ECE 137 B: Notes Set 12
Negative Feedback & Bandwidth:
Root Locus

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Feedback with ideal op-amp

Assume ideal op-amp (Z_{in}=∞ Ω, Z_{out}=0 Ω, CMRR=∞)

$$V_{out} = A_D(V^+ - V^-)$$

$$A_D = A_{OL} = \text{open-loop gain}$$

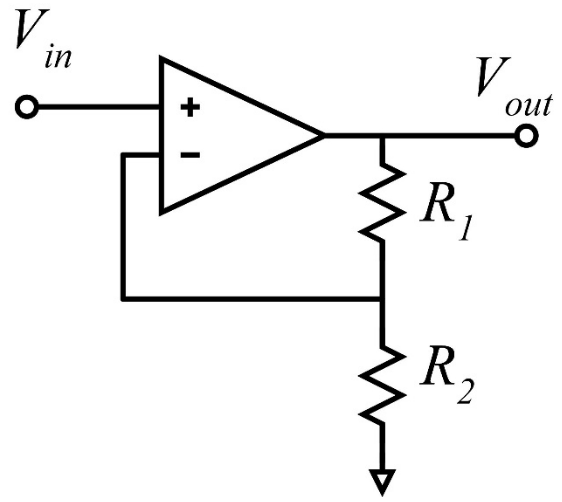
$$V^+ = V_{in}$$

$$V^- = \beta V_{out}$$

$$\beta = \text{feedback factor} = \frac{R_2}{R_1 + R}$$

$$T = \text{loop transmission} = A_D\beta = A_{OL}\beta$$

$$A_{CL} = \frac{V_{out}}{V_{in}} = \text{closed-loop gain} = \frac{1}{\beta}\frac{T}{1+T}$$



Feedback: systems representation

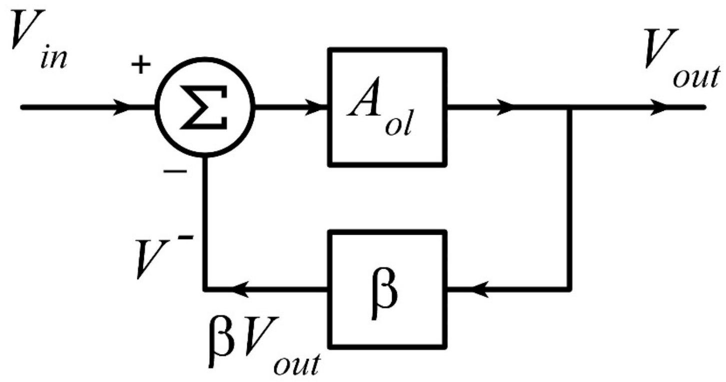
$$V_{out} = A_{OL}(V_{in} - V^-) = A_{OL}(V_{in} - \beta V_{out})$$

$$A_{CL}(s) = \frac{V_{out}(s)}{V_{in}(s)} = \text{closed-loop gain} = \frac{A_{OL}(s)}{1 + A_{OL}(s)\beta(s)} = \frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}$$

$\beta(s) =$ feedback factor

$A_{OL}(s) =$ open-loop gain

$T(s) =$ loop transmission $= A_{OL}(s)\beta(s)$



This might represent a physical system with feedback:

Car anti-lock braking

Electronic feedback control of aircraft roll, pitch, or yaw

Car anti-rollover protection

Fuel/air mixture control in car engine

Robotics: position of mechanical arm

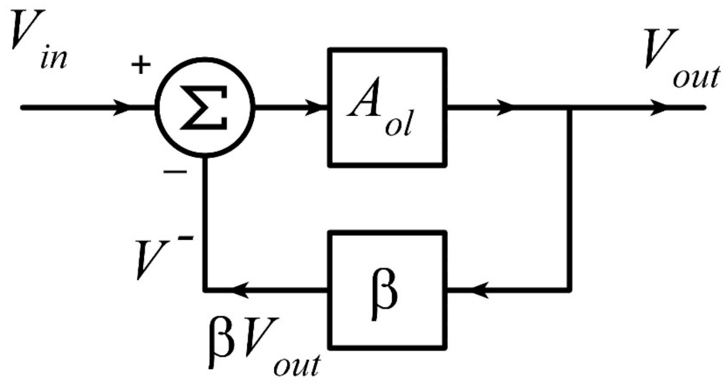
....

It is widely stated that feedback increases bandwidth...

once we ensure that the feedback is stable, this may or may not be true.

Effect of Feedback on Single-Pole System (1)

$$A_{CL}(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{A_{OL}(s)}{1+A_{OL}(s)\beta(s)}=\frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}$$



Consider an open-loop gain with a single real-axis pole:

$$A_{OL}(s)=\frac{A_{OL,DC}}{1+s/\omega_{OL}}$$

$A_{OL,DC}$ = open-loop gain at DC

ω_{OL} = one real-axis pole in the open-loop gain

Consider a frequency-independent feedback factor :

$\beta(s)=\beta_0$;

$$\rightarrow T(s)=A_{OL}(s)\beta(s)=\frac{A_{OL,DC}\beta_0}{1+s/\omega_{OL}}=\frac{T_0}{1+s/\omega_{OL}}=\frac{N_T(s)}{D_T(s)}$$

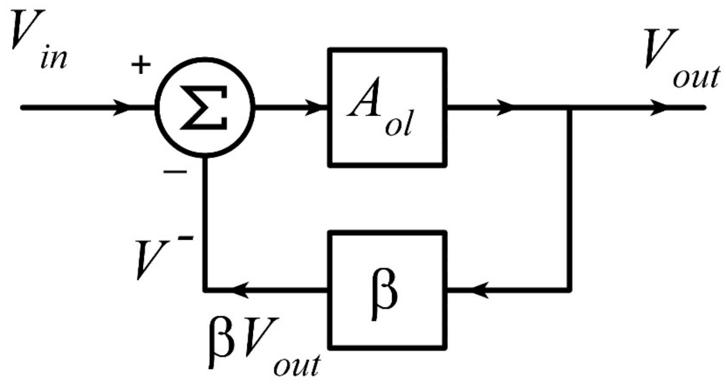
$T_0=A_{OL,DC}\beta_0$ = loop transmission at DC

Effect of Feedback on Single-Pole System (2)

$$A_{CL}(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}=\frac{1}{\beta(s)}\frac{N_T/D_T}{1+N_T/D_T}=\frac{1}{\beta(s)}\frac{N_T}{N_T+D_T}$$

$$=\frac{1}{\beta_0}\frac{T_0}{T_0+(1+s/\omega_{OL})}=\frac{1}{\beta_0}\frac{T_0}{1+T_0+s/\omega_{OL}}$$

$$=\frac{1}{\beta_0}\frac{T_0}{1+T_0}\frac{1}{1+\left(\frac{s}{(1+T_0)\omega_{OL}}\right)}$$



So:

$$A_{CL}(s)=A_{CL,DC}\frac{1}{1+s/\omega_{CL}}$$

where $A_{CL,DC}=\frac{1}{\beta_0}\frac{T_0}{1+T_0}=$ closed-loop gain at DC

and $\omega_{CL}=(1+T_0)\omega_{OL}=$ bandwidth (pole frequency) of closed-loop gain.

