega_n)}{s^2/\omega_n^2 + s(2\zeta/\omega_n) + 1}$$

re-arrange into 2 useful forms:

$$Z(s) = R\cdot\frac{s\,(2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$\zeta\omega_n = \alpha$

$\omega_d^2 = (1-\zeta^2)\,\omega_n^2$

this form for inverse Laplace transform

Poles, Zeros

$$V(s)/I(s) = Z(s) = R\,\frac{s(2\zeta/\omega_n)}{s^2/\omega_n^2 + s(2\zeta/\omega_n) + 1}$$

$$Z(s) = R\cdot\frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n - j\omega_d)(s+\zeta\omega_n + j\omega_d)}$$

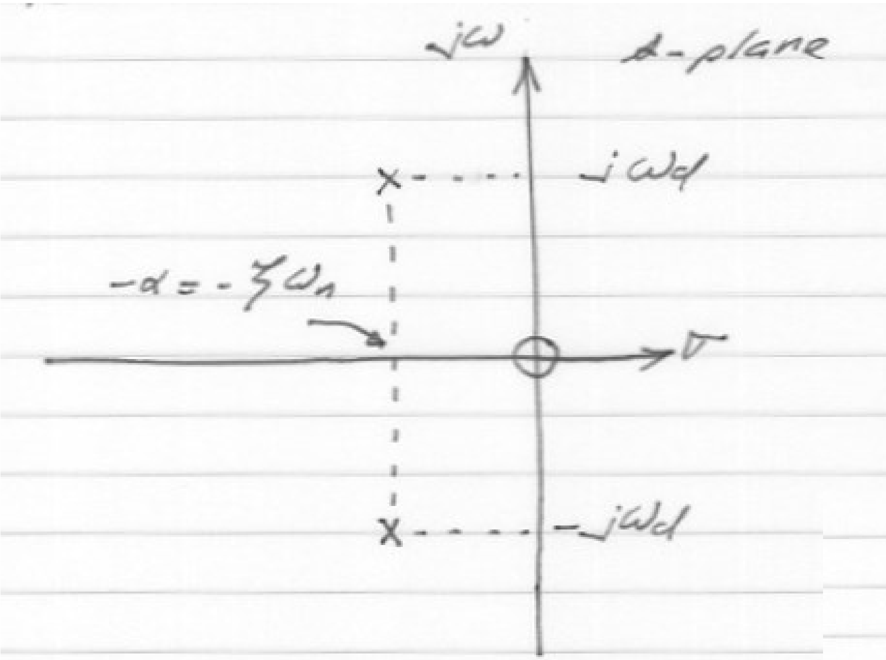
zero at DC.

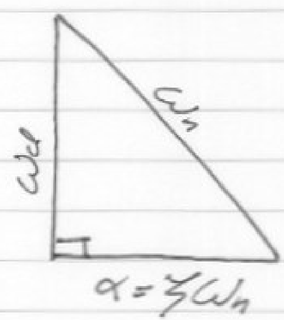
$$= R\,\frac{s(2\zeta\omega_n)}{(s-s_{p1})(s-s_{p2})}$$

pole frequencies

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

Root Constellation





$$\omega_d^2 + (\zeta\omega_n)^2 = \omega_n^2$$

$$\Downarrow$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

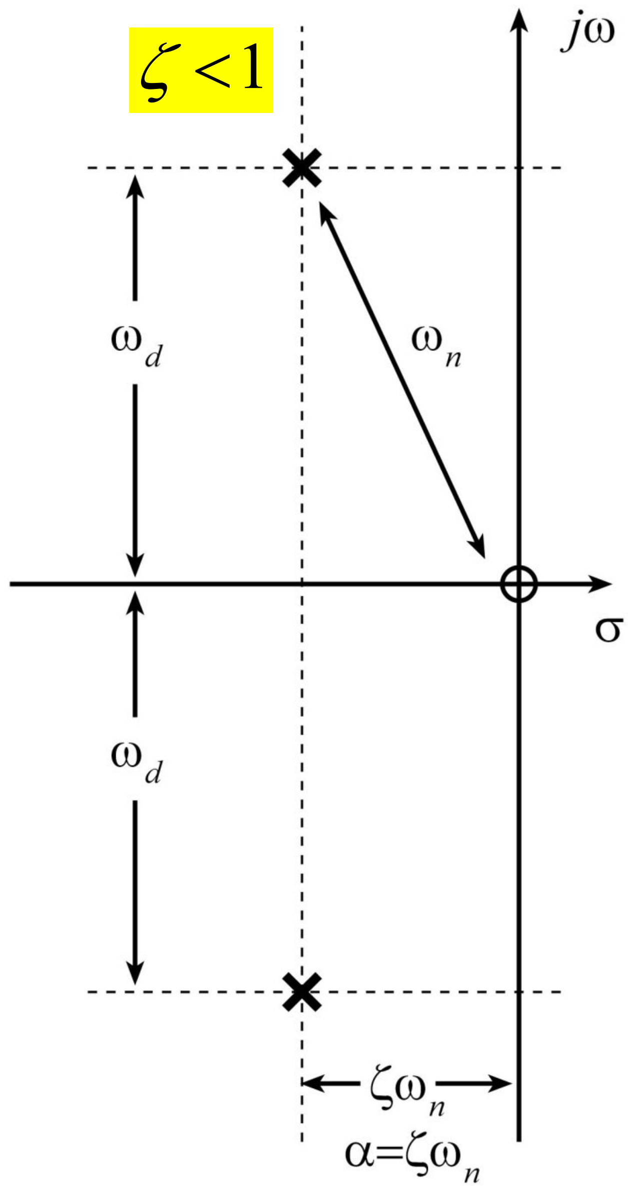
Very important

Root Constellation: For Underdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$

$$H(s) = \frac{s(2\zeta/\omega_n)}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$



Root Constellation: For Overdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

but if $\zeta > 1$,

then $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2}$ is complex.

