Transmission lines in the frequency domain and the Smith chart.

Time-domain analysis is intellectually clearer, the picture being forward & reverse waves propagating, reflecting, and re-reflecting.

This analysis becomes intractable as soon as we introduce reactive (\( -j \) & \( j \)) impedances as multiple convolutions will be required for time-domain reflector analysis.

To analyze in the frequency domain instead this is the classical approach.
\[ V_s = \text{Re}\{V_0 e^{-jwt}\} \]

\[ V_0 \text{ is complex: } V_0 = |V_0| \exp\{i \phi_0\} \]

\[ o_3(t) = |V_0| \exp\{i \phi_0\} \cos(\omega t + \phi_0) \]

On a transmission line, waves travel as \((z \pm vt)\) or equivalently \((t \pm z/v)\)

For a wave traveling at velocity \(v\),

\[ \cos(\omega t + \phi_0) \rightarrow \cos(\omega (t \pm z/v) + \phi_0) \]

\[ = \cos(\omega t + \phi_0 \pm z \cdot \omega/v) \]

\[ = \cos(\omega t + \phi_0 \pm \beta z) \]

where \(\beta\) is the phase constant \(\beta = \omega/v\)
$\beta = \omega / v = 2\pi / \lambda$

... where $\lambda$ is the wavelength

For exponential waves, the form will thus be

\[ v_0 e^{-i\phi_0} e^{i\beta z} \]

and, as always with phasor notation, the $e^{j\omega t}$ time dependence is taken as implicit:

Voltage on the transmission line:

\[ V(z) = V^+(z) + V^-(z) \]

\[ = v^+(0) e^{-i\beta z} + v^-(0) e^{i\beta z} \]

\[ \zeta_0 I(z) = V^+(z) - V^-(z) \]

\[ = v^+(0) e^{-i\beta z} - v^-(0) e^{i\beta z} \]
Wave parameters, again:

define wave amplitudes such that \( a = 1 \)

then wave power = 1 watt

\[ a_q = \frac{V_0}{\sqrt{2}} \] forward wave amplitude

\[ b_q = \frac{V_0}{\sqrt{2}} \] reverse wave amplitude.

Power in forward wave = \( a a^* \)

"" reverse "" = \( b b^* \)

It is assumed throughout that we are using R.M.S. quantities.

*Note that the relationships must be generalized if \( E_0 \) is complex.
Reflection in the frequency domain:

\[ V'(0) = V^+(0) \Gamma_2 \]

\[ \Gamma_2 = \frac{Z_2 - 1}{Z_2 + 1} \text{ where} \]

\[ Z_2 = \frac{Z_L}{Z_0} \] is the normalized load impedance.
From a generator:

\[ V^+(0) = I_S V^-(0) + T_S V_s \]

\[ T_S = \text{source transmission coefficient} \]

\[ T_S = \frac{Z_0}{Z_s + Z_0} \]

\[ I_S = \frac{Z_s - 1}{Z_s + 1} \text{ source reflection coefficient} \]

\[ Z_s = Z_s / Z_0 \text{ normalized source impedance.} \]
Movement of reference plane

\[ v(3) = v^+(3) + v^-(3) \]

\[ = v^+(3) \left[ 1 + \Gamma(3) \right] \]

where \( \Gamma(3) \equiv \frac{v^-(3)}{v^+(3)} \) is the position-dependent impedance.

Reflection coefficient

\[ v(3) = v^{+(0)} e^{-j/3} \left[ 1 + \Gamma(0) e^{+2j/3} \right] \]

Because

\[ \Gamma(3) = \frac{v^-(3)}{v^+(3)} = \frac{v^{+(0)} e^{+j/3}}{v^{+(0)} e^{-j/3}} = \Gamma(0) e^{2j/3} \]
So:
\[ \Gamma(-l) = \text{the reflection coefficient at a distance } l \text{ from the load} \]
\[ = \Gamma(0) e^{-2j\beta l} \]

So, the reflection coefficient has gone through a phase shift of
\[ \text{minus } \frac{l}{\lambda} \cdot 2 \cdot 2\pi \text{ radians} \]
or
\[ \text{minus } \frac{2\beta l}{\lambda} \text{ radians} \]
or
\[ \text{minus } \frac{l}{\lambda} \cdot 2 \cdot 360 \text{ degrees} \]

... reflection coefficient changes phase with position.
Impedance vs. position:

\[ Z(z) = \frac{V(z)}{I(z)} \]

\[ = \left[ V^+(z) + V^-(z) \right] / \left[ \frac{1}{\varepsilon_0} \left( V^+(z) - V^-(z) \right) \right] \]

\[ = \frac{\varepsilon_0}{1 + \Gamma(z)} \frac{1}{1 - \Gamma(z)} \]

Normalized impedance at any point:

\[ \frac{Z(z)}{Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \]

Input impedance of line at \( z = -L \):

\[ Z(-L) = \frac{1 + \Gamma(-L)}{1 - \Gamma(-L)} \text{ normalized} \]

\[ Z(-L) = \varepsilon_0 \frac{Z_0}{Z_0} \frac{Z(-L)}{Z_0} \text{ unnormalized} \]
So: Impedance depends upon position.

