ECE 145B / 218B, notes set 11a: Oscillator phase noise

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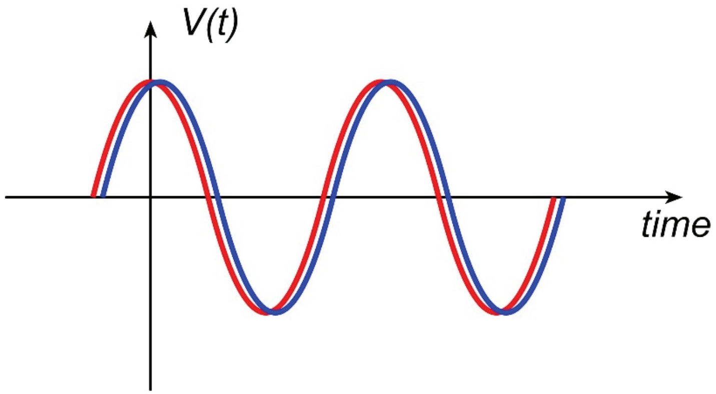
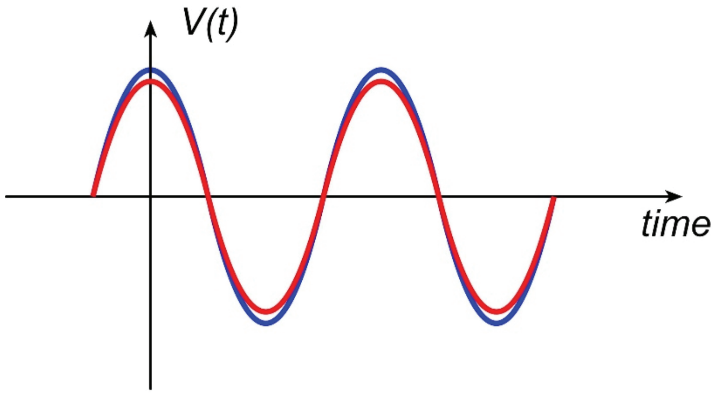
Oscillator Amplitude and Phase Modulation

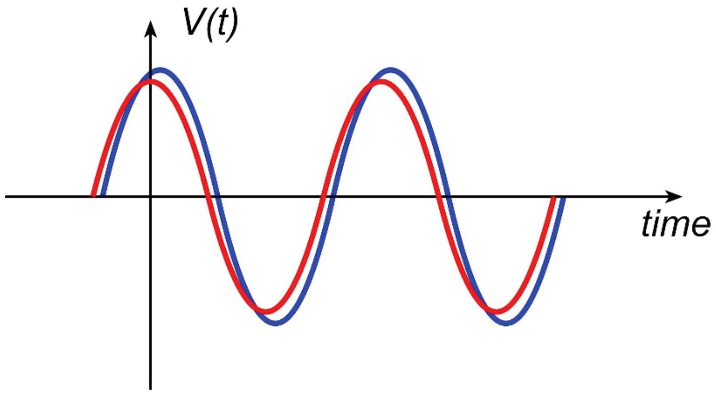
$v(t) = v_o \cos(\omega_0 t + \theta(t))$ where $\theta(t)$ are phase fluctuations



$v(t) = v_o (1 + A(t)) \cos(\omega_0 t)$ where $A(t)$ are normalized amplitude fluctuations



$v(t) = v_o (1 + A(t)) \cos(\omega_0 t + \theta(t))$ with both amplitude and phase fluctuations



Oscillator Phase Modulation Only

$$v(t)=v_o\cos(\omega_0 t+\theta(t)) \rightarrow\ 2v(t)/v_0=\exp(j(\omega_0 t+\theta(t)))+\cancel{\exp(-j(\omega_0 t+\theta(t)))}$$

let's not worry about the negative frequency components (they will be complex conjugates of the positive ones)

Small angle approximation if $\theta(t)<<1$ (radian):

$$\exp(j(\omega_0 t+\theta(t)))=\exp(j\omega_0 t)\cdot\exp(j\theta(t))\cong\exp(j\omega_0 t)\cdot\left(1+j\theta(t)-\theta^2(t)/2\right)\approx\exp(j\omega_0 t)\cdot\left(1+j\theta(t)\right)$$

$$2v(t)/v_0=\exp(j\omega_0 t)+\exp(j\omega_0 t)\cdot j\theta(t)\ \text{multiplication !}$$

Now suppose:

$$2\theta(t)=2\theta_c\cos(\omega_m t)+2\theta_s\sin(\omega_m t)=\theta_c e^{j\omega_m t}+\theta_c e^{-j\omega_m t}-j\theta_s e^{j\omega_m t}+j\theta_s e^{-j\omega_m t}$$

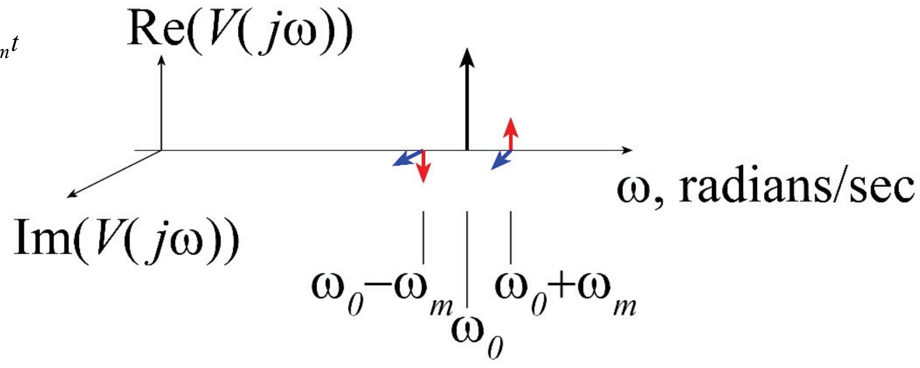
$$2v(t)/v_0=e^{j\omega_0 t}+e^{j\omega_0 t}\cdot j\left(\frac{\theta_c}{2}e^{j\omega_m t}+\frac{\theta_c}{2}e^{-j\omega_m t}-j\frac{\theta_s}{2}e^{j\omega_m t}+j\frac{\theta_s}{2}e^{-j\omega_m t}\right)$$

$$2v(t)/v_0=e^{j\omega_0 t}+j\frac{\theta_c}{2}e^{j(\omega_0+\omega_m)t}+j\frac{\theta_c}{2}e^{j(\omega_0-\omega_m)t}+\frac{\theta_s}{2}e^{j(\omega_0+\omega_m)t}-\frac{\theta_s}{2}e^{j(\omega_0-\omega_m)t}$$

Oscillator Phase Modulation Only

$$2v(t)/v_0 = \exp(j(\omega_0 t + \theta(t)))$$

where $2\theta(t) = 2\theta_c\cos(\omega_m t) + 2\theta_s\sin(\omega_m t) = \theta_c e^{j\omega_m t} + \theta_c e^{-j\omega_m t} - j\theta_s e^{j\omega_m t} + j\theta_s e^{-j\omega_m t}$

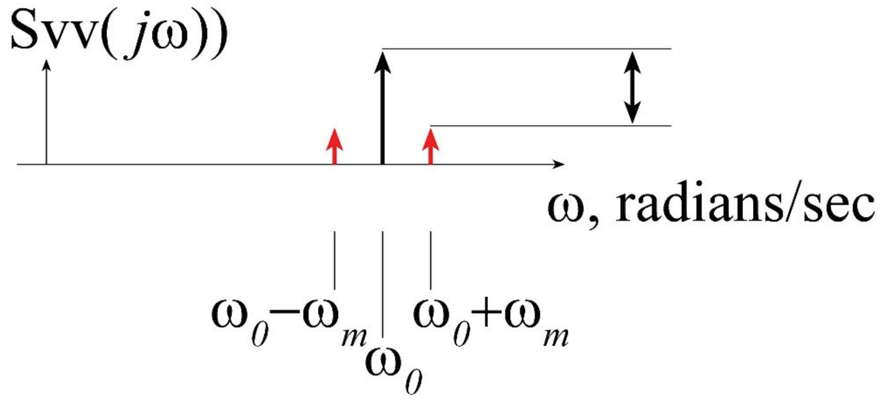


$$2v(t)/v_0 = e^{j\omega_0 t} + j\frac{\theta_c}{2}e^{j(\omega_0+\omega_m)t} + j\frac{\theta_c}{2}e^{j(\omega_0-\omega_m)t} + \frac{\theta_s}{2}e^{j(\omega_0+\omega_m)t} - \frac{\theta_s}{2}e^{j(\omega_0-\omega_m)t}$$

Phase modulation produces upper and lower modulation sidebands.

Phase modulation sidebands are *skew Hermetian* (+/- real parts, equal imaginary parts)

$$\frac{\text{total sideband power}}{\text{total carrier power}} = \left(\frac{\theta_c}{2}\right)^2 + \left(\frac{\theta_c}{2}\right)^2 + \left(\frac{\theta_s}{2}\right)^2 + \left(\frac{\theta_s}{2}\right)^2 = \frac{\theta_c^2 + \theta_s^2}{2}$$



$$\frac{\text{total sideband power}}{\text{total carrier power}} = (\text{total RMS phase modulation})^2.$$

Oscillator Amplitude and Phase Modulation

$$v(t)=v_o(1+A(t))\exp(j(\omega_0 t+\theta(t))); \qquad \theta(t)=\text{phase fluctuations}; \; A(t)=\text{amplitude fluctuations}$$

$$2\theta(t)=2\theta_c\cos(\omega_m t)+2\theta_s\sin(\omega_m t)$$

$$2A(t)=2A_c\cos(\omega_m t)+2A_s\sin(\omega_m t)$$

$$v(j\omega)/2\pi=\delta(\omega-\omega_0)+C_{upper}\delta(\omega-\omega_0-\omega_m)+C_{lower}\delta(\omega-\omega_0+\omega_m)$$

$$C_{upper}=C_{amplitude}+C_{phase}$$

$$C_{lower}=C^*_{amplitude}-C^*_{phase}$$

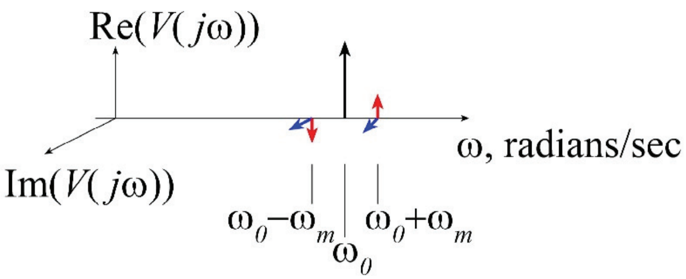
$$C_{amplitude}=A_c/2+jA_s/2$$

$$C_{phase}=-\theta_s/2+j\theta_c/2$$

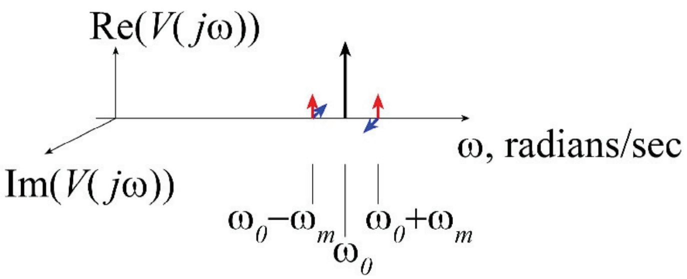
$$C_{amplitude}=(C_{upper}+C^*_{lower})/2$$

$$C_{phase}=(C_{upper}-C^*_{lower})/2$$

phase modulation



amplitude modulation



Amplitude modulation produces Hermetian sidebands (real equal, imaginary +/-)
Phase modulation produces skew Hermetian sidebands (real +/-, imaginary equal)

$$\text{For amplitude modulation: } \frac{\text{power in sidebands}}{\text{power in carrier}}=\left\langle A^2(t)\right\rangle$$

$$\text{For phase modulation: } \frac{\text{power in sidebands}}{\text{power in carrier}}=\left\langle \theta^2(t)\right\rangle$$

M. J. W. Rodwell, D. M. Bloom and K. J. Weingarten, "Subpicosecond laser timing stabilization," in *IEEE Journal of Quantum Electronics*, vol. 25, no. 4, pp. 817-827, April 1989, doi: 10.1109/3.17346.