If $\zeta > 1$, then use

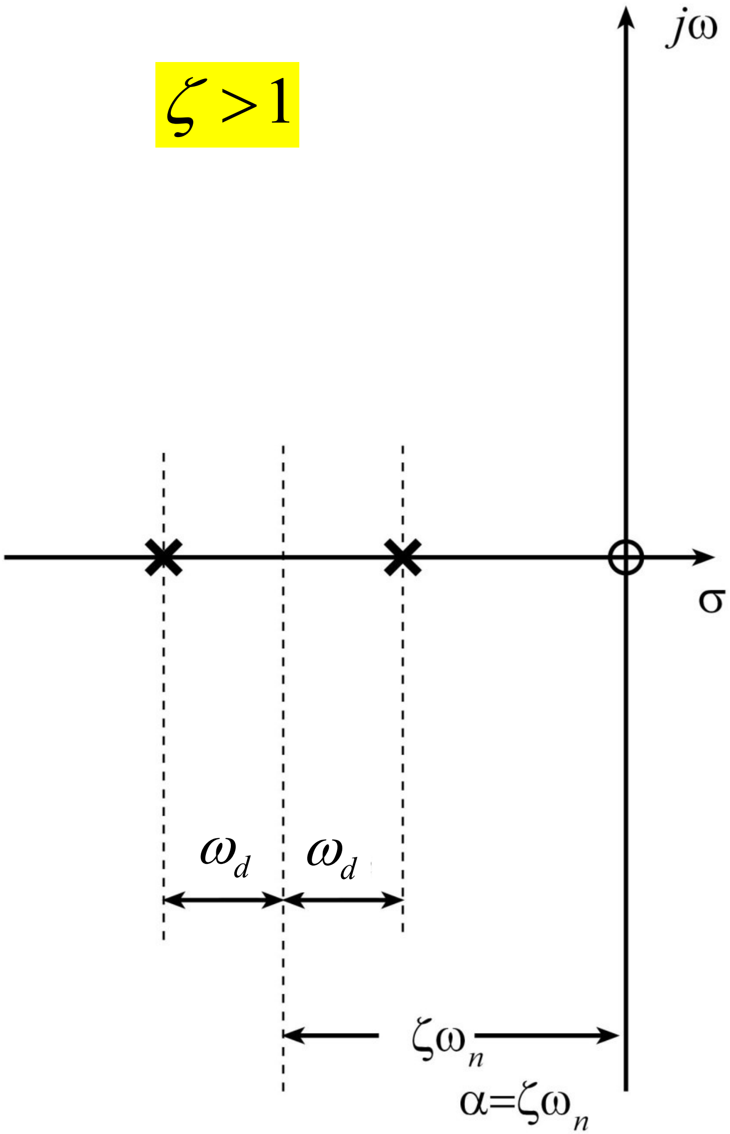
$$j\omega_d = j\omega_n \cdot \sqrt{1-\zeta^2} = \omega_n \cdot \sqrt{\zeta^2 - 1}$$

So

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$

$$= -\zeta\omega_n \pm \omega_n \cdot \sqrt{\zeta^2 - 1}$$

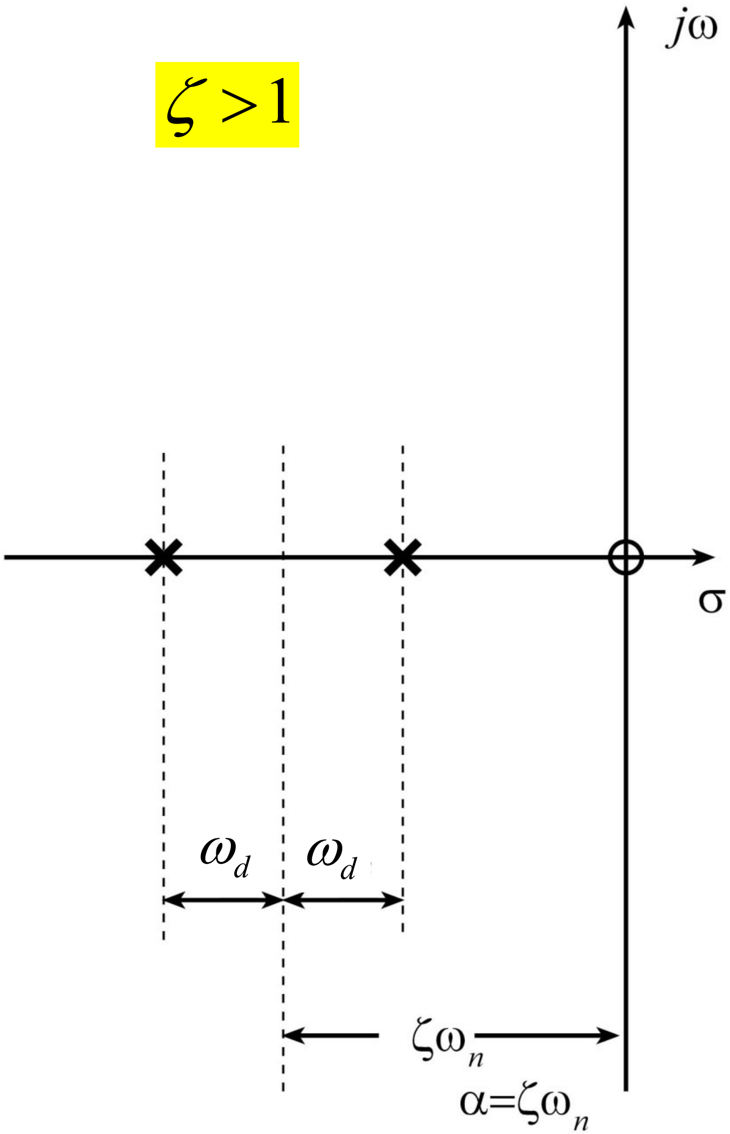
$\zeta > 1$



Root Constellation: For Overdamped System

$$s_{p1,2} = -\zeta\omega_n \pm j\omega_d$$
$$= -\zeta\omega_n \pm \omega_n \cdot \sqrt{\zeta^2 - 1}$$