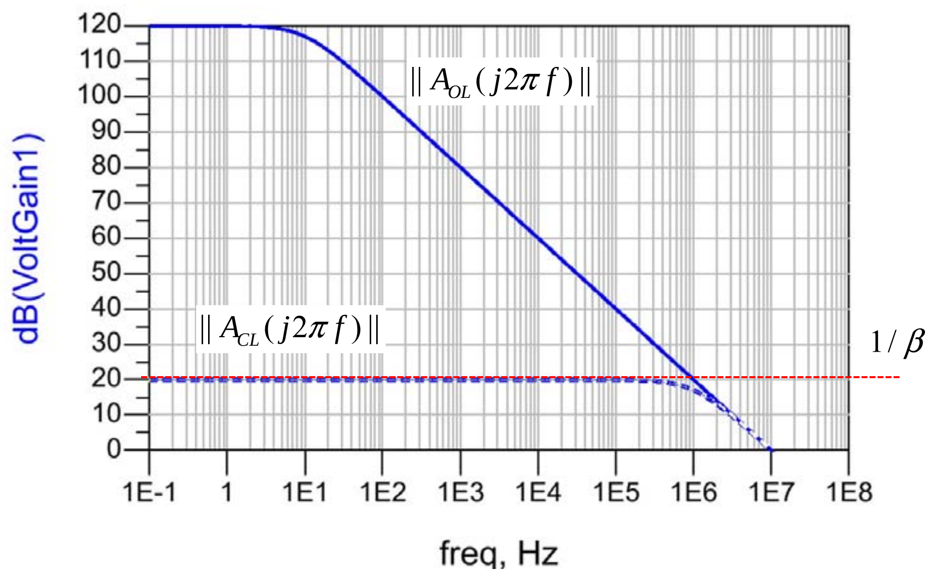
The bandwidth has increased by the factor $(1+T_0)$

Effect of Feedback on Single-Pole System (3)

Example:

$$A_{OL}(j2\pi f) = \frac{A_{OL,DC}}{1 + jf / f_{OL}} = \frac{10^6}{1 + jf / 10 \text{ Hz}}$$

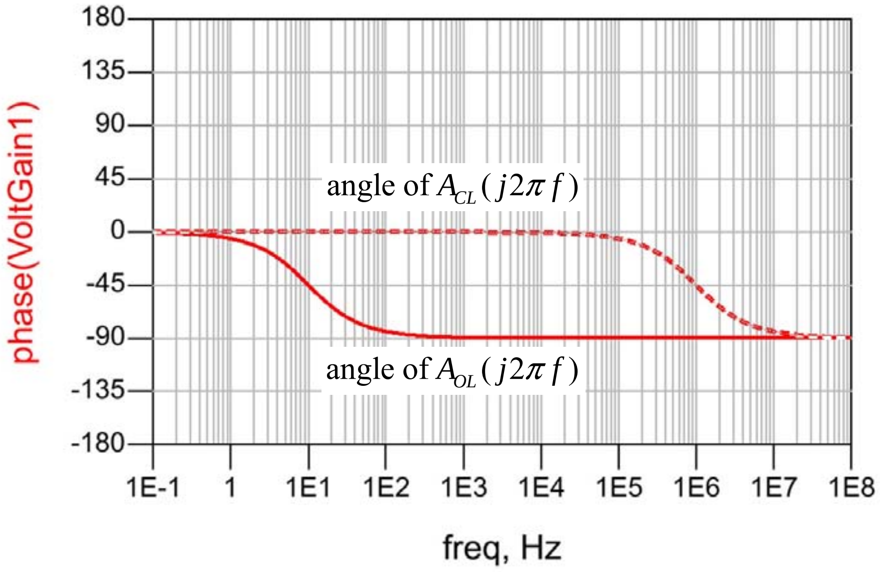
$$\beta(j2\pi f) = \beta_0 = 1/10$$



$$A_{CL}(j2\pi f) = \frac{1}{\beta(j2\pi f)} \frac{T(j2\pi f)}{1 + T(j2\pi f)} = A_{CL,DC} \frac{1}{1 + jf / f_{CL}}$$

where $A_{CL,DC} = \dfrac{1}{\beta_0} \dfrac{T_0}{1 + T_0} = 10 \dfrac{10^6 / 10}{1 + 10^6 / 10} \simeq 10$

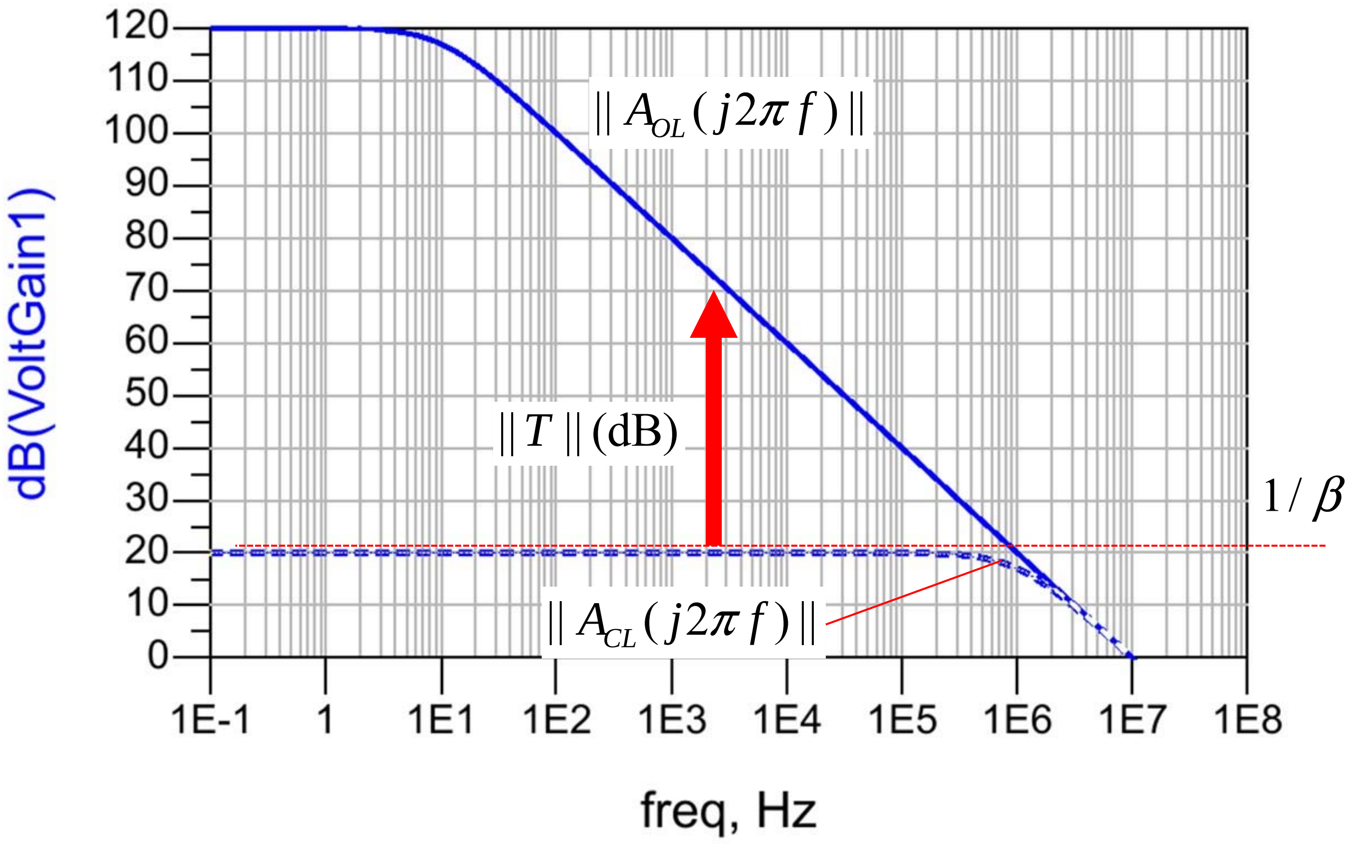
and $f_{CL} = (1 + T_0) f_{OL} = (1 + 10^5)(10 \text{ Hz}) = 1 \text{ MHz}.$