Oscillator Phase Noise

Now suppose $\theta(t)$ is a random process.

$R_{\theta\theta}(\tau) = E\left[\theta(t)\theta(t+\tau)\right]$ autocorrelation function

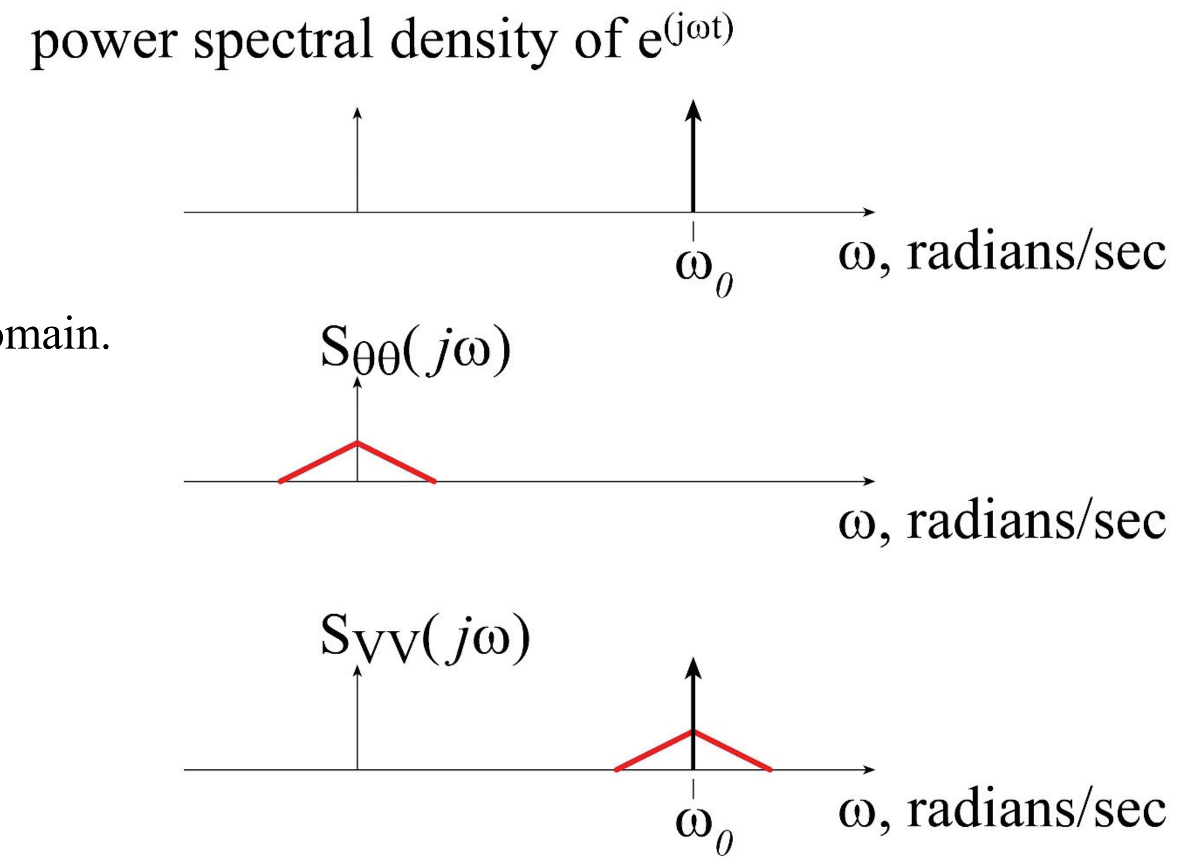
$S_{\theta\theta}(j\omega) = \mathcal{F}\left[R_{\theta\theta}(\tau)\right]$ power spectral density

$$2v(t)/v_0 = \exp(j(\omega_0 t + \theta(t))) \approx e^{j\omega_0 t} + e^{j\omega_0 t} \cdot j\theta(t)$$

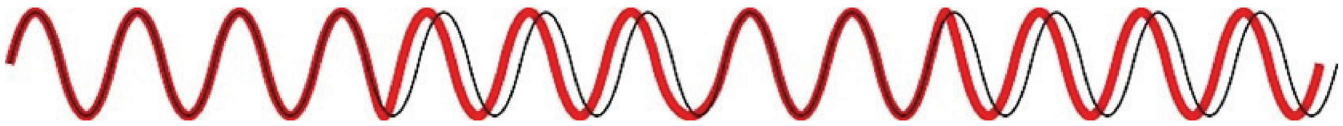
Multiplication in the time domain is convolution in the frequency domain.

$$\frac{4}{v_0^2} S_{VV}(j\omega) = 2\pi\delta(\omega - \omega_0) + S_{\theta\theta}(\omega - \omega_0)$$

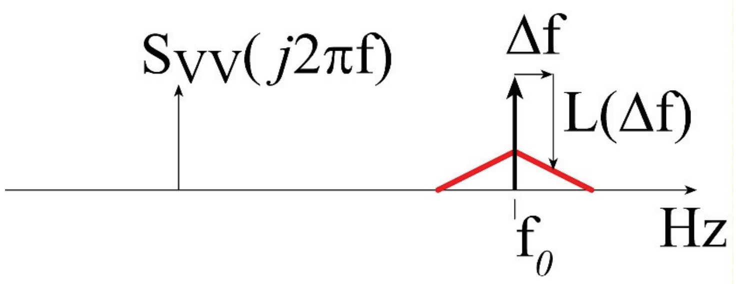
The spectrum of the phase deviations become modulation sidebands



$$\frac{\text{total sideband power}}{\text{total carrier power}} = \left(\text{total RMS phase modulation}\right)^2.$$



Oscillator Phase Noise Spectral Density



$$L(\Delta f)=\frac{\text{noise power, in 1 Hz bandwidth, at frequency }(f_0+\Delta f)}{\text{power in carrier}}$$

In the small angle approximation (but not otherwise)

$$L(\Delta f)=S_{\theta\theta}(\Delta f)$$

In linear units $L(\Delta f)$ is expressed as $(\text{radians})^2/\text{Hz}$ or just 1/Hz, as radians are dimensionless

In dB units $L(\Delta f)$ is expressed as dB relative to carrier, in a 1 Hz bandwidth, i.e.

$L(\Delta f=100\text{ Hz})=10^{-10}\ (1/\text{Hz})$ is written as

$$L(\Delta f=100\text{ Hz})=10\cdot\log_{10}\left(10^{-10}\ (1/\text{Hz})*1\text{ Hz}\right)=-100\text{ dB}_\text{c}\ (1\text{ Hz})$$

$$\frac{\text{total sideband power}}{\text{total carrier power}}=\left(\text{total RMS phase modulation}\right)^2.$$

Phase noise examples

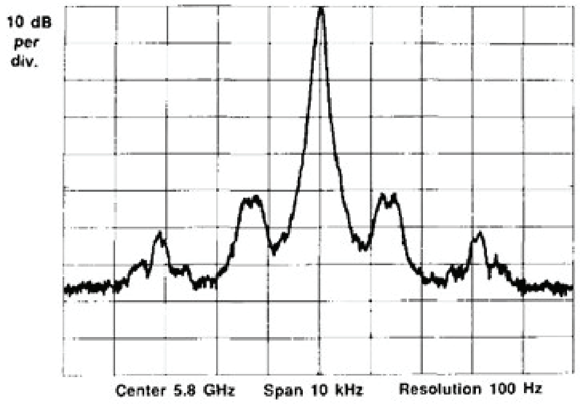


Fig. 3. Coherent Antares 1.06 μm Nd:YAG laser phase-noise sidebands, 500 Hz-5 kHz, measured on the 70th harmonic of the 70 MHz pulse repetition rate.

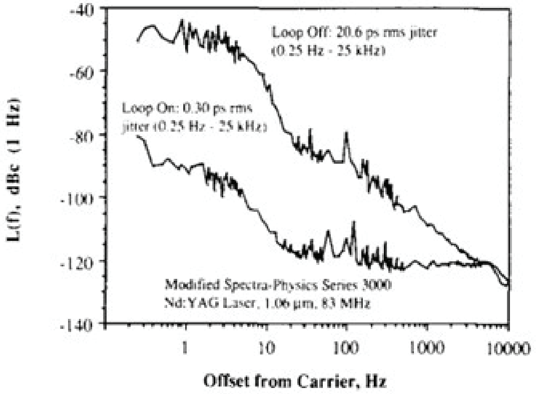


Fig. 12. Phase-noise spectral density, stabilizer on and off, as measured at the output of the pulse compressor. With the stabilizer on, phase-noise measurements below 30 Hz are dominated by amplitude noise.

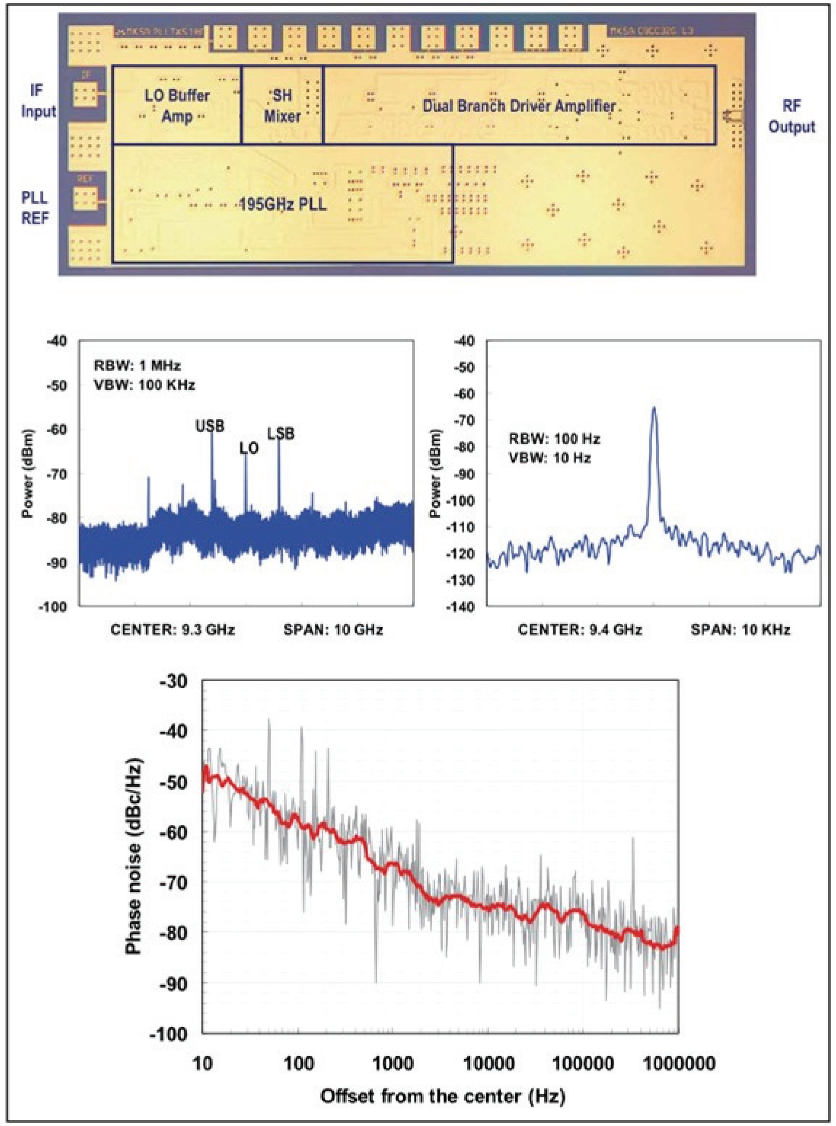


***Fig. 20.** Chip photograph of integrated 590-GHz transmitter circuit with integrated PLL LO source (top). Downconverted transmitter output spectrum (middle). Measured transmitter phase noise (bottom). Chip dimensions: 1.95×0.7 mm^2.*

M. J. W. Rodwell, D. M. Bloom and K. J. Weingarten, "Subpicosecond laser timing stabilization," in IEEE Journal of Quantum Electronics, vol. 25, no. 4, pp. 817-827, April 1989, doi: 10.1109/3.17346.