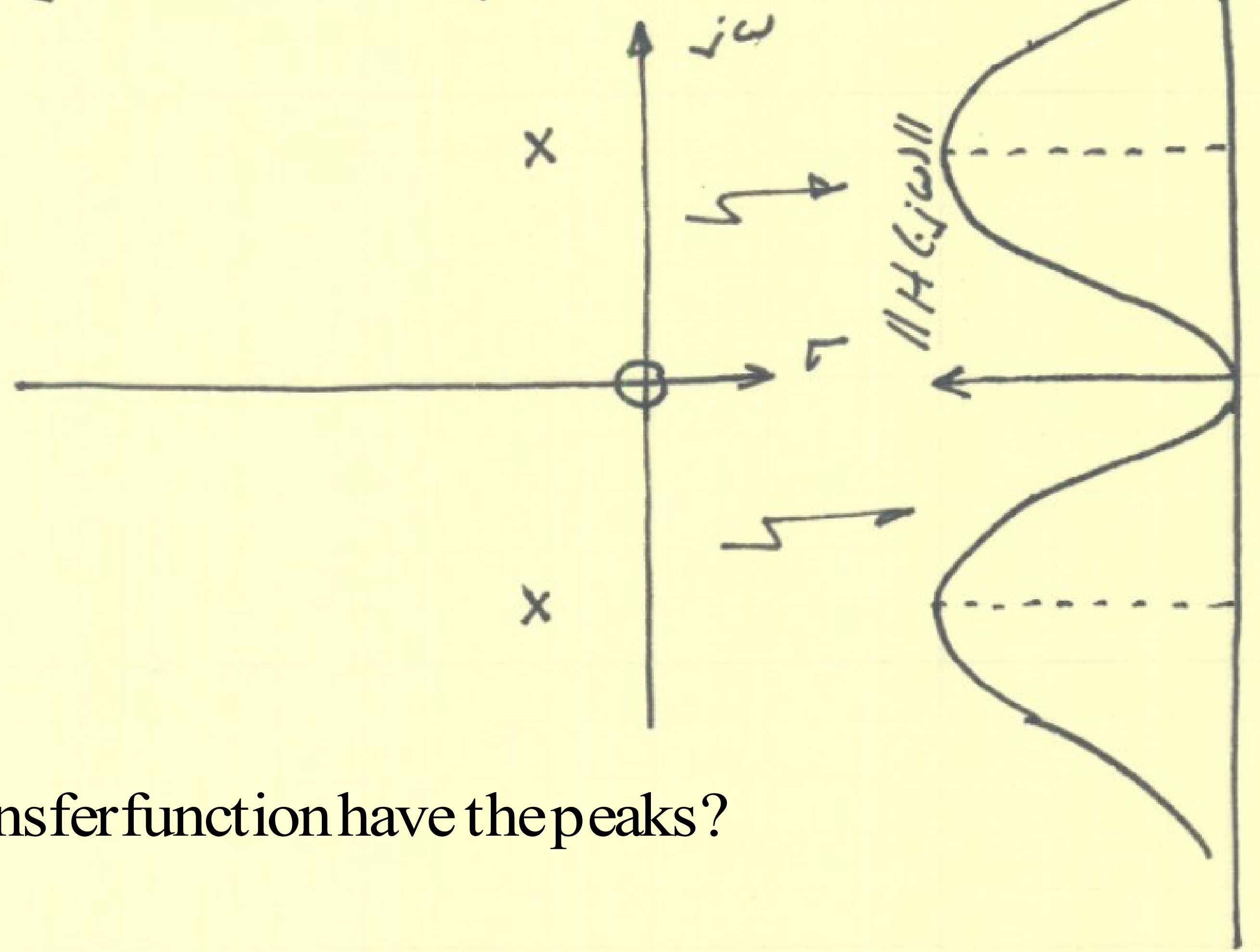
$\zeta > 1$



$$H(s) = \frac{s(2\zeta / \omega_n)}{1 + s(2\zeta / \omega_n) + s^2 / \omega_n^2}$$