Effect of Feedback on Single-Pole System (4)

$$T = A_{OL}\beta \rightarrow T(\text{dB}) = A_{OL}(\text{dB}) - (1/\beta)(\text{dB})$$

$$A_{CL} = \frac{1}{\beta}\frac{T}{1+T} = \frac{A_{OL}}{1+T} = \begin{cases} \dfrac{1}{\beta} & \|T\| >> 1 \\ \dfrac{1}{\beta}\dfrac{e^{j\theta_T}}{1+e^{j\theta_T}} & \|T\| = 1 \\ A_{OL} & \|T\| << 1 \end{cases}$$



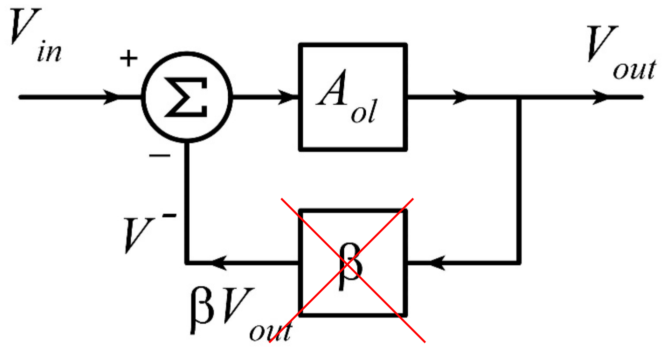
Example:

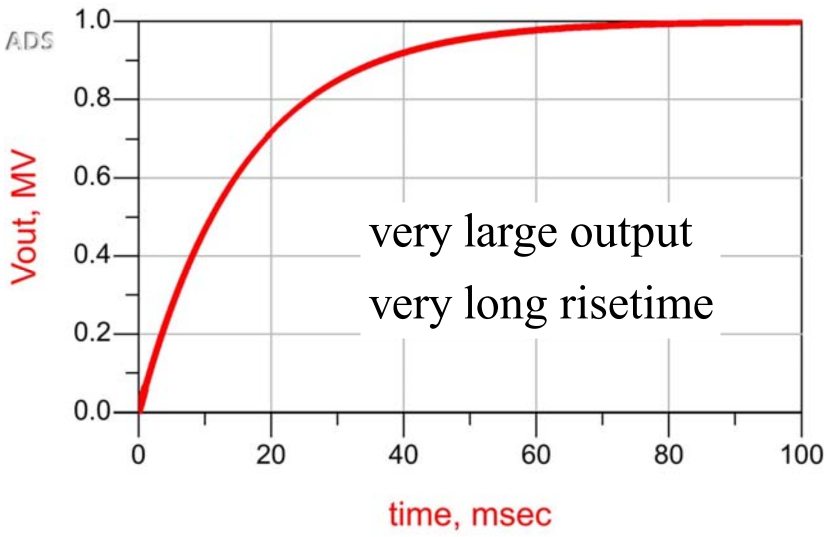
$$A_{OL}(j2\pi f) = \frac{A_{OL,DC}}{1+jf/f_{OL}} = \frac{10^6}{1+jf/10\text{ Hz}}$$

$$\beta(j2\pi f) = \beta_0 = 1/10$$

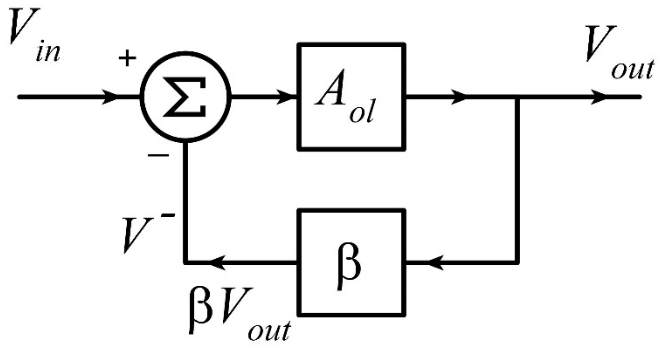
Effect of Feedback on Single-Pole System (5)

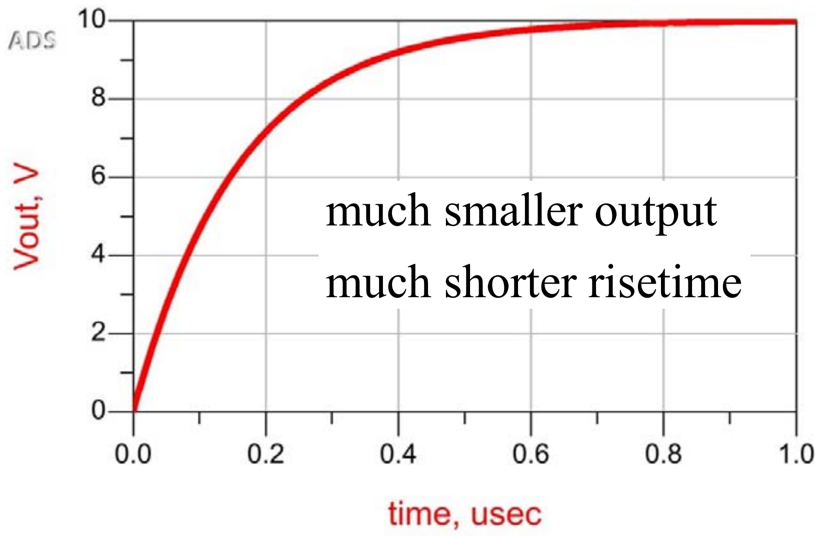
Step response, no feedback





Step response, with feedback





Example:

$$A_{OL}(j2\pi f) = \frac{A_{OL,DC}}{1 + jf / f_{OL}} = \frac{10^6}{1 + jf / 10 \text{ Hz}}$$

$$\beta(j2\pi f) = \beta_0 = 1/10$$

Effect of Feedback on Single-Pole System (6)