M. Urteaga, Z. Griffith, M. Seo, J. Hacker and M. J. W. Rodwell, "InP HBT Technologies for THz Integrated Circuits," in Proceedings of the IEEE, vol. 105, no. 6, pp. 1051-1067, June 2017, doi: 10.1109/JPROC.2017.2692178.

Leeson's phase noise formula

Leeson's equation - Wikipedia

https://en.wikipedia.org/wiki/Leeson's_equation

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Leeson's equation is an empirical expression that describes an oscillator's phase noise spectrum.

Leeson's expression[1] for single-sideband (SSB) phase noise in dBc/Hz (decibels relative to output level per hertz) and augmented for flicker noise:[2]

$$L(f_m) = 10\log\left[\frac{1}{2}\left(\left(\frac{f_0}{2Q_l f_m}\right)^2 + 1\right)\left(\frac{f_c}{f_m}+1\right)\left(\frac{FkT}{P_s}\right)\right]$$

where f_0 is the output frequency, Q_l is the loaded quality factor, f_m is the offset from the output frequency (Hz), f_c is the $1/f$ corner frequency, F is the noise factor of the amplifier, k is Boltzmann's constant in joules/kelvin, T is absolute temperature in kelvins, and P_s is the available power at the sustaining amplifier input.[3]

There is often misunderstanding around Leeson's equation, even in text books. In the 1966 paper, Leeson stated correctly that "P_s is the signal level at the oscillator active element input" (often referred to as the power through the resonator now, strictly speaking it is the available power at the amplifier input). F is the device noise factor, however this does need to be measured at the operating power level. The common misunderstanding, that P_s is the oscillator output level, may result from derivations that are not completely general. In 1982, W. P. Robins (IEEE Publication "Phase noise in signal sources") correctly showed that the Leeson equation (in the -20 dB/decade region) is not just an empirical rule, but a result that follows from a linear analysis of an oscillator circuit. However, a used constraint in his circuit was that the oscillator output power was approximately equal to the active device input power.

The Leeson equation is presented in various forms. In the above equation, if f_c is set to zero the equation represents a linear analysis of a feedback oscillator in the general case (and flicker noise is not included), it is for this that Leeson is most recognised, showing a -20 dB/ decade of offset frequency slope. If used correctly, the Leeson equation gives a useful prediction of oscillator performance in this range. If a value for f_c is included, the equation also shows a curve fit for the flicker noise. The f_c for an amplifier depends on the actual configuration used, because radio-frequency and low-frequency negative feedback can have an effect on f_c. So for accurate results, f_c must be determined from added noise measurements on the amplifier using R.F., with the actual circuit configuration to be used in the oscillator.

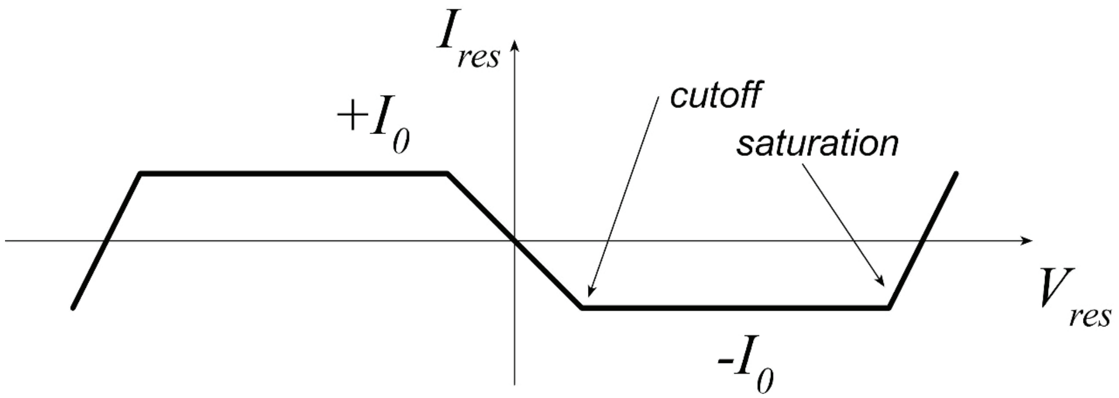
Evidence that P_s is the amplifier input power (often contradicted or very unclear in text books) can be found in the derivation in further reading which also shows experimental results, Enrico Rubiola, The Leeson Effect also shows this in a different form.

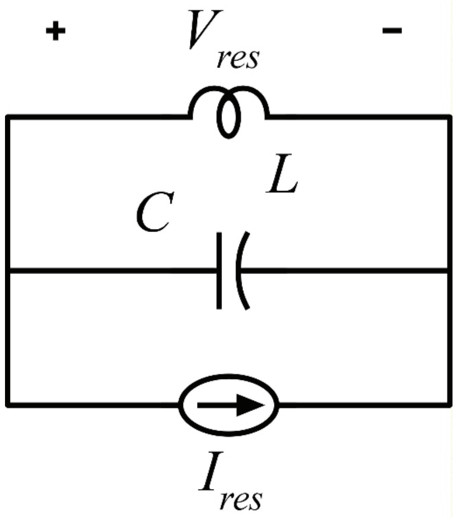
Let us see if we can derive this...

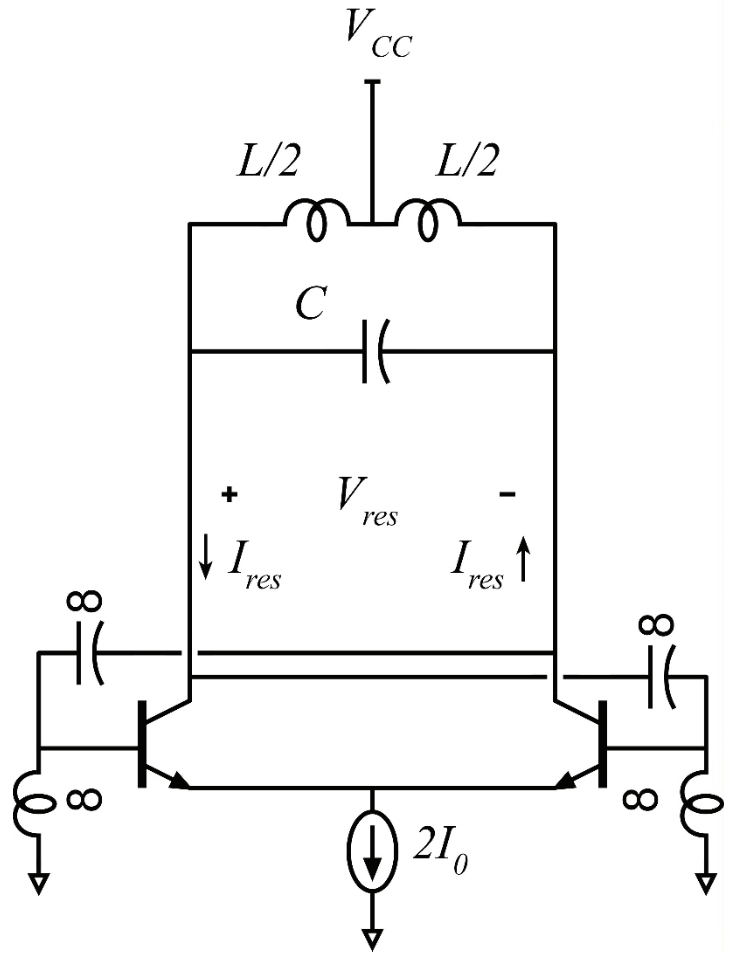
See also: W.P. Robins, "Phase Noise in Signal Soures", Peter Peregrinus Press, London, 1982 ISBN 978-0-86341-026-0

Oscillator model

Note the strong nonlinearity in the negative conductance provided by the transistors





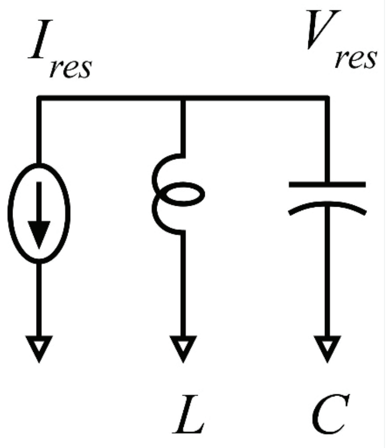


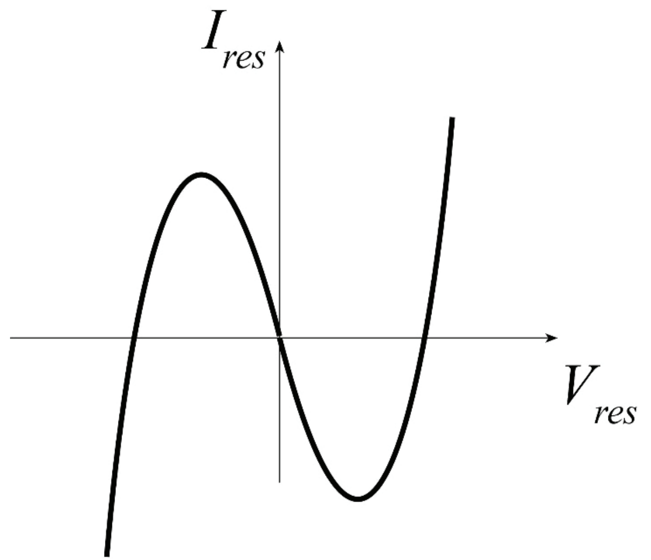
Oscillator model: idealized

Transistor (or other active element) clipping is modelled by a cubic I-V curve

$$I_{res}(v) = a_1 v_{res} + a_3 v_{res}^3$$

What oscillation voltage amplitude ?





$$V_{res}(t) = V_o \cos(\omega_0 t)$$

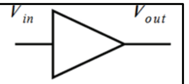
$$I_{res}(t) = a_1 V_0 \cos(\omega_0 t) + a_3 V_0^3 \cos^3(\omega_0 t)$$

$$I_{res}(t) = a_1 V_0 \cos(\omega_0 t) + 3a_3 (V_0/2)^3 \cdot 2\cos(\omega_0 t) + \cancel{a_3 (V_0/2)^3 \cdot 2\cos(3\omega_0 t)} = 0$$

$$a_1 V_0 + 3a_3 (V_0/2)^3 \cdot 2 = 0$$

$$V_0 = 2 \cdot \sqrt{-a_1/3a_3}$$

$$V_{in}(t) = V_0 \cos(\omega_0 t) = (V_0/2)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) = (V_0/2)\left(z + z^{-1}\right)$$



$$V_{out} = a_0 + a_1 \cdot (\delta V_{in}) + a_2 \cdot (\delta V_{in})^2 + a_3 \cdot (\delta V_{in})^3 + \ldots$$

$$(\delta V_{in})^3 = (V_0/2)^3 \left(z + z^{-1}\right)^3 = (V_0/2)^3 \left(z^3 z^{-0} + 3z^2 z^{-1} + 3z^{-2} z^1 + z^0 z^{-3}\right)$$

$$(\delta V_{in})^3 = (V_0/2)^3 \left(z^3 + z^{-3}\right) + 3(V_0/2)^3 \left(z + z^{-1}\right)$$

$$(\delta V_{in})^3 = (V_0/2)^3 \cdot 2\cos(3\omega_0 t) + 3(V_0/2)^3 \cdot 2\cos(\omega_0 t)$$

Cubic repsonse generates response at fundamental and the 3nd harmonic

Amplitude stabilizes when the fundamental component of $I_{res}(t)$ is zero.