Root Constellation and Frequency Response: Complex Poles

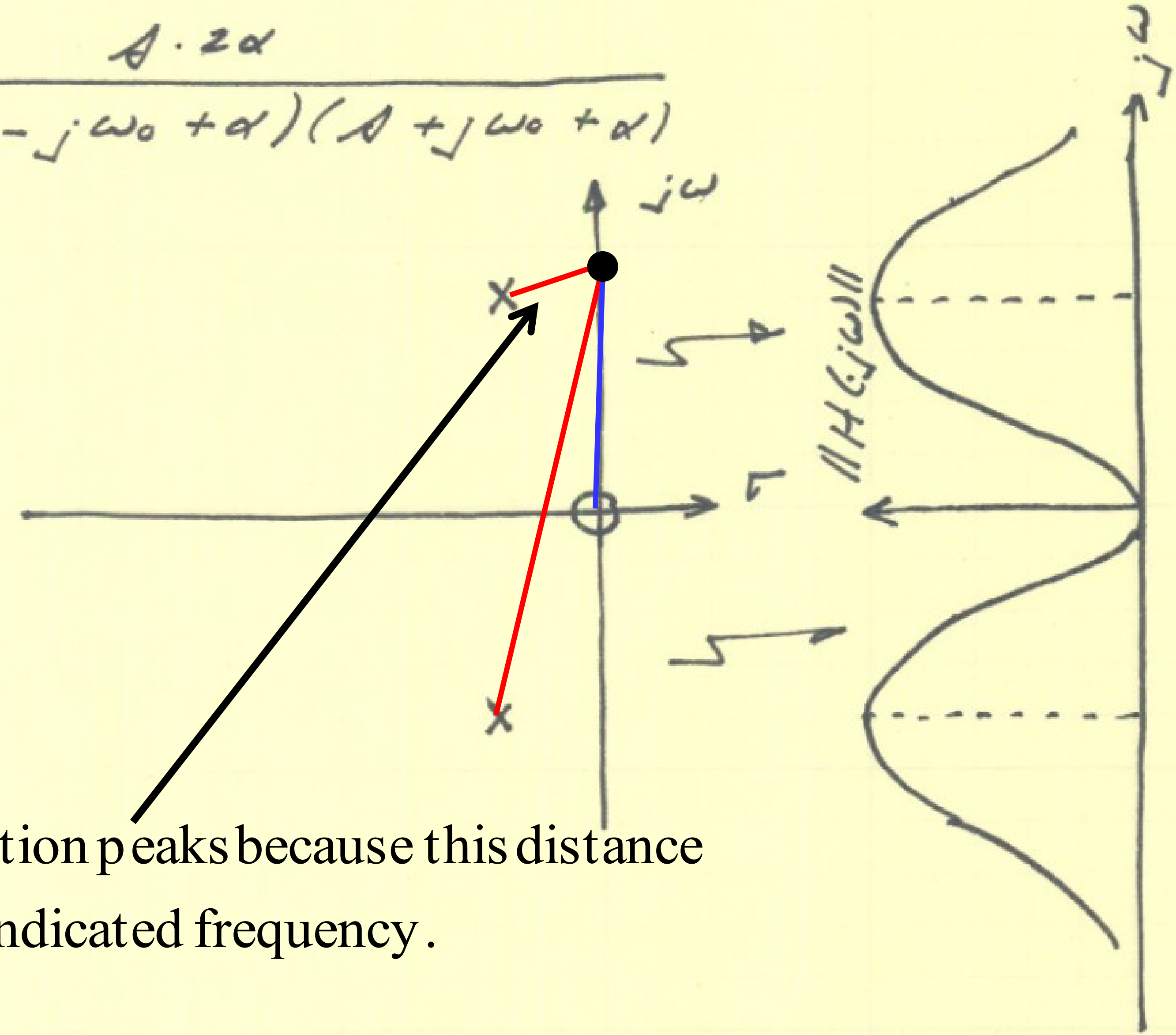
$$H(j\omega) = \frac{s \cdot 2\alpha}{(s - j\omega_0 + \alpha)(s + j\omega_0 + \alpha)}$$



Why does the transfer function have the peaks?

Root Constellation and Frequency Response: Complex Poles

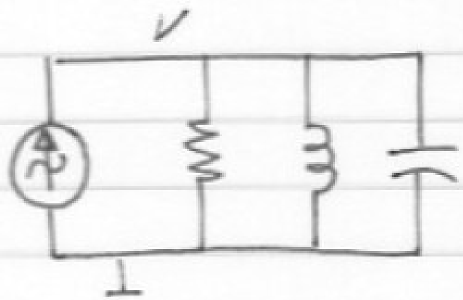
$$H(j\omega) = \frac{s \cdot 2\alpha}{(s - j\omega_0 + \alpha)(s + j\omega_0 + \alpha)}$$



The transfer function peaks because this distance gets small at the indicated frequency.

Quality factor of an Energy Storage Element

$$Q = 2\pi \cdot \frac{\text{Maximum energy stored}}{\text{energy dissipated in one cycle}}$$



$I = I_0 \cos \omega_n t$, $\omega_n = 1/\sqrt{LC}$

$v = \underbrace{R I_0}_{V_0} \cos \omega_n t$ because $\omega = \omega_n$

Energy in Capacitor

$$E_C = 1/2\, C v^2 = 1/2\, C V_0^2 \cos^2(\omega_n t)$$

Energy in inductor

$$I_L = (1/L) \int v(t)\, dt = (V_0/L)(\omega_n)^{-1} \sin(\omega_n t)$$

$$E_L = 1/2\, L I_L^2 = (V_0^2/2L\omega_n^2) \sin^2(\omega_n t) = 1/2\, C V_0^2 \sin^2(\omega_n t)$$

Quality factor of an Energy Storage Element

Total Energy stored: $E_T = E_C + E_L = 1/2\, C V_0^2$!

Energy Dissipated in 1 cycle

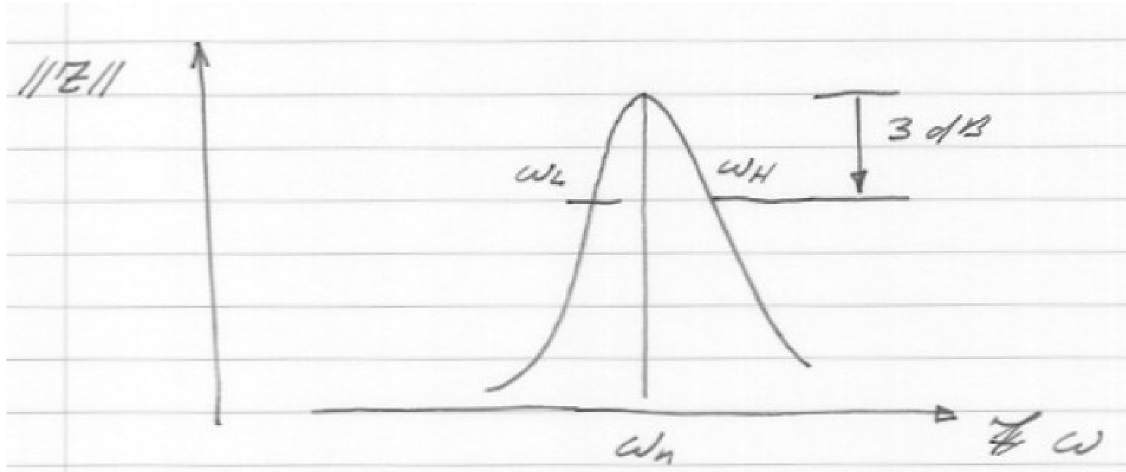
$$E = P \cdot T = (V_0^2/2R)\left(\frac{1}{f}\right) = (V_0^2/2R)\left(\frac{2\pi}{\omega_n}\right)$$

Q factor.