Summarizing:

\[ V(z) = V^+(z) + V^-(z) \]  \{ waves \}

\[ Z_0 I(z) = V^+(z) - V^-(z) \]

\[ R(z) = \frac{V^-(z)}{V^+(z)} \]  \{ reflection coefficient \}

\[ = \Gamma(0) e^{2 j \beta z} \]

\[ Z(z) = \frac{1 + R(z)}{1 - R(z)} \]  \{ normalized impedance \}

while this is conceptually very simple, it is a great deal of math.

⇒ A graphical representation would help

   this is the Smith chart.
The relationship

\[ \frac{y}{y} = \frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \]

is key. It is a 1:1 mapping between complex numbers, and is in fact an analytic function and a conformal transformation. Review your complex analysis if necessary and interested.

In the 2-dimensional plane of values of \( z \) - the \( \mathbb{C} \)-plane - a reflection coefficient \( \Gamma \) is represented by a point.
as we move away from the load by a distance \( l \) on the transmission line, \( \Gamma \) rotates by an angle \( \Delta \theta \)

\[
\Delta \theta = -2\beta l
\]

\[
= -360^\circ \times \frac{l}{\lambda} \times 0.2
\]

= one whole rotation in the \( \Gamma \) plane for each half-wavelength movement on the line.
\[ \hat{Z} = \frac{1 + \Gamma}{1 - \Gamma^2} \]

This is a 1:1 relationship between \( \hat{Z} \) (magnitude & phase) and \( \hat{Z} \) (real & imaginary).

Note the units of normalized impedance on the \( \hat{Z} \)-plane!

\[ \mathcal{Z} = R + jX \]
\[ \hat{Z} = \Gamma + jX \]

This is the Smith chart.
So, given some load impedance \( \bar{z} \),
we can calculate \( I \) just by looking
up the \( z \)-coordinates of the point and
reading off the magnitude and phase of \( I \)
with a straightedge and protractor.

Change of impedance vs position: just
rotate the \( I \)-vector clockwise around
the chart for the distance \( l \), at a
rate of one wavelength for every
half-wavelength of movement.

- Then read off impedance from the chart.
Impedance - Admittance Chart

Impedance
\[ Z = R + jX \]
- Resistance
- Reactance

Normalized Impedance
\[ \tilde{Z} = \frac{Z}{Z_0} = r + jx \]

Admittance
\[ Y = \frac{1}{Z} = G + jB \]
- Conductance
- Susceptance

Normalized admittance
\[ \tilde{Y} = \frac{Y}{Y_0} = \frac{Y}{Z_0} = g + j\beta \]

We can plot the values of
- \( r \)
- \( x \)
- \( g \)
- \( \beta \)
on the \( \tilde{Y} \)-plane. This is the impedance - admittance chart.
Among several advantages,

\[
E = 50 + j50 \alpha \\
E_0 = 50 \alpha
\]

so that \( \bar{z} = 1 + j1 \)

then \( Y = ? \)

while we could use complex math, we can just use the plot:

draw the point \( \bar{z} = 1 + j1 \) and read off the normalized admittance as \( Y = 0.5 - j0.5 \)

un-normalize: \( Y = \frac{0.5}{50} \)

This is convenient.
What if there is no transmission line?

We can still use the chart, just define whatever $Z_0$ is convenient.

\[ \begin{align*}
    & \quad -1.75 \Omega \\
    \text{Example:} & \quad \frac{1}{3} + j1.2 \Omega \\
    0 & \quad \frac{1}{3} + j1.2 \Omega
\end{align*} \]

* Solution on Smith chart (next page) gives $Z_{in} = 75 + j10 \ \Omega$.

* This is also an introductory example of impedance matching.
First statement of impedance matching

\[ Z_s = R_s + jX_s \]

Maximum power delivered to load if

\[ Z_L = Z_s^* \quad \text{(Conjuate matching)} \]

\[ P_{\text{max}} = \frac{1}{4} \frac{\|V_{\text{gen}}\|^2}{\Re[Z_s]} \]

\[ = \frac{1}{4} \frac{\|V_{\text{gen}}\|^2}{\Re[R_s]} \]

F.M.S. quantities are assumed.
This is called the power available from the generator.
NORMALIZED IMPEDANCE AND ADMITTANCE COORDINATES
Impedance Matching

Matching Network

Generator

Transistor Match Transistor

Diagram of an electrical circuit involving impedance matching and a transistor match.