Steady-state amplitude (zero noise)

$$V_{res}(t) = V_o \cos(\omega_0 t) \text{ where } V_0 = \sqrt{-4a_1/3a_3}$$

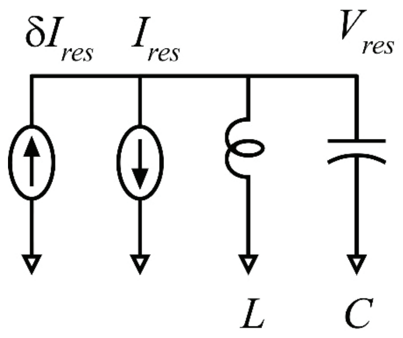
Voltage perturbation and resulting current change (1)

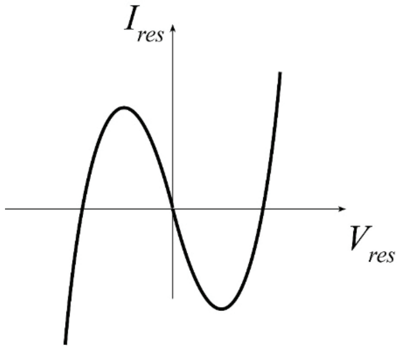
Now assume a perturbation in the oscillation amplitude

$$V_{res,0}(t) = V_o \cos(\omega_0 t) \text{ where } V_0 = \sqrt{-4a_1/3a_3}$$

$$V_{res}(t) = V_{res,0}(t) + \delta V_c \text{ where } \delta V_c = \delta V_{c0}\cos(\omega_0 t)$$

$$I_{res}(t) = a_1 V_{res}(t) + a_3 V_{res}^3(t) = a_1\left(V_{res,0}(t) + \delta V_c\right) + a_3\left(V_{res,0}(t) + \delta V_c\right)^3$$





$$I_{res}(t) = a_1 V_{res,0}(t) + a_1\delta V_c + a_3\left(V_{res,0}^3(t) + 3V_{res,0}^2(t)\delta V_c + 3V_{res,0}^1(t)(\delta V_c)^2 + (\delta V_c)^3\right)$$

$$I_{res}(t) = a_1 V_{res,0}(t) + a_3 V_{res,0}^3(t) + a_1\delta V_c + a_3\cdot 3V_{res,0}^2(t)\delta V_c + O(\delta V_c)^2$$

but $a_1 V_{res,0}(t) + a_3 V_{res,0}^3(t) = 0$, so to leading order

$$I_{res}(t) = a_1\delta V_c + a_3\cdot 3V_{res,0}^3(t)\delta V_c$$

$$I_{res}(t) = a_1\delta V_c + a_3\cdot 3V_{res,0}^2(t)\delta V_c$$

$$I_{res}(t) = a_1\delta V_{c0}\cos(\omega_0 t) + a_3\delta V_{c0}\cdot 3V_0^2\left(\cos(\omega_0 t)\right)^3$$

$$I_{res}(t) = a_1\delta V_{c0}\cos(\omega_0 t) + a_3\delta V_{c0}\cdot\frac{3V_0^2}{8}\left(2\cos(3\omega_0 t) + 6\cos(\omega_0 t)\right)$$

$$I_{res}(t) = a_1\delta V_{c0}\cos(\omega_0 t) + \frac{9a_3 V_0^2}{4}\delta V_{c0}\cos(\omega_0 t)$$

$$I_{res}(t) = \left(a_1 + \frac{9a_3 V_0^2}{4}\right)\cdot\delta V_{c0}\cos(\omega_0 t)$$

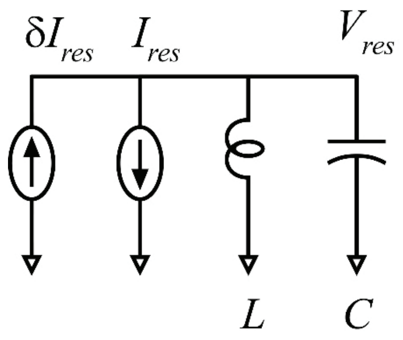
$$I_{res}(t) = \left(a_1 + \frac{9a_3}{4}\frac{-4a_1}{3a_3}\right)\cdot\delta V_{c0}\cos(\omega_0 t)$$

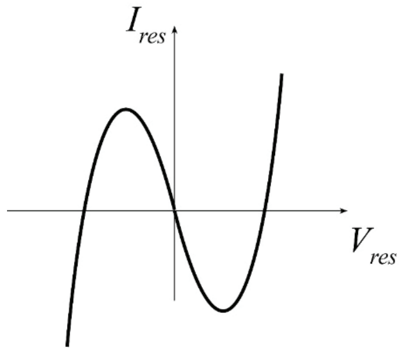
$$I_{res}(t) = -2a_1\cdot\delta V_{c0}\cos(\omega_0 t)$$

$$I_{res}(t) = -2a_1\cdot\delta V_{c0}\cos(\omega_0 t) \text{ given } \delta V_c = \delta V_{c0}\cos(\omega_0 t)$$

So, an in-phase voltage perturbation sees a positive conductance $-a_1$ from the non-reactive elements $I_{res}(t)$

Voltage perturbation and resulting current change (2)





Now assume a perturbation in the oscillation phase

$V_{res,0}(t) = V_o \cos(\omega_0 t)$ where $V_0 = \sqrt{-4a_1/3a_3}$

$V_{res}(t) = V_{res,0}(t) + \delta V_s$ where $\delta V_s = \delta V_{s0}\sin(\omega_0 t)$

$$I_{res}(t) = a_1 V_{res}(t) + a_3 V_{res}^3(t) = a_1\left(V_{res,0}(t) + \delta V_s\right) + a_3\left(V_{res,0}(t) + \delta V_s\right)^3$$

$I_{res}(t) = {*0*} \cdot \delta V_{s0}\sin(\omega_0 t)$ given $\delta V_s = \delta V_{s0}\sin(\omega_0 t)$

So, a quadrature-phase voltage perturbation sees a conductance of *zero* from the non-reactive elements $I_{res}(t)$

$$I_{res}(t) = a_1 V_{res,0}(t) + a_1\delta V_s + a_3\left(V_{res,0}^3(t) + 3V_{res,0}^2(t)\delta V_{sc} + 3V_{res,0}^1(t)(\delta V_s)^2 + (\delta V_{sc})^3\right)$$

$$I_{res}(t) = a_1 V_{res,0}(t) + a_3 V_{res,0}^3(t) + a_1\delta V_s + a_3\cdot 3V_{res,0}^2(t)\delta V_s + O(\delta V_s)^2$$

but $a_1 V_{res,0}(t) + a_3 V_{res,0}^3(t) = 0$, so to leading order

$$I_{res}(t) = a_1\delta V_s + a_3\cdot 3V_{res,0}^3(t)\delta V_s$$

$$I_{res}(t) = a_1\delta V_s + a_3\cdot 3V_{res,0}^2(t)\delta V_s$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot 3V_0^2\left(\cos(\omega_0 t)\right)^2\sin(\omega_0 t)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot\frac{3V_0^2}{j8}\left(z_0 + 1/z_0\right)^2\left(z_0 - 1/z_0\right)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot\frac{3V_0^2}{j8}\left(z_o^2 + 2 + 1/z_o^2\right)\left(z_0 - 1/z_0\right)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot\frac{3V_0^2}{j8}\left(z_o^3 - z_0 + 2z_0 - 2/z_0 + 1/z_0 - 1/z_0^3\right)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot\frac{3V_0^2}{j8}\left(z_o^3 - 1/z_0^3 + z_0 - 1/z_0\right)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + a_3\delta V_{s0}\cdot\frac{3V_0^2}{4}\left(\sin(3\omega_0 t) + \sin(\omega_0 t)\right)$$

$$I_{res}(t) = a_1\delta V_{s0}\sin(\omega_0 t) + \frac{3a_3 V_0^2}{4}\delta V_{s0}\sin(\omega_0 t)$$

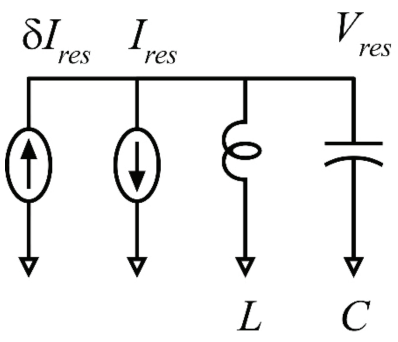
$$I_{res}(t) = \left(a_1 + \frac{3a_3 V_0^2}{4}\right)\cdot\delta V_{s0}\sin(\omega_0 t)$$

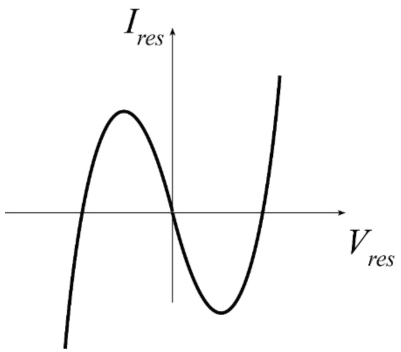
$$I_{res}(t) = \left(a_1 - \frac{3a_3}{4}\frac{4a_1}{3a_3}\right)\cdot\delta V_{s0}\sin(\omega_0 t)$$

$$I_{res}(t) = a_1\left(1 - \frac{3}{4}\frac{4}{3}\right)\cdot\delta V_{s0}\sin(\omega_0 t) = 0\ !$$