$$Q = \frac{2\pi\,(1/2\, C V_0^2)}{2\pi\,(1/2\, V_0^2/R\omega_n)} = CR\omega_n = R\sqrt{\frac{L}{C}} = \frac{1}{2\zeta}$$

$$\boxed{Q = \omega_n/2\alpha = 1/2\zeta}$$

-3 dB (Half-Power) Bandwidth



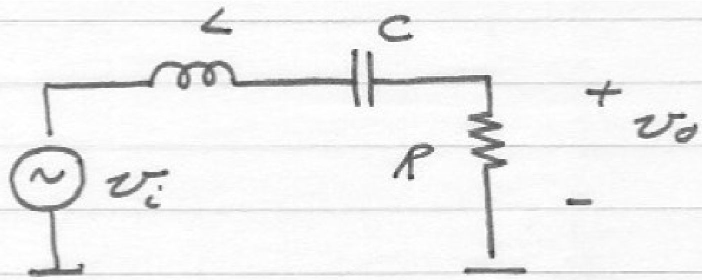
After some math:

$$\omega_L = \omega_n\left[\sqrt{1+\zeta^2} - \zeta\right]$$

$$\omega_H = \omega_n\left[\sqrt{1+\zeta^2} + \zeta\right]$$

$$3\text{ dB bandwidth} = \omega_H - \omega_L = 2\zeta\omega_n = \omega_n/Q$$

Series RLC Circuit



$$\frac{v_o}{v_i} = \frac{R}{R + sL + 1/sC} = \frac{sRC}{1 + sRC + s^2LC}$$

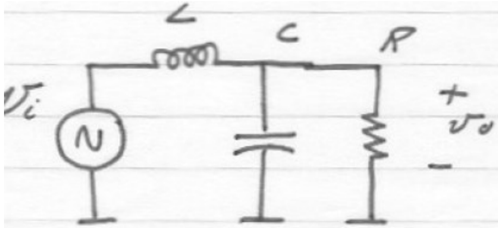
$$= \frac{s \cdot 2\zeta/\omega_n}{1 + s \cdot 2\zeta/\omega_n + s^2/\omega_n^2}$$

Canonical form.

$$\omega_n = 1/\sqrt{LC}$$

$$2\zeta/\omega_n = 2\zeta\left[\sqrt{LC}\right] = RC \Rightarrow \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Another Resonant Circuit



$$v_o/r + v_o sC + v_o/sL - v_i/sL = 0$$

$$\Rightarrow \frac{v_o}{v_i} = \frac{1/sL}{1/R + sC + 1/sL}$$

$$= \frac{1}{1 + s(L/R) + s^2 LC}$$

$$= \frac{1}{1 + (2\zeta/\omega_n)s + s^2/\omega_n^2}$$

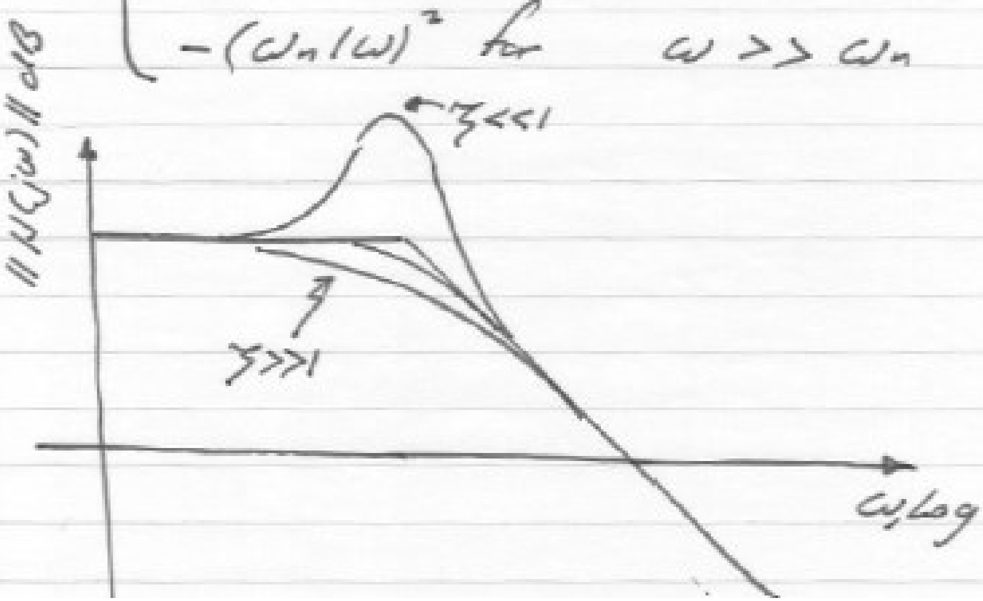
$$\omega_n = 1/\sqrt{LC}$$

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Another Resonant Circuit

$$H(j\omega) = \frac{1}{1 + (2\zeta/\omega_n)\cdot j\omega - \omega^2/\omega_n^2}$$

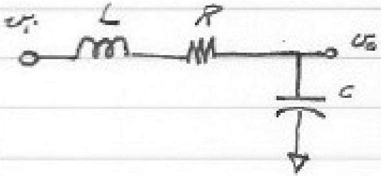
$$= \begin{cases} 1 & \text{for } \omega \ll \omega_n \\ 1/2\zeta & \text{for } \omega = \omega_n \\ -(\omega_n/\omega)^2 & \text{for } \omega \gg \omega_n \end{cases}$$



The case $\zeta \geq 1/\sqrt{2}$ gives no peaking

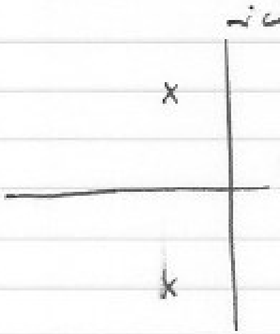
Peaking gets stronger for small ζ.