$$A_{OL}(s)=\frac{A_{OL,DC}}{1+s/\omega_{OL}}$$

$$\beta(s)=\beta_0$$

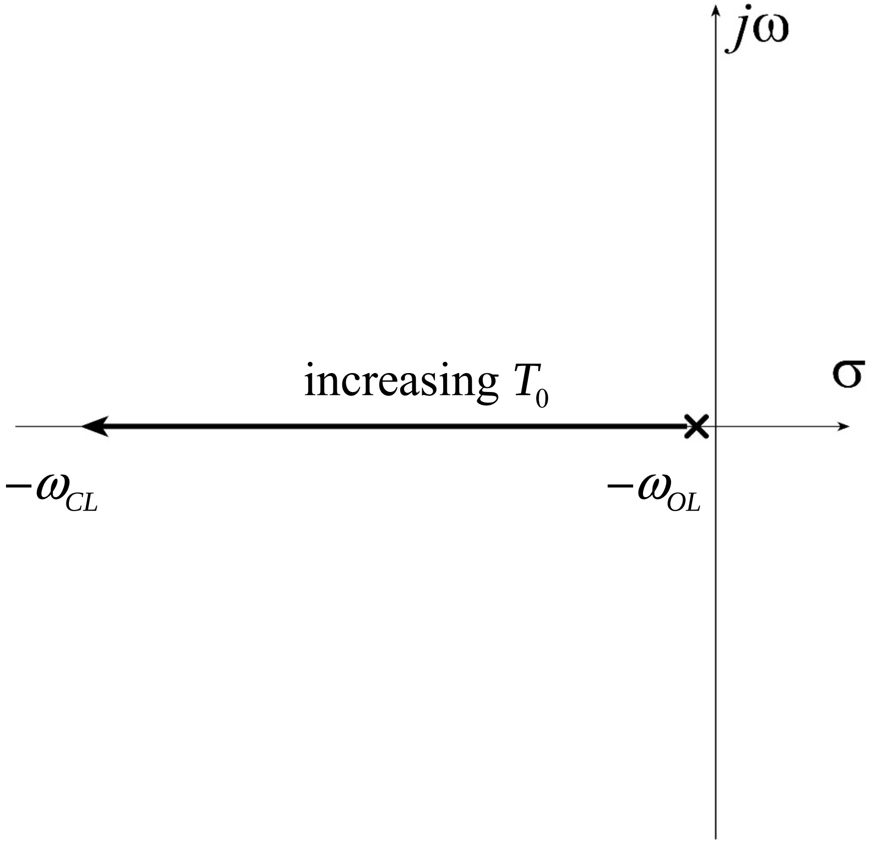
$$T(s)=A_{OL}(s)\beta_0$$

$$T(s)=\frac{A_{OL,DC}\beta_0}{1+s/\omega_{OL}}$$

$$A_{CL}(s)=\frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}$$

As we apply increasing amounts of negative feedback, the closed-loop plot moves to the left

This plot, showing the movement of the pole positions with feedback, is called a root locus plot.



Effect of Feedback on Two-Pole System (1)

$$A_{OL}(s)=\frac{A_{OL,DC}}{(1+s/\omega_{OL1})(1+s/\omega_{OL2})}=\frac{A_{OL,DC}}{1+s(1/\omega_{OL1}+1/\omega_{OL2})+s^2/\omega_{OL1}\omega_{OL2}}$$

$$=\frac{A_{OL,DC}}{1+a_1s+a_2s^2} \text{ where } a_1=(1/\omega_{OL1}+1/\omega_{OL2}) \text{ and } a_2=1/\omega_{OL1}\omega_{OL2}$$

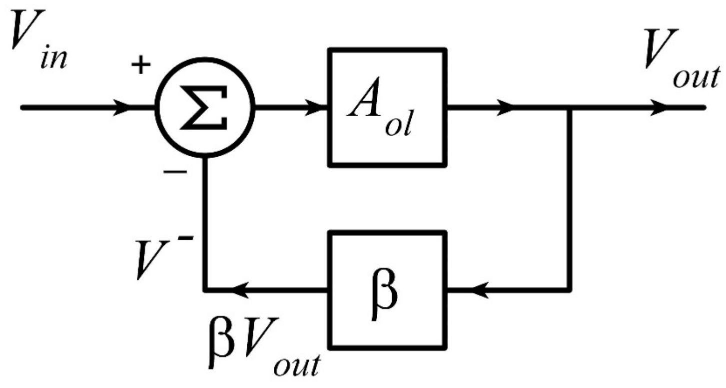
$$\beta(s)=\beta_0$$

$$T(s)=A_{OL}(s)\beta(s)=\frac{A_{OL,DC}\beta_0}{1+a_1s+a_2s^2}=\frac{T_0}{1+a_1s+a_2s^2}=\frac{N_T(s)}{D_T(s)}$$

$$A_{CL}(s)=\frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}=\frac{1}{\beta(s)}\frac{N_T/D_T}{1+N_T/D_T}=\frac{1}{\beta(s)}\frac{N_T}{N_T+D_T}$$

$$A_{CL}(s)=\frac{1}{\beta_0}\frac{T_0}{T_0+1+a_1s+a_2s^2}=\frac{1}{\beta_0}\frac{T_0}{T_0+1}\frac{1}{1+\left(\frac{a_1}{1+T_0}\right)s+\left(\frac{a_2}{1+T_0}\right)s^2}$$

$$A_{CL}(s)=A_{CL,DC}\frac{1}{1+s\left(2\zeta/\omega_n\right)+s^2/\omega_n^2}$$



Effect of Feedback on Two-Pole System (2)

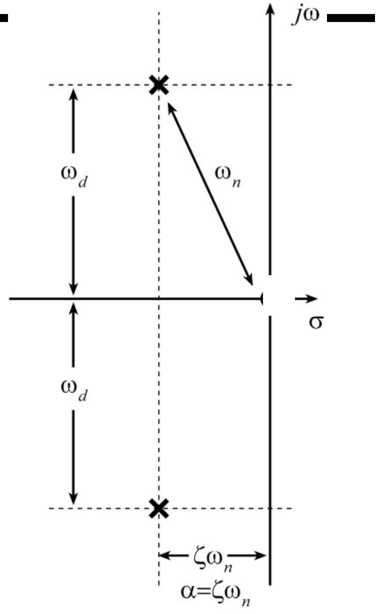
$$\frac{1}{\omega_n^2}=\left(\frac{a_2}{1+T_0}\right) \text{ so } \omega_n=\sqrt{\frac{1+T_0}{a_2}}$$

$$\frac{2\zeta}{\omega_n}=\left(\frac{a_1}{1+T_0}\right) \text{ so } \zeta=\left(\frac{a_1}{1+T_0}\right)\frac{\omega_n}{2}=\frac{1}{2}\left(\frac{a_1}{1+T_0}\right)\sqrt{\frac{1+T_0}{a_2}}=\frac{1}{2}\frac{a_1}{\sqrt{a_2}}\sqrt{\frac{1}{1+T_0}}$$

$\zeta<1$

$$s_{p1,2}=-\zeta\omega_n \pm j\omega_d$$

where $\omega_d=\omega_n\cdot\sqrt{1-\zeta^2}$



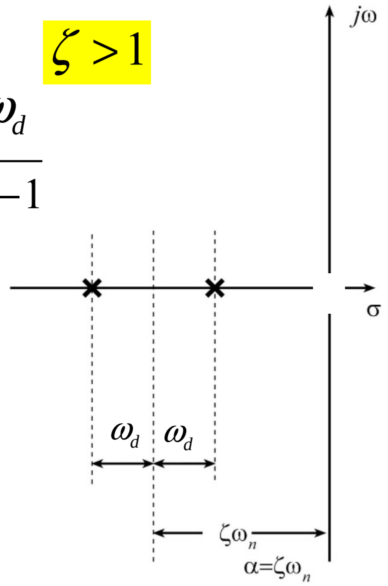
$$\omega_n=\sqrt{\frac{1+T_0}{a_2}} \text{ more feedback} \rightarrow \text{ higher natural resonant frequency}$$

$$\zeta=\frac{1}{2}\frac{a_1}{\sqrt{a_2}}\sqrt{\frac{1}{1+T_0}} \text{ more feedback} \rightarrow \text{ lower damping}$$