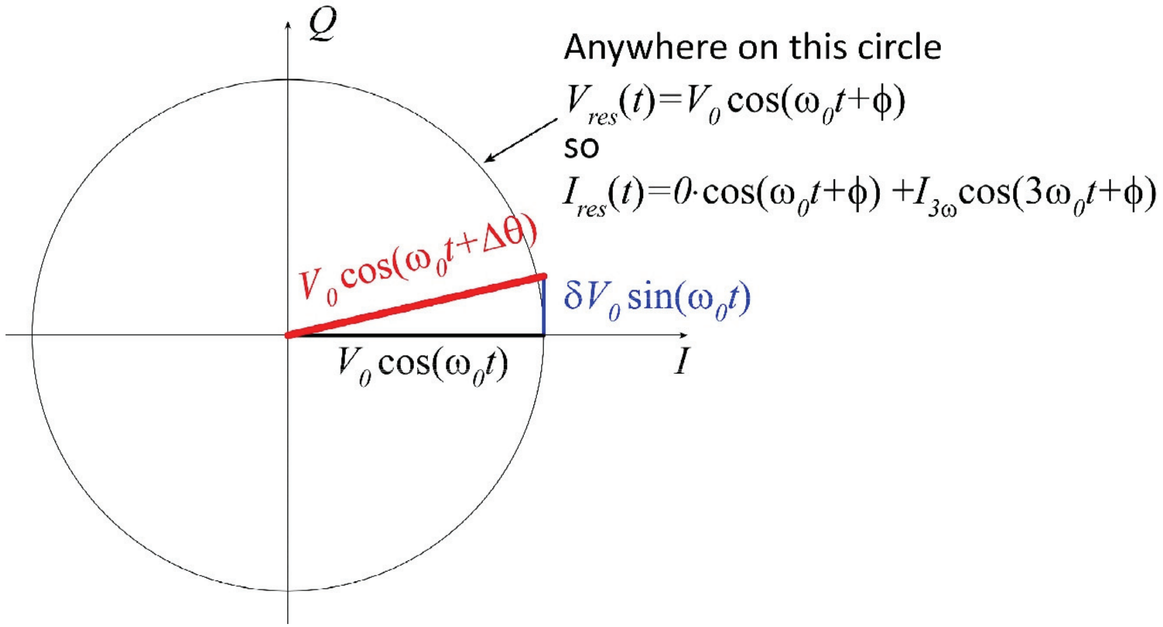
$$I_{res}(t) = 0$$

Voltage perturbation and resulting current change (3)





Finding this result without calculation:



Now assume a perturbation in the oscillation phase

$V_{res,0}(t) = V_o \cos(\omega_0 t)$ where $V_0 = \sqrt{-4a_1 / 3a_3}$

$V_{res}(t) = V_{res,0}(t) + \delta V_s$ where $\delta V_s = \delta V_{s0} \sin(\omega_0 t)$

$$I_{res}(t) = a_1 V_{res}(t) + a_3 V_{res}^3(t) = a_1 \left(V_{res,0}(t) + \delta V_s\right) + a_3 \left(V_{res,0}(t) + \delta V_s\right)^3$$

$I_{res}(t) = {*}0{*} \cdot \delta V_{s0} \sin(\omega_0 t)$ given $\delta V_s = \delta V_{s0} \sin(\omega_0 t)$

So, a quadrature-phase voltage perturbation sees
a conductance of *zero*
from the non-reactive elements $I_{res}(t)$

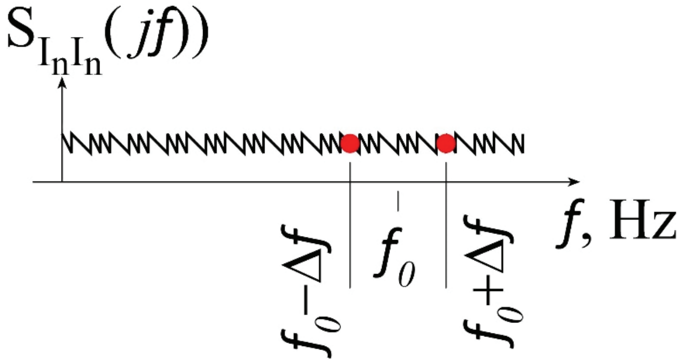
Adding oscillator noise current

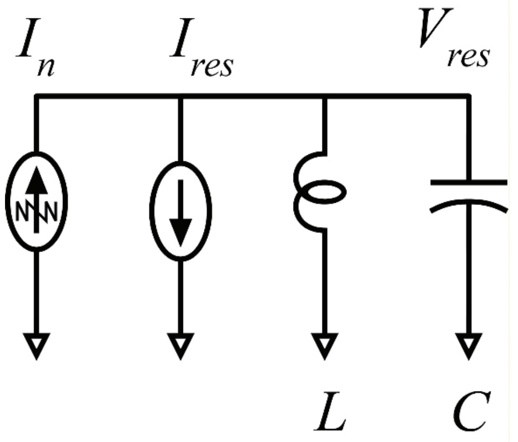
Add noise current with spectral density $S_{I_n I_n}(jf)$

Without noise, the oscillation is $V(t) = \mathrm{Re}\left(V_0 e^{j\omega_0 t}\right)$

Consider the noise current at frequencies $f_0 \pm \Delta f$,

at offsets $\pm\Delta f$ from the oscillation frequency f_0.





These noise currents will contribute sidebands to the oscillation.

Half the power of these sidebands will be in-phase with the carrier (AM).

$$I_n(t) = I_n \cos\left((\omega_0 + \Delta\omega)t\right) + I_n \cos\left((\omega_0 - \Delta\omega)t\right) = \mathrm{Re}\left(I_n e^{j(\omega_0+\Delta\omega)t} + I_n e^{j(\omega_0-\Delta\omega)t}\right)$$

Half the power of these sidebands will be quadrature-phase with the carrier (PM).

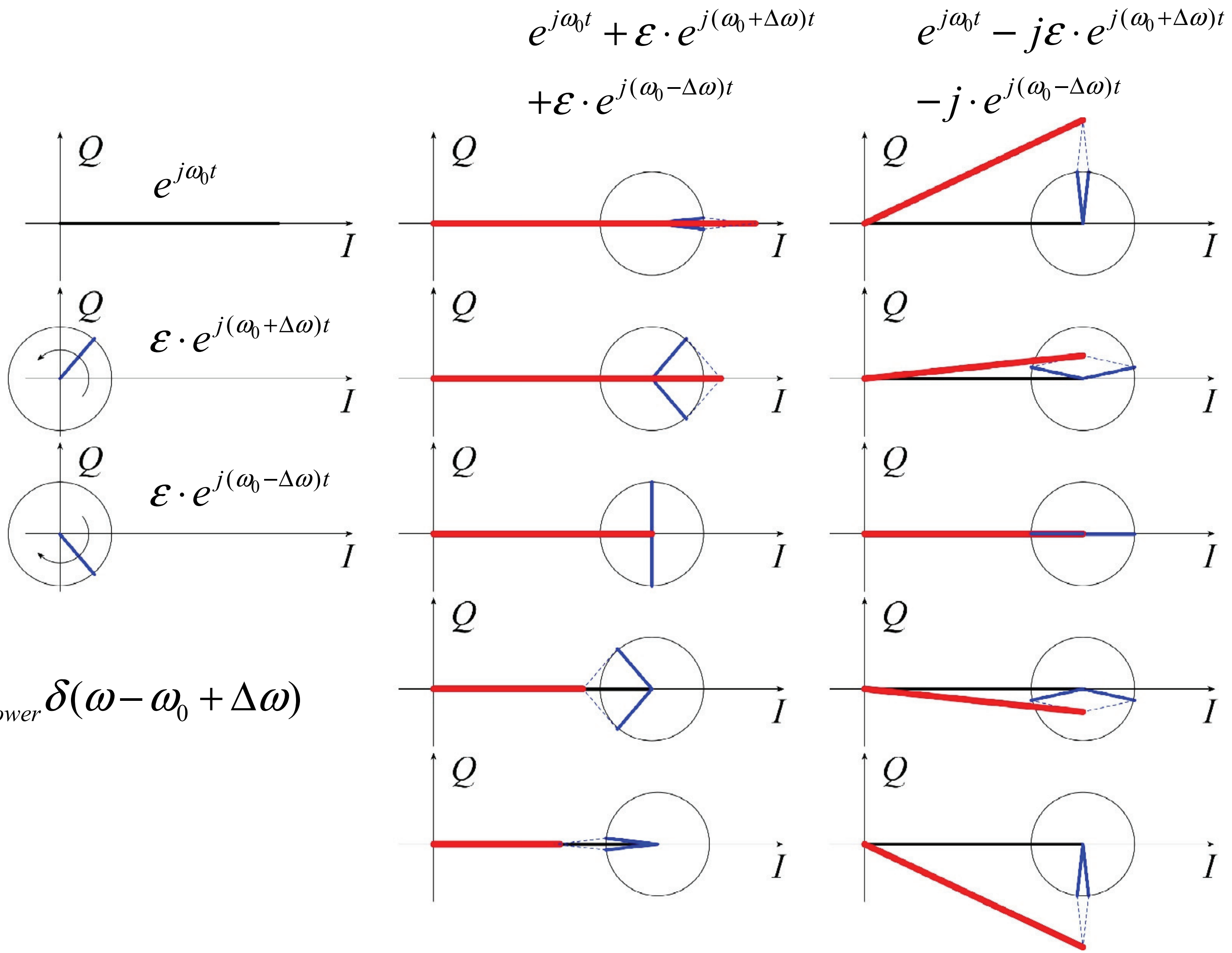
$$I_n(t) = I_n \sin\left((\omega_0 + \Delta\omega)t\right) + I_n \sin\left((\omega_0 - \Delta\omega)t\right) = \mathrm{Re}\left(-jI_n e^{j(\omega_0+\Delta\omega)t} - jI_n e^{j(\omega_0-\Delta\omega)t}\right)$$

In-phase (AM) vs. quadrature-phase (PM) sidebands

Stated approximately,

In-phase sidebands $\rightarrow$ AM

Quadrature-phase sideband $\rightarrow$ PM



Earlier in this note set, we had shown how to decompose sidebands into AM and PM sidebands:

$$v(j\omega)/2\pi=\delta(\omega-\omega_0)+C_{upper}\delta(\omega-\omega_0-\Delta\omega)+C_{lower}\delta(\omega-\omega_0+\Delta\omega)$$

$$C_{amplitude}=(C_{upper}+C^*_{lower})/2$$

$$C_{phase}=(C_{upper}-C^*_{lower})/2$$

Decomposing current noise into AM vs. PM

PM current noise at offset $\Delta\omega$ from ω_0 :

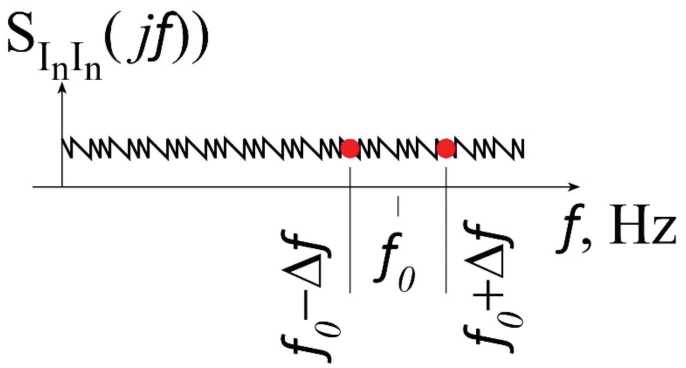
$$I_{phase}(\Delta\omega) = \frac{1}{2} I_n(\omega_0 + \Delta\omega) - \frac{1}{2} I_n^*(\omega_0 - \Delta\omega)$$