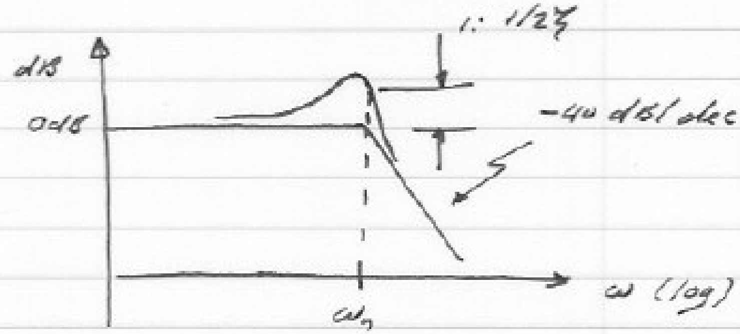
Low-Pass, 2nd-Order Filters



$$\frac{v_o}{v_i} = \frac{1/sC}{1/sC + R + sL} = \frac{1}{1 + sRC + s^2 LC}$$

$$= \frac{1}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$



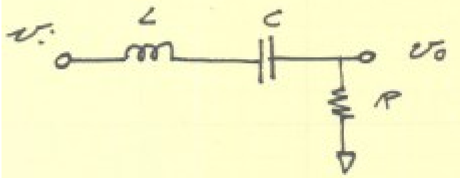


$$\frac{v_o}{v_i} = \begin{cases} 1 & \omega \ll \omega_n \\ 1/2\zeta & \omega = \omega_n \\ \omega_n^2/\omega^2 & \omega \gg \omega_n \end{cases}$$

Low Pass form

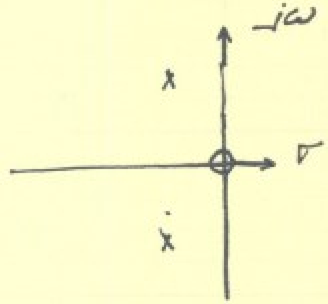
~~one~~ no zero, 2 complex poles

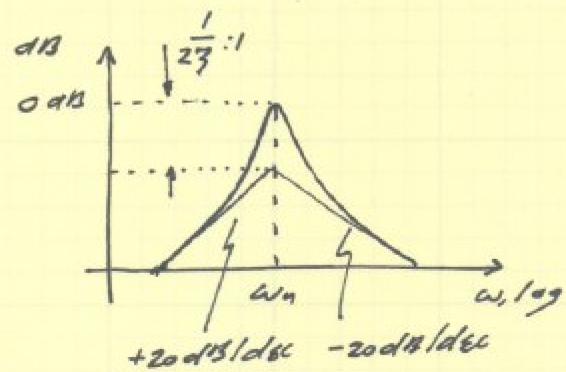
Band-Pass 2nd-Order Filters



$$\frac{v_o}{v_i} = \frac{R}{1/sC + R + sL} = \frac{sRC}{1 + sRC + s^2LC}$$

$$= \frac{s(2\zeta/\omega_n)}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$



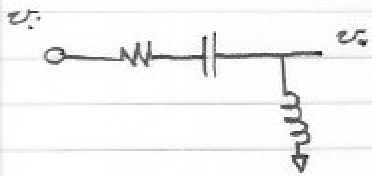


band pass form of "resonator"

one zero, two complex poles.

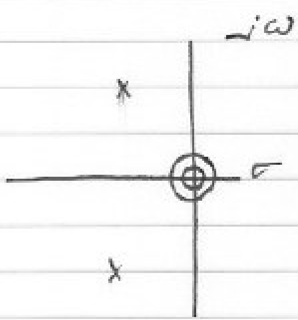
$$v_o/v_i = \begin{cases} 2\zeta \cdot \omega/\omega_n & (+20\text{ dB/dec}) \text{ for } \omega \ll \omega_n \\ 1 & \text{for } \omega = \omega_n \\ 2\zeta \cdot \omega_n/\omega & (-20\text{ dB/dec}) \text{ for } \omega \gg \omega_n \end{cases}$$

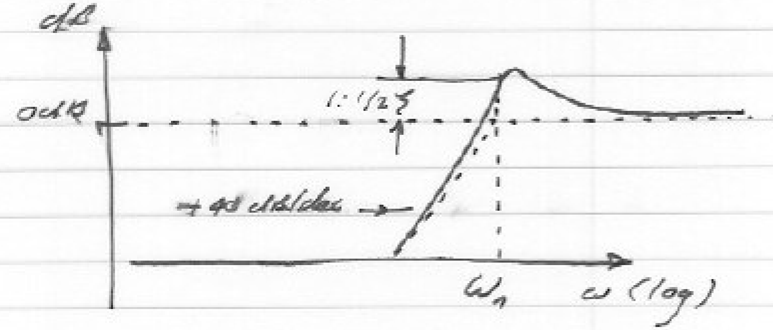
High-Pass 2nd-Order Filters



$$\frac{v_o}{v_i} = \frac{sL}{1/sC + R + sL} = \frac{s^2LC}{1 + sRC + s^2LC}$$

$$= \frac{s^2/\omega_n^2}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$



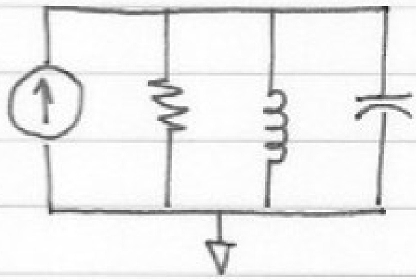


$$v_o/v_i = \begin{cases} \omega^2/\omega_n^2 \quad (+40\ \text{dB/dec}) & \text{for } \omega \ll \omega_n \\ 1/2\zeta & \omega = \omega_n \\ 1 & \omega \gg \omega_n \end{cases}$$