$\zeta>1$

$$s_{p1,2}=-\zeta\omega_n \pm \omega_d$$

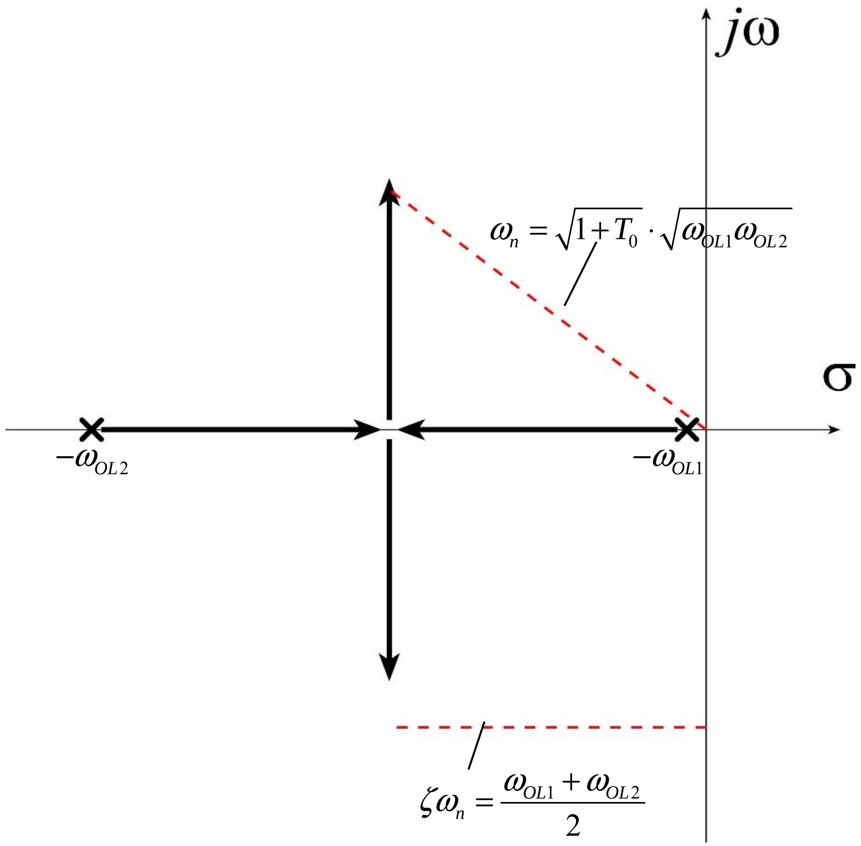
where $\omega_d=\omega_n\cdot\sqrt{\zeta^2-1}$



Effect of Feedback on Two-Pole System (3)

$$\omega_n = \sqrt{\frac{1+T_0}{a_2}} = \sqrt{1+T_0} \cdot \sqrt{\omega_{OL1}\omega_{OL2}}$$

$$\zeta\omega_n = \frac{1}{2}\frac{a_1}{\sqrt{a_2}}\sqrt{\frac{1}{1+T_0}}\sqrt{\frac{1+T_0}{a_2}} = \frac{1}{2}\frac{a_1}{a_2} = \frac{1}{2}\frac{1/\omega_{OL1}+1/\omega_{OL2}}{1/\omega_{OL1}\omega_{OL2}} = \frac{\omega_{OL1}+\omega_{OL2}}{2}$$



Here again is the root locus

Increasing feedback $\rightarrow$ poles move towards each other

Eventually, they will meet.

Further increased feedback $\rightarrow$ complex poles

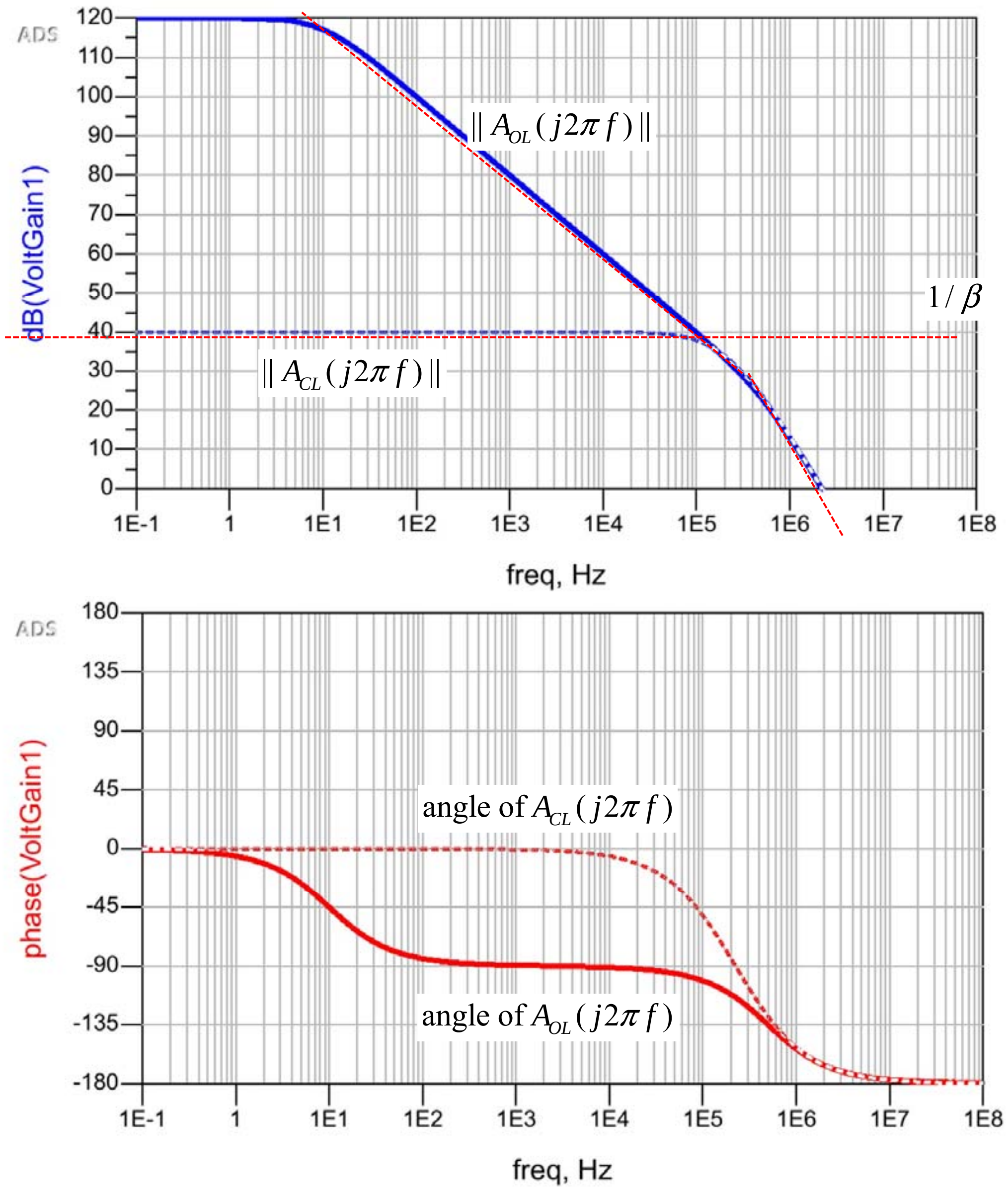
Effect of Feedback on Two-Pole System (3)