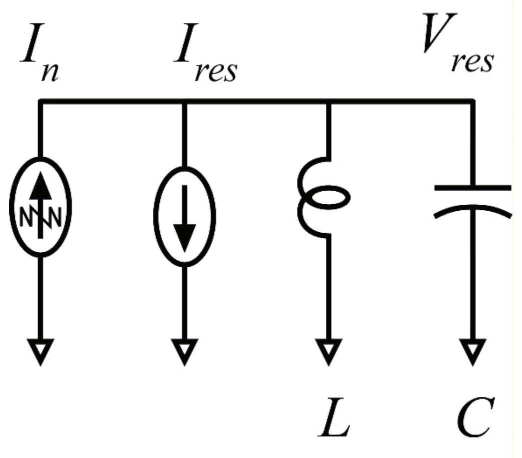
$$S_{I_{phase}}(\Delta\omega) = \frac{S_{I_n I_n}(\omega_0 + \Delta\omega)}{4} + \frac{S_{I_n I_n}(\omega_0 - \Delta\omega)}{4}$$

AM current noise at offset $\Delta\omega$ from ω_0 :

$$I_{amplitude}(\Delta\omega) = \frac{1}{2} I_n(\omega_0 + \Delta\omega) + \frac{1}{2} I_n^*(\omega_0 - \Delta\omega)$$

$$S_{I_{amplitude}}(\Delta\omega) = \frac{S_{I_n I_n}(\omega_0 + \Delta\omega)}{4} + \frac{S_{I_n I_n}(\omega_0 - \Delta\omega)}{4}$$





The AM current noise sees a low impedance

$$Z_{load,AM} = (-a_1) \,||\, j\omega L \,||\, (1/j\omega C)$$

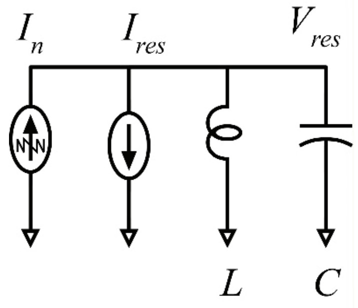
This will produce only a small AM voltage, generally neglected.

The PM current noise sees much higher impedance

$$Z_{load,PM} = j\omega L \,||\, (1/j\omega C)$$

This will produce a substantial PM voltage, producing phase noise

Computing the PM noise voltage and Phase noise (1)



Phase component of current noise:

$$I_{phase}(\Delta\omega)=\frac{1}{2}I_n(\omega_0+\Delta\omega)-\frac{1}{2}I_n^*(\omega_0-\Delta\omega)$$

Resonator impedance, where $\omega_0=L^{-1/2}C^{-1/2}$ and $Z_r=L^{1/2}C^{-1/2}$

$$Z_{res}(j\omega)=\frac{(j\omega L)(1/j\omega C)}{(j\omega L)+(1/j\omega C)}=\frac{j\omega L}{1-\omega^2LC}=\frac{j(\omega/\omega_0)Z_r}{1-\omega^2/\omega_0^2}=jZ_r\frac{1+\Delta\omega/\omega_0}{1-(1+\Delta\omega/\omega_0)^2}$$

$$Z_{res}(j\omega)=\frac{-jZ_r}{2}\frac{1+\Delta\omega/\omega_0}{\Delta\omega/\omega_0+(\Delta\omega/\omega_0)^2}\cong\frac{-jZ_r}{2}\frac{\omega_0}{\Delta\omega}\text{ for }\Delta\omega<<\omega_0$$

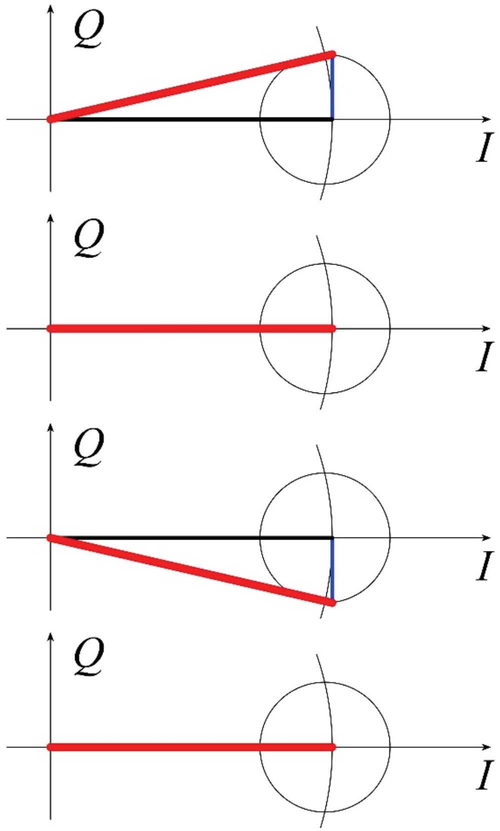
Resulting quadrature voltage

$$V_{phase}(\Delta\omega)=\frac{1}{2}I_n(\omega_0+\Delta\omega)Z_{res}(\omega_0+\Delta\omega)-\frac{1}{2}I_n^*(\omega_0-\Delta\omega)Z_{res}(\omega_0-\Delta\omega)$$

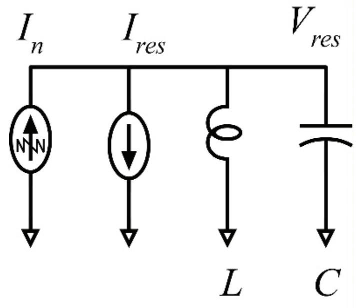
$$V_{phase}(\Delta\omega)=\frac{-jZ_r}{4}\frac{\omega_0}{\Delta\omega}I_n(\omega_0+\Delta\omega)-\frac{-jZ_r}{4}\frac{\omega_0}{-\Delta\omega}I_n^*(\omega_0-\Delta\omega)$$

Resulting phase deviation

$$\theta_n(\Delta\omega)=\frac{V_{phase}(\Delta\omega)}{V_0}=\frac{-jZ_r}{4V_0}\frac{\omega_0}{\Delta\omega}I_n(\omega_0+\Delta\omega)-\frac{-jZ_r}{4V_0}\frac{\omega_0}{-\Delta\omega}I_n^*(\omega_0-\Delta\omega)$$



Computing the PM noise voltage and Phase noise (2)

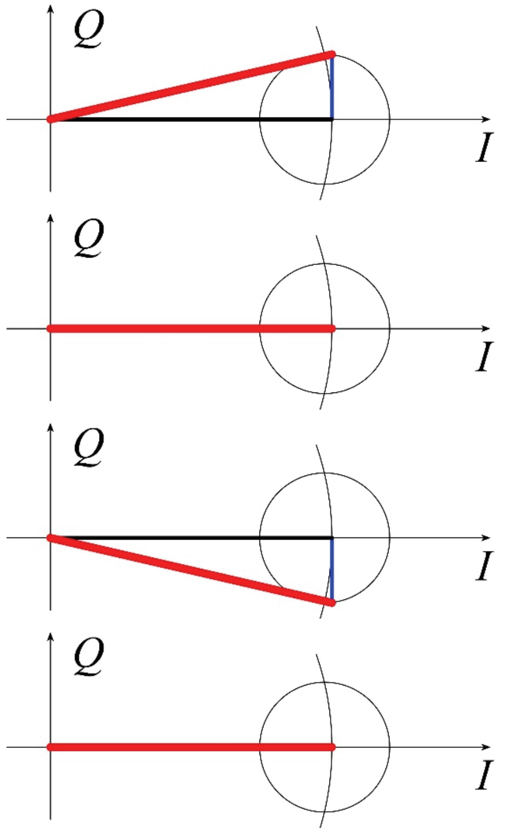


$$\theta_n(\Delta\omega)=\frac{V_{phase}(\Delta\omega)}{V_0}=\frac{-jZ_r}{4V_0}\frac{\omega_0}{\Delta\omega}I_n(\omega_0+\Delta\omega)-\frac{-jZ_r}{4V_0}\frac{\omega_0}{-\Delta\omega}I_n^*(\omega_0-\Delta\omega)$$

$\theta_n(\Delta\omega)(1\text{ Hz})$ is the RMS phase deviation, in a 1 Hz bandwidth, at an offset $\Delta\omega$ from carrier.

The oscillator's upper and lower modulation sideband amplitudes are each $\langle\theta_n\rangle^2/2$, hence

$$L(\Delta\omega)=\frac{1}{2}\left(\frac{Z_r}{4V_0}\right)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_nI_n}(\omega_0+\Delta\omega)+\frac{1}{2}\left(\frac{Z_r}{4V_0}\right)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_nI_n}(\omega_0-\Delta\omega)$$



Now assume that the noise current is constant for frequencies near ω_0.

$S_{I_nI_n}(\omega_0+\Delta\omega)=S_{I_nI_n}(\omega_0-\Delta\omega)=S_{I_nI_n}(\omega)$ hence

$$L(\Delta\omega)=\left(\frac{Z_r}{4V_0}\right)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_nI_n}(\omega)$$

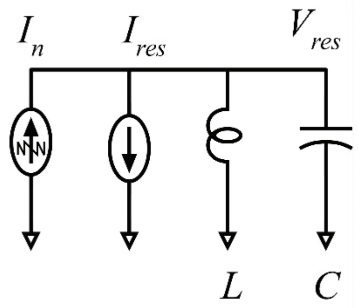
$$L(\Delta\omega)=\left(\frac{Z_r}{4V_0}\right)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_nI_n}(\omega)$$

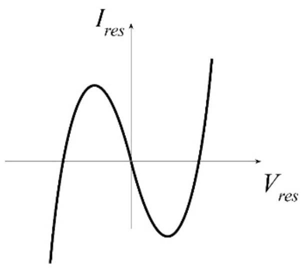
where $I_{res}(v)=a_1v_{res}+a_3v_{res}^3$, $V_0=\sqrt{-4a_1/3a_3}$ and $Z_{res}=\sqrt{L/C}$

Our phase noise model: key features

$$L(\Delta\omega) = \left(\frac{Z_r}{4V_0}\right)^2 \left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_n I_n}(\omega)$$

where $I_{res}(v) = a_1 v_{res} + a_3 v_{res}^3$, $V_0 = \sqrt{-4a_1/3a_3}$ and $Z_{res} = \sqrt{L/C}$

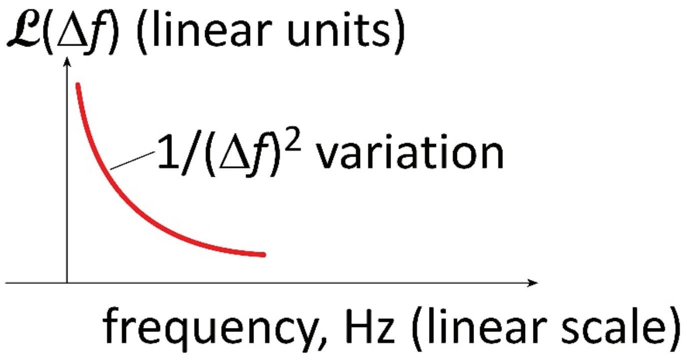


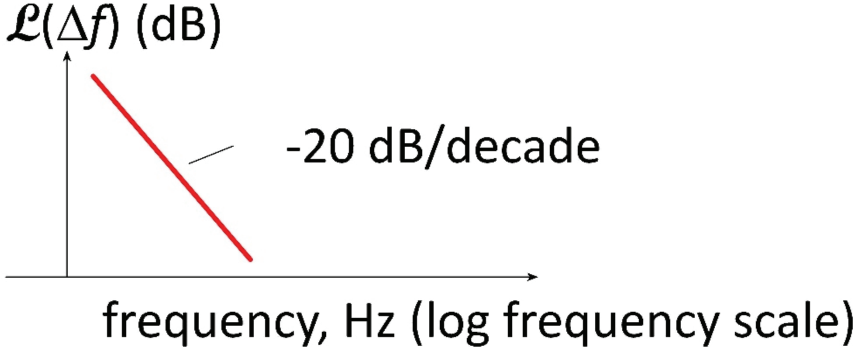


Phase noise varies as $1/(\Delta f)^2$.