High pass form

two zeros, 2 complex poles

Impulse Response



$$i(t) = q_0 \cdot \delta(t)$$

$$V(s)/I(s) = R \frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

but $I(s) = q_0$

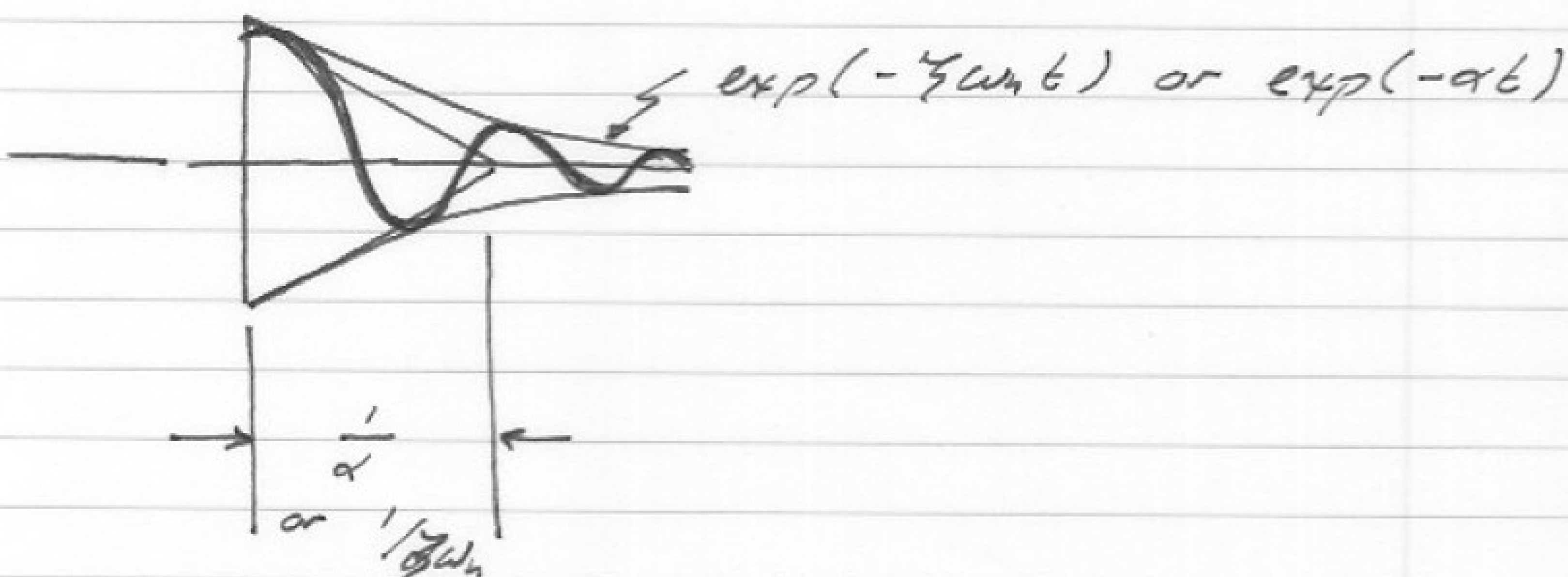
$$\Rightarrow V(s) = q_0 R \frac{s(2\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$= q_0 R 2\zeta\omega_n \left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{(-\zeta\omega_n/\omega_d)\cdot\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} \right]$$

but $(\omega_n/\omega_d) = 1/\sqrt{1-\zeta^2}$

Impulse Response

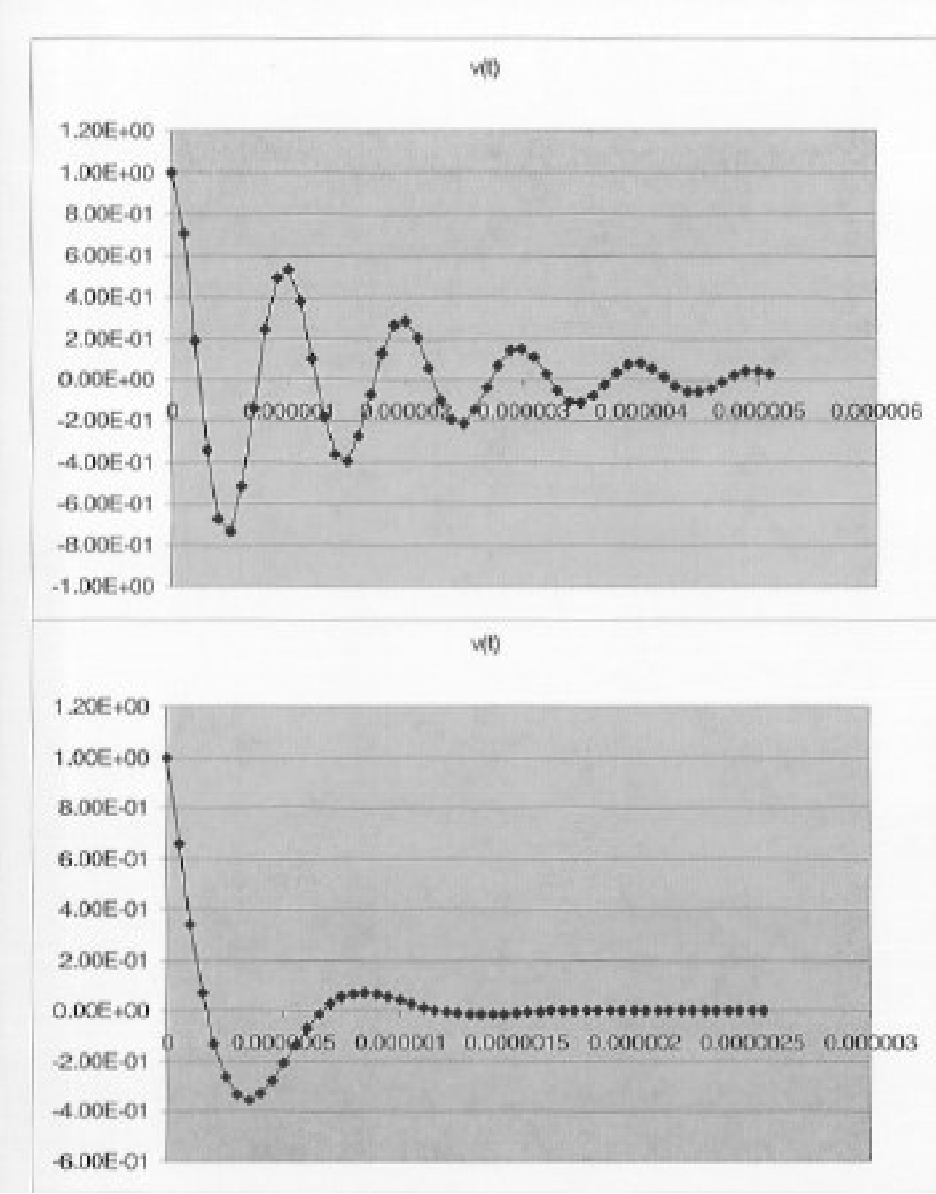
$$v(t) = q_0 R(2\zeta\omega_n)\left[e^{-\zeta\omega_n t}\right]\left[\cos(\omega_d t) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$



