APPENDIX W1

FOURIER TRANSFORMS

In Chapter 13 we studied the Fourier series which resolves a periodic waveform into an infinite series of sinusoids at discrete harmonic frequencies nf_0, $n = 1, 2, \ldots\infty$. The Fourier series allows us to construct the line spectrum of a periodic waveform from the distribution amplitudes and phase angles across the harmonic frequencies. In this appendix we introduce Fourier transforms that extend the idea of a frequency spectrum to aperiodic waveforms that occur only once. The Fourier transform of an aperiodic signal and the Fourier series of a periodic waveform both involve infinitely many frequencies. A key difference is that the frequencies in a Fourier series are integer multiples of f_0 while the frequencies in a Fourier transform span a continuum of values.

W1-1 DEFINITION OF FOURIER TRANSFORMS

Like Laplace transforms, Fourier transforms involve two domains: (1) the time domain in which the signal is represented by its *waveform* $f(t)$, and (2) the frequency domain in which the signal is characterized by it Fourier *transform* $F(\omega)$. In general $F(\omega)$ is a complex-valued function of the *real frequency variable* ω. The magnitude and phase angle of $F(\omega)$ are called the *amplitude spectrum* and *phase spectrum*