Example:

$$A_{OL}(j2\pi f)=\frac{A_{OL,DC}}{(1+jf/f_{OL1})(1+jf/f_{OL2})}$$

$$=\frac{10^6}{(1+jf/10\text{ Hz})(1+jf/500\text{ kHz})}$$

$$\beta(j2\pi f)=\beta_0=1/100$$



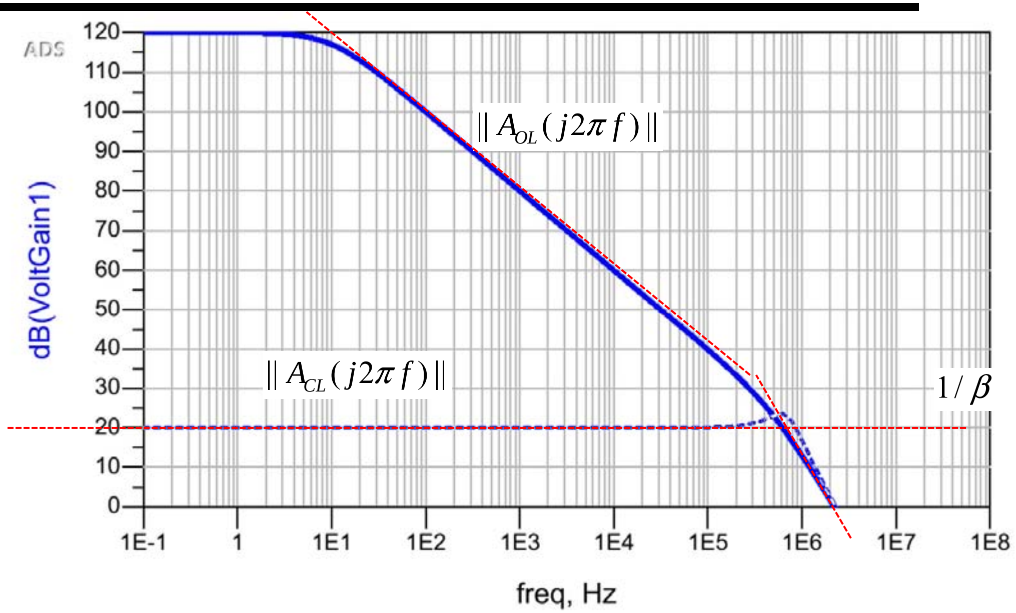
Effect of Feedback on Two-Pole System (5)

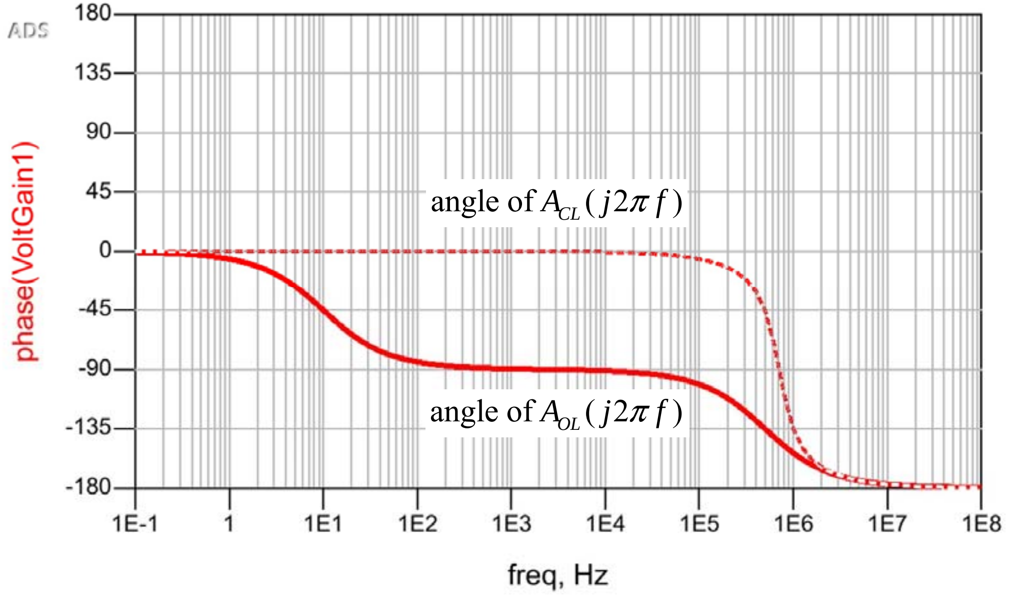
Example:

$$A_{OL}(j2\pi f)=\frac{A_{OL,DC}}{(1+jf/f_{OL1})(1+jf/f_{OL2})}$$

$$=\frac{10^6}{(1+jf/10\text{ Hz})(1+jf/500\text{ kHz})}$$

$$\beta(j2\pi f)=\beta_0=1/10$$





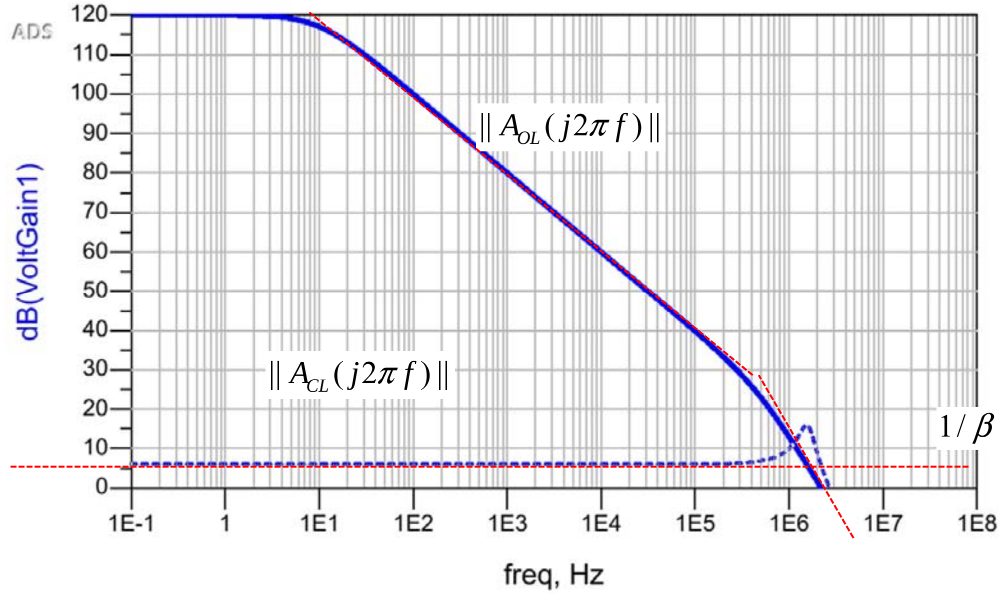
Effect of Feedback on Two-Pole System (5)