Lower phase noise given:

- Lower noise current
- Larger oscillation amplitude (voltage)
- Lower (parallel) resonator impedance.





Our phase noise model compared to Leeson (1)

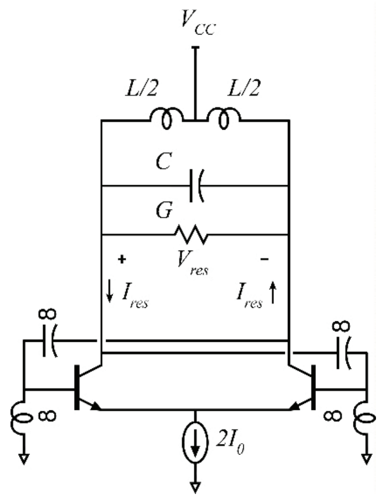
$$L(\Delta\omega)=\left(\frac{Z_r}{4V_0}\right)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2 S_{I_nI_n}(\omega)$$

where $I_{res}(v)=a_1v_{res}+a_3v_{res}^3$, $V_0=\sqrt{-4a_1/3a_3}$ and $Z_{res}=\sqrt{L/C}$

Add parallel resonator conductance G with noise I_{nG}; $S_{I_{nG}I_{nG}}=4kTG$.

Change active element A such that circuit is not changed: $I_{res}=(a_1-G)V_{res}+a_3V_{res}^3$; $S_{I_{nA}I_{nA}}=S_{I_nI_n}-S_{I_{nG}I_{nG}}$

...in which case the above $\mathcal{L}(\Delta f)$ expression remains correct.



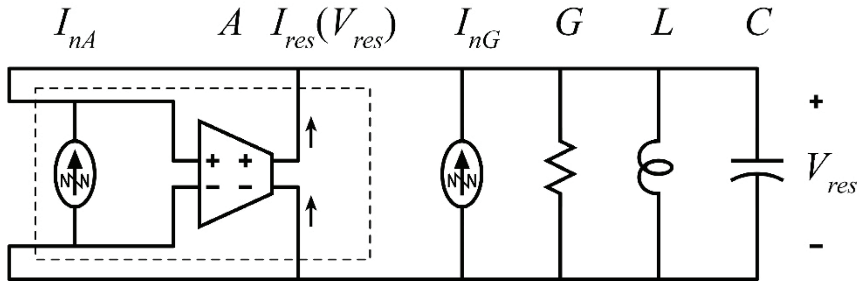
We now have

$$Q=\frac{1}{G\sqrt{L/C}}=\frac{1}{Z_rG}\rightarrow Z_r=\frac{1}{QG}$$

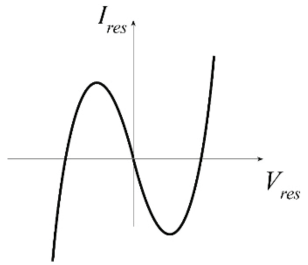
$$F=1+S_{I_{nA}I_{nA}}/S_{I_{nG}I_{nG}}=S_{I_nI_n}/S_{I_{nG}I_{nG}}=S_{I_nI_n}/kTG\rightarrow S_{I_nI_n}=kTFG$$

$$P_{res}=GV_0^2/2\rightarrow V_0^2=2P_{sig}/G$$

$$L(\Delta\omega)=\left(\frac{1}{16G^2Q^2P_{sig}/G}\right)\left(\frac{\omega_0}{\Delta\omega}\right)^2 kTFG+\left(\frac{1}{16G^2Q^28P_{sig}/G}\right)\left(\frac{\omega_0}{\Delta\omega}\right)^2 kTFG$$

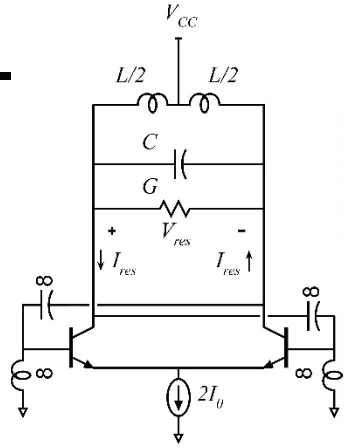


$A=$ active element



$$L(\Delta\omega)=\left(\frac{kTF}{8Q^2P_{sig}}\right)\left(\frac{\omega_0}{\Delta\omega}\right)^2$$ which is a (simplified) Leeson expression

Our phase noise model compared to Leeson (2)



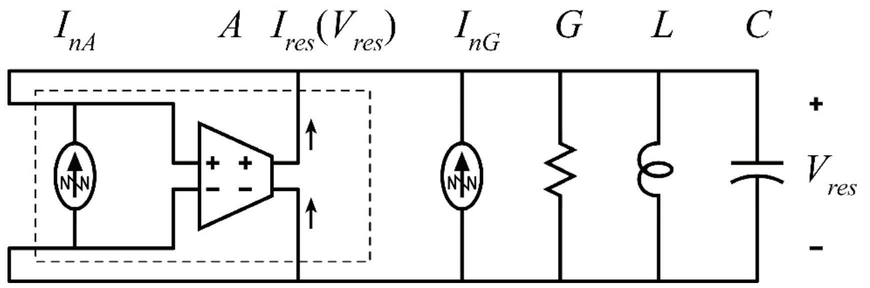
Our Leeson-like expression

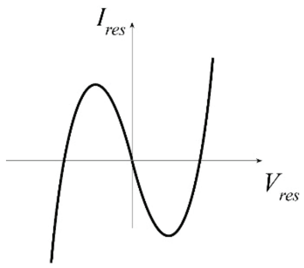
$$L(\Delta\omega) = \left(\frac{kT}{8}\right)\left(\frac{F}{Q^2 P_{res}}\right)\left(\frac{\omega_0}{\Delta\omega}\right)^2 \text{ where } Q = \frac{1}{G\sqrt{L/C}} \; ; \; P_{res} = \frac{GV_0^2}{2} \; ; \; F = \frac{S_{I_n I_n}}{4kTG} = 1 + \frac{S_{I_{nA} I_{nA}}}{4kTG}$$

Critical point: separating the losses, conductivities, and noise of the active and passive circuits is ambiguous; we can assign e.g. parasitic interconnect losses to either the resonator or the active element, changing the values of Q, P_{sig}, and F. Consequently, the values of Q, P_{sig}, and F are not uniquely defined for the oscillator, but only become precisely defined when we define the boundary between the active gain element and the passive resonator.

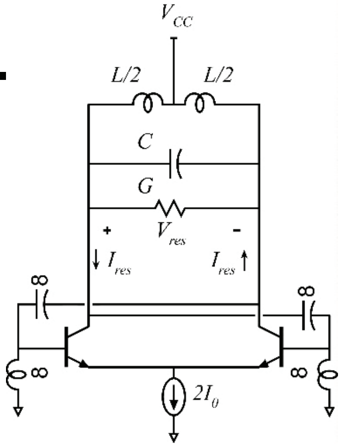
The Leeson formula works because $(F / Q^2 P_{sig})$ is independent of where we draw this boundary; we must just use the values of F, Q, P_{sig} that are consistent with how we have drawn the boundary.

Also critical: P_{res} is the RF power dissipated in the resonator (given how G has been defined), not the output power.





Our phase noise model compared to Leeson (3)

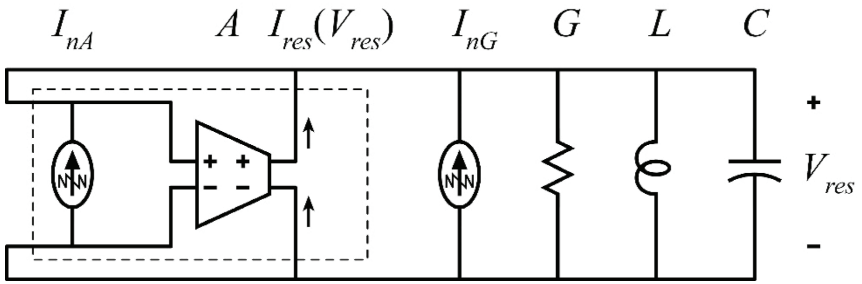


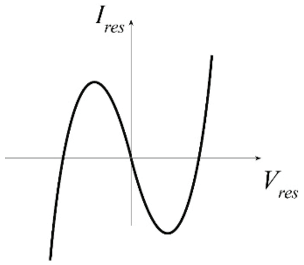
We have not yet carefully considered the amplifier input and output impedances

Modelling the amplifier $Y_{ij} = G_{ij} + jB_{ij}$,

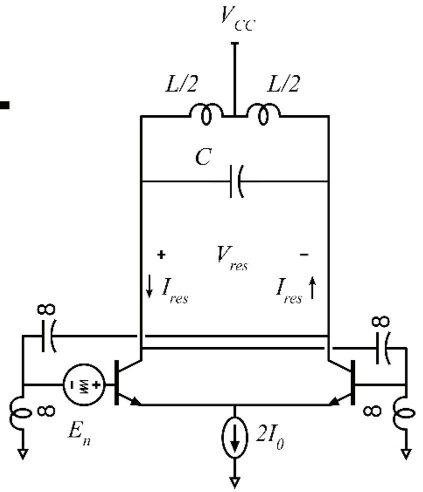
1) The currents $(jB_{11} + jB_{12} + jB_{21} + jB_{22})V_{res}$ have been included in the resonator, i.e. the resonator capacitance includes those of the amplifier.

2) In the expression $I_{res} = (a_1 - G)V_{res} + a_3 V_{res}^3$, the term $(a_1 - G)V_{res}$ includes the currents $(G_{11} + G_{12} + G_{21} + G_{22})V_{res}$





Adding active element (1/f) noise (1)



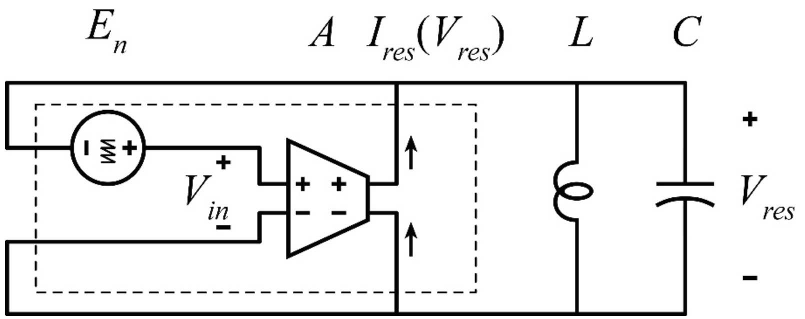
Transistors typically have low frequency noise voltage E_n with spectral density $S_{E_nE_n}(f) \approx S_{E,0}(1+f_x/f)$, i.e. increasing at low frequencies. Mathematical difficulties* can arise if this function is extended to DC. This would given an output noise current $S_{E_nE_n}(f) \approx a_1^2 S_{E,0}(1+f_x/f)$, approximately $a_1^2 S_{E,0}$ near the oscillation frequency f_0.