- rings at a frequency ω_d

- decays at a rate $\exp(-\alpha t)$

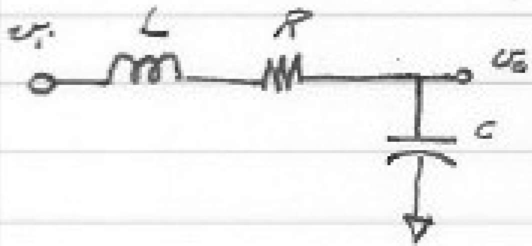
Impulse Response



$\zeta = 0.1$

$\zeta = 0.5$

Step Response of 2nd Order Systems



$$\frac{v_o}{v_i} = \frac{1/sC}{1/sC + R + sL} = \frac{1}{1 + sRC + s^2LC}$$

$$= \frac{1}{1 + s(2\zeta/\omega_n) + s^2/\omega_n^2}$$

What would be the step response?

Step Response of 2nd Order Systems

$$H(s) = \frac{1}{1 + js(2\zeta/\omega_n) + s^2/\omega_n^2}$$

$$= \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

so, if $v_{in}(t) = v_k \cdot u(t) \rightarrow V_{in}(s) = v_k/s$

then: $V_{out}(s) = V_k \dfrac{\omega_n^2/s}{(s + \zeta\omega_n)^2 + \omega_d^2}$

$$\frac{1/s}{(s + \zeta\omega_n)^2 + \omega_d^2} = \frac{a}{s} + \frac{bs + c}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$= a/s + \frac{bs + c}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\rightarrow 1 = a \cdot [s^2 + 2\zeta\omega_n s + \omega_n^2] + bs^2 + cs$$

s^0: $1 = a\omega_n^2 \rightarrow a = 1/\omega_n^2$

s^1: $0 = 2\zeta\omega_n a + c \rightarrow c = -2\zeta/\omega_n$

s^2: $0 = a + b \rightarrow b = -1/\omega_n^2$

Step Response of 2nd Order Systems

so: $$V_o(s) = \frac{V_k}{s} + \frac{-V_k\cdot s - 2\zeta\omega_n V_k}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$= \frac{V_k}{s} - V_k\frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

note that: $$\frac{V_k\cdot(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} \Leftrightarrow V_k\cdot e^{-\zeta\omega_n t}\cos\omega_d t\cdot u(t)$$

and that: $$\frac{V_k\cdot\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \rightleftharpoons V_k\zeta\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t\cdot u(t)$$

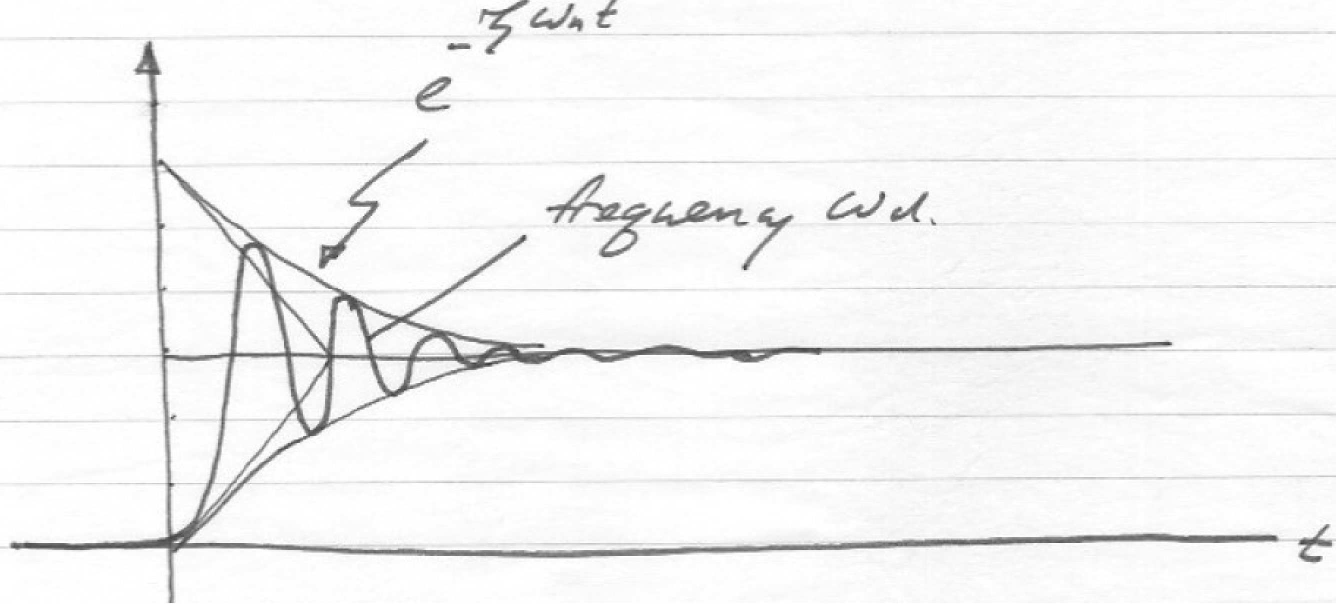
this used: $\omega_n = \omega_d\cdot\sqrt{1-\zeta^2}$

Low-Pass, 2nd-Order Filters: Its step response:

So:

$$v_o(t) = V_k \cdot u(t)$$

$$\times \left[1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \zeta \sqrt{1-\zeta^2} \sin \omega_d t \right) \right]$$



Low-Pass, 2nd-Order Filters: Its step response:

The***key observation****