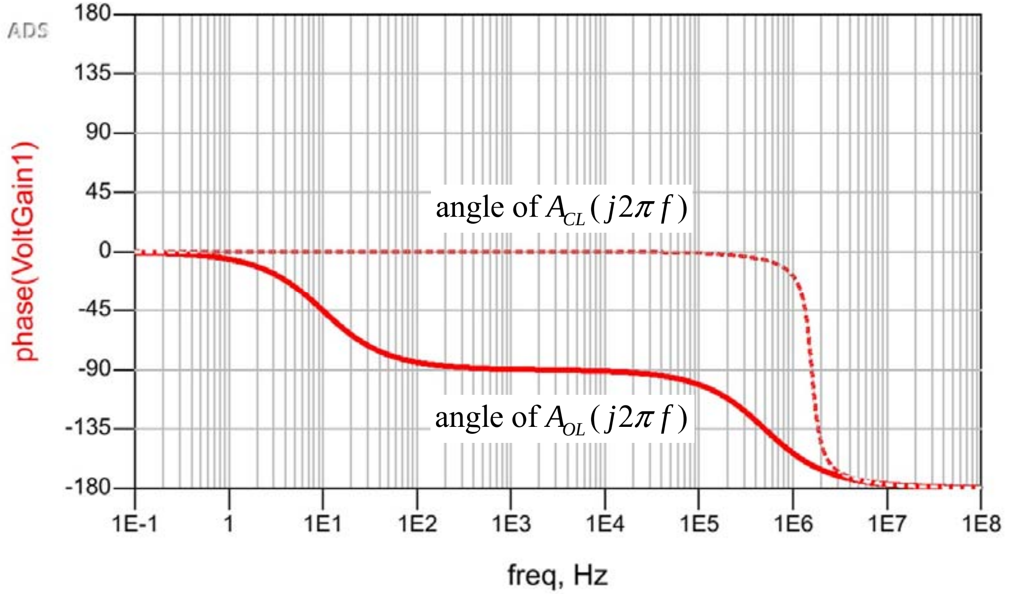
Example:

$$A_{OL}(j2\pi f) = \frac{A_{OL,DC}}{(1+jf/f_{OL1})(1+jf/f_{OL2})}$$

$$= \frac{10^6}{(1+jf/10\text{ Hz})(1+jf/500\text{ kHz})}$$

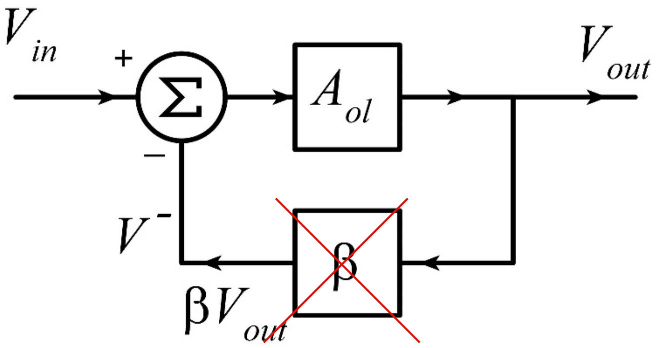
$$\beta(j2\pi f) = \beta_0 = 1/2$$

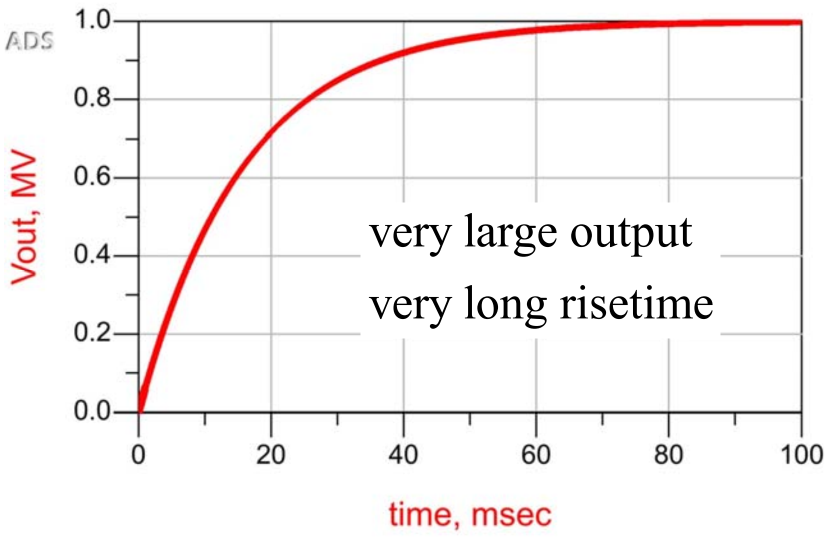




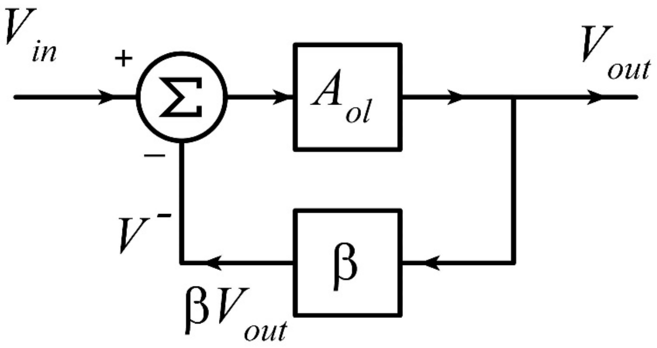
Effect of Feedback on Two-Pole System (6)

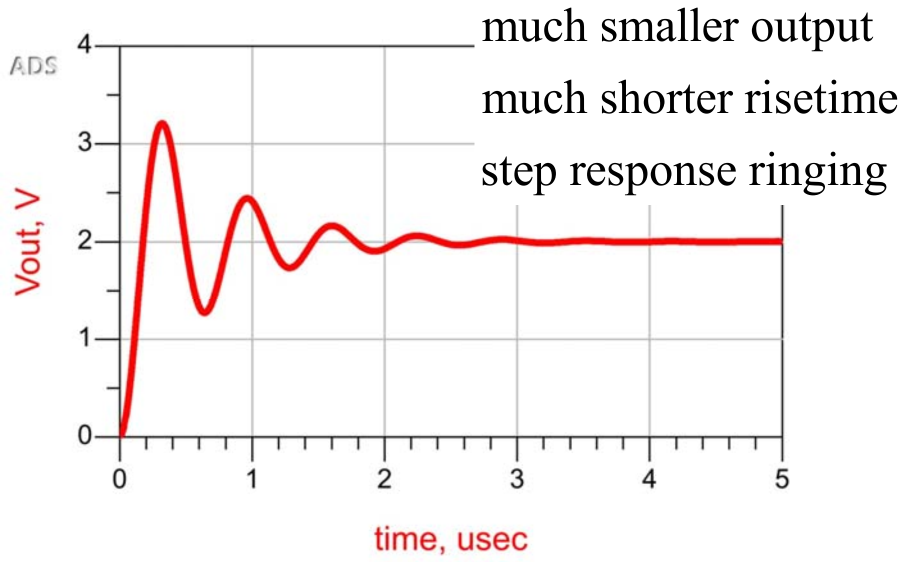
Step response, no feedback





Step response, with feedback





$$A_{OL}(j2\pi f)=\frac{10^6}{(1+jf/10\text{ Hz})(1+jf/500\text{ kHz})}$$

$$\beta(j2\pi f)=1/2$$

Effect of Feedback on Three-Pole System (1)

$$A_{OL}(s)=\frac{A_{OL,DC}}{(1+s/\omega_{OL1})(1+s/\omega_{OL2})(1+s/\omega_{OL3})}$$

$$=\frac{A_{OL,DC}}{1+a_1s+a_2s^2+a_3s^3}$$; you can work out a_1, a_2, and a_3.