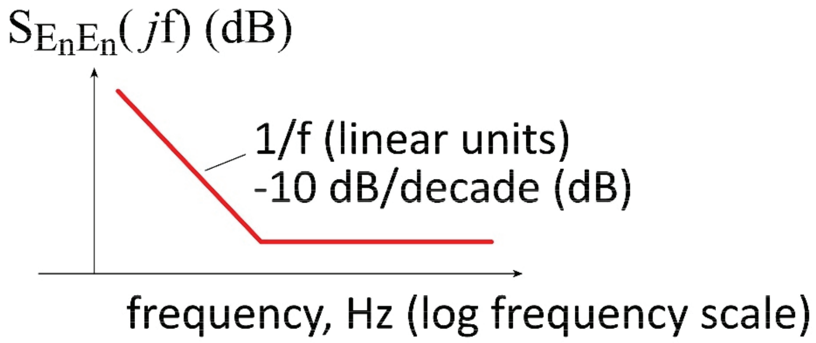
Assume $I_{res}(t) = a_1 V_{in}(t) + a_2 V_{in}^2(t) + a_3 V_{in}^3(t)$, with $a_2 \neq 0$, then:

$$I_{res}(t) = a_1\left(E_n(t)+V_{res}(t)\right) + a_2\left(E_n(t)+V_{res}(t)\right)^2 + a_3\left(E_n(t)+V_{res}(t)\right)^3$$

$$=\ldots+2a_2E_n(t)V_{res}(t)+\ldots.=\ldots+2a_2E_n(t)V_0\cos(\omega_0 t)+\ldots.=$$

The term $E_n(t)V_{res}(t) = E_n(t)V_0\cos(\omega_0 t)$ is a mixing term.





*M. S. Keshner, "1/f noise," in *Proceedings of the IEEE*, vol. 70, no. 3, pp. 212-218, March 1982, doi: 10.1109/PROC.1982.12282.

Adding active element (1/f) noise (2)

The mixing of $E_n(t)$ with $V_0\cos(\omega_0 t)$ will shift a component of the spectrum of the 1/f noise current to be centered around f_0.

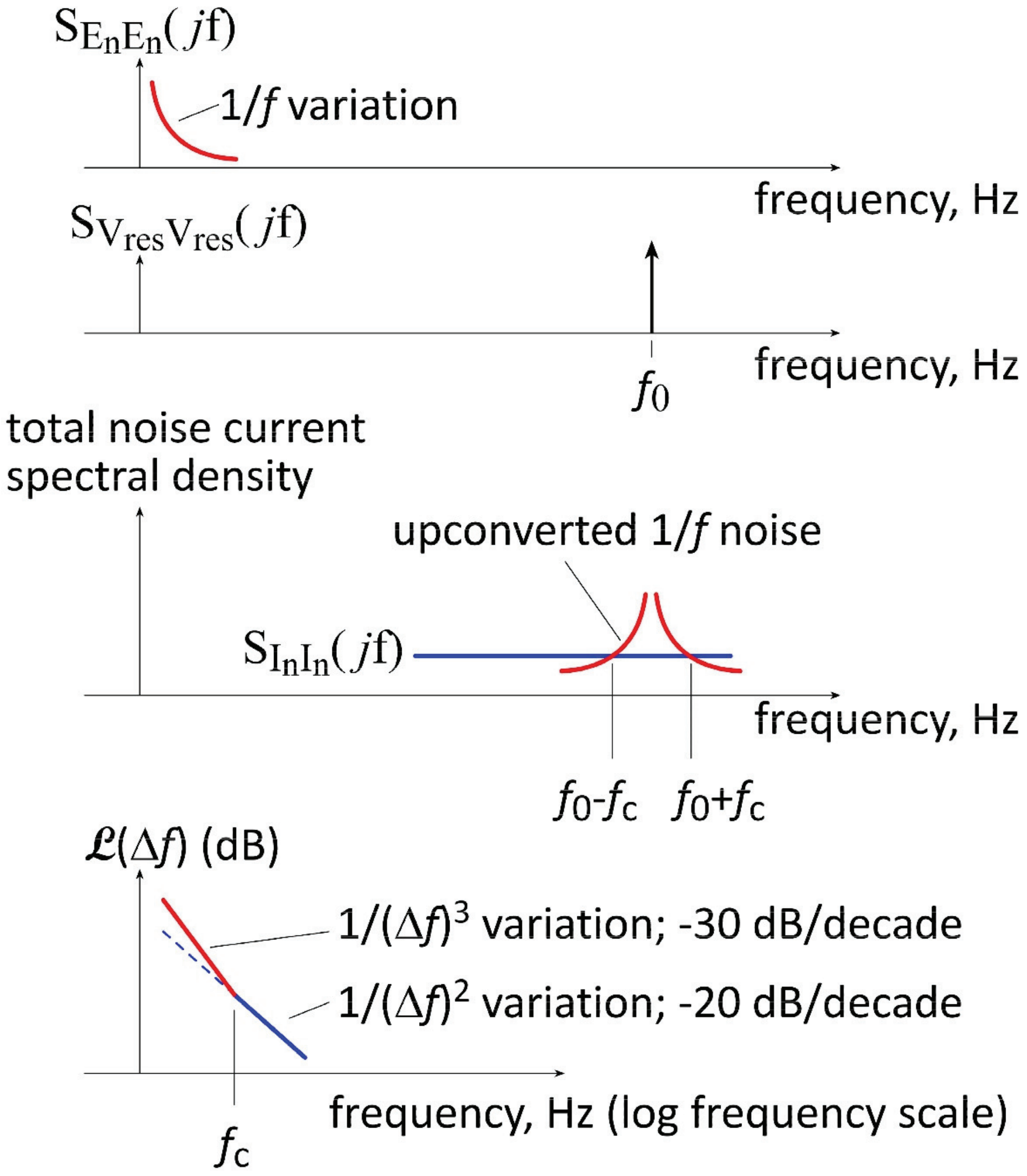
This shifted 1/f component is proportional to a_2.

The noise current is now $S_{I_nI_n}(f)=S_{I_{n0}I_{n0}}\left(1+\dfrac{f_c}{|f-f_0|}\right)$

Consequently

$$L(\Delta f)=\frac{1}{8}\left(\frac{kTF}{Q^2P_{res}}\right)\left(\frac{f_0}{\Delta f}\right)^2\left(1+\frac{f_c}{\Delta f}\right)$$

Variation is as $|1/\Delta f^3|$, i.e. -30 dB/decade, close to carrier.



*M. S. Keshner, "1/f noise," in *Proceedings of the IEEE*, vol. 70, no. 3, pp. 212-218, March 1982, doi: 10.1109/PROC.1982.12282.

Leeson Model: optimum oscillator design ?

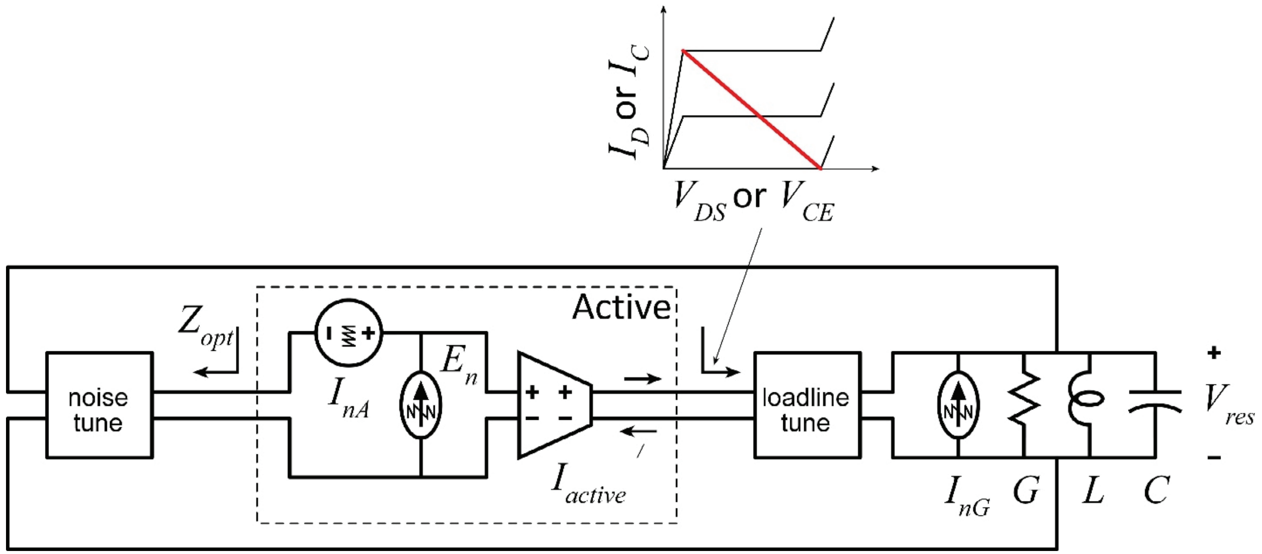
$$L(\Delta\omega) = \frac{1}{8}\left(\frac{kTF}{Q^2 P_{res}}\right)\left(\frac{\omega_0}{\Delta\omega}\right)^2$$

The Leeson expression suggests that we

1) Obtain the highest feasible unloaded* resonator Q,

2) Tune the active element output for maximum P_{res}.

2) Tune the active element input for minimum F.

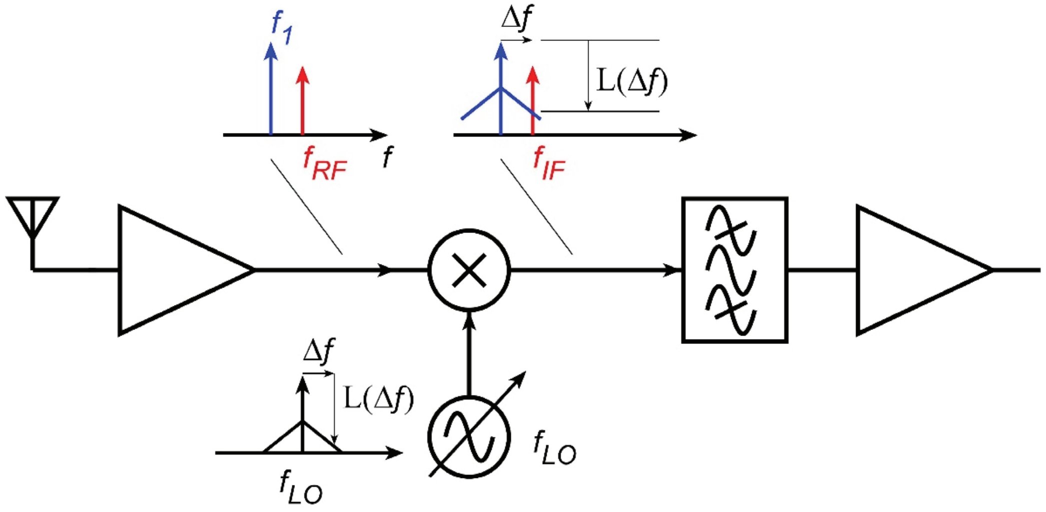


*Note that, by our definitions, G is that of the resonator, independent of the active element input and output impedances. Consequently, Q in this expression is independent of the active element input and output impedances.

Why phase noise matters in receivers & transmitters

In a receiver, with desired signal at f_{RF} and interferer at f_1, mixing of f_1 with the local oscillator will produce sidebands around f_1 that may fall into the bandwidth of the desired signal.

This will add to the noise background, degrading sensitivity.



In a transmitter, mixing of the IF with the local oscillator will produce sidebands around the RF signal that may fall outside the allocated modulation bandwidth of the desired signal.

This will interfere with communications in adjacent frequency bands.