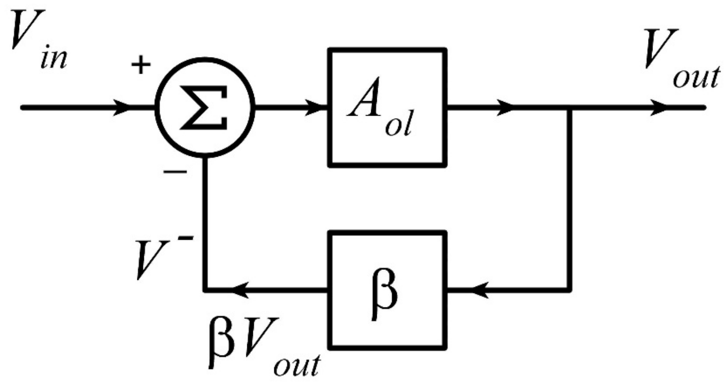
$$\beta(s)=\beta_0$$

$$T(s)=A_{OL}(s)\beta(s)=\frac{T_0}{1+a_1s+a_2s^2+a_3s^3}=\frac{N_T(s)}{D_T(s)}$$ where $T_0=A_{OL,DC}\beta_0$

$$A_{CL}(s)=\frac{1}{\beta(s)}\frac{T(s)}{1+T(s)}=\frac{1}{\beta(s)}\frac{N_T/D_T}{1+N_T/D_T}=\frac{1}{\beta(s)}\frac{N_T}{N_T+D_T}$$

$$A_{CL}(s)=\frac{1}{\beta_0}\frac{T_0}{T_0+1+a_1s+a_2s^2+a_3s^3}$$

$$A_{CL}(s)=\frac{1}{\beta_0}\frac{T_0}{T_0+1}\frac{1}{1+\left(\frac{a_1}{1+T_0}\right)s+\left(\frac{a_2}{1+T_0}\right)s^2+\left(\frac{a_3}{1+T_0}\right)s^3}$$



Finding the roots (poles) of this cubic equation is hard work,
but we can quickly make some key observations.

Effect of Feedback on Three-Pole System (2)

poles: $1+\left(\dfrac{a_1}{1+T_0}\right)s_p+\left(\dfrac{a_2}{1+T_0}\right)s_p^2+\left(\dfrac{a_3}{1+T_0}\right)s_p^3=0$

Suppose that $(1+T_0)$ is very large.

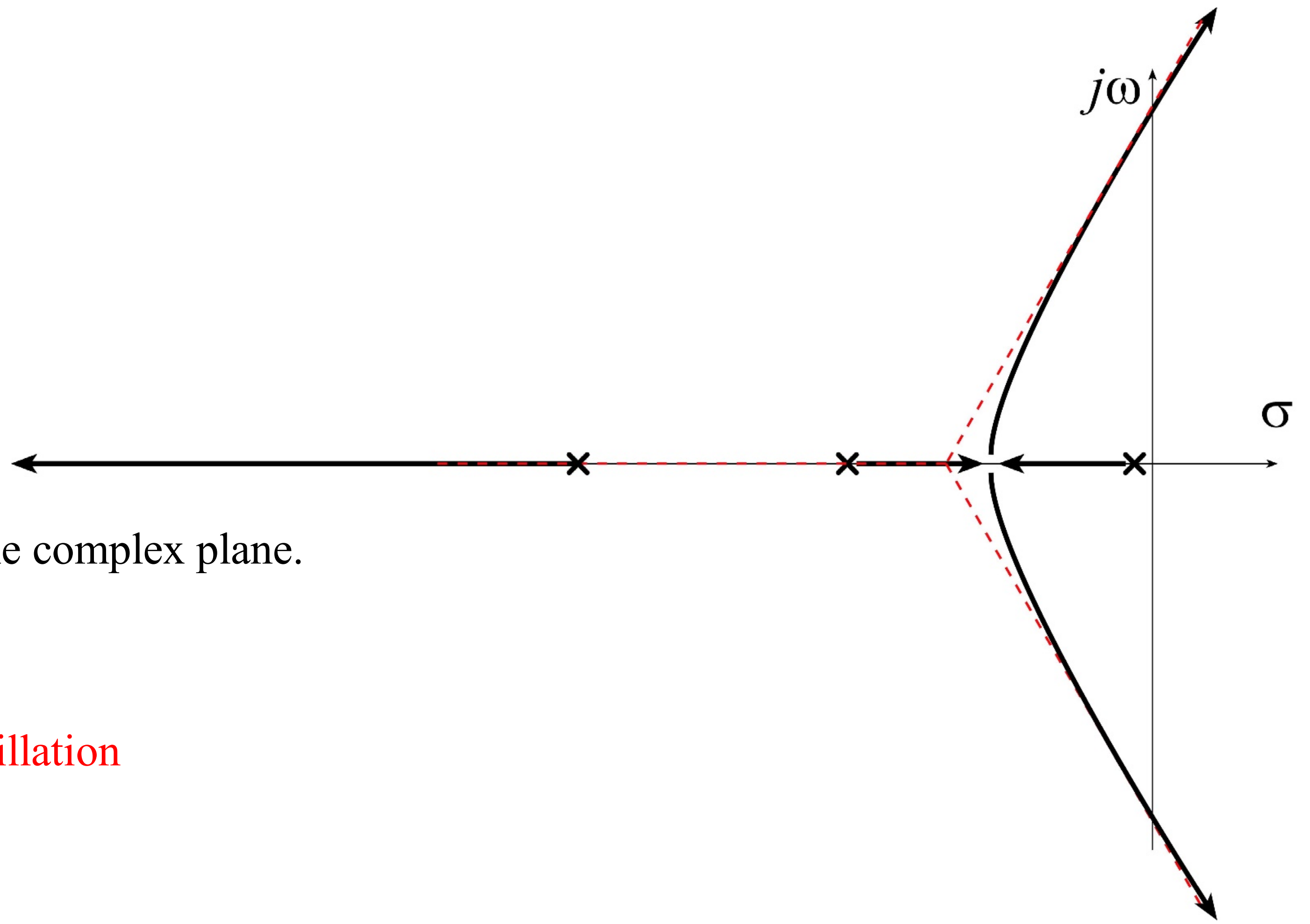
Then $|s_p|$ must be very large.

So $|s_p^3| \;\;>>\;\; |s_p^2| \;\;>>\;\; |s_p^1|$

so $s_p \simeq \left(-\dfrac{1+T_0}{a_3}\right)^{1/3}$ if $\|T_0\|>>1$

$(-1)^{1/3}$ has 3 roots, at angles of 60^0, -60^0, and 180^0 in the complex plane.

Root locus is therefore as sketched.



With large T_0, poles move into right half plane $\rightarrow$ oscillation

Comments

Feedback is used in transistor circuits.

Feedback is used in far, far more things than transistor circuits.
In general, the subject is called "control system theory".

There are 10-week undergraduate courses in control systems.

There's enough material for 4-8 Ph.D. -level courses in control systems.

We are just learning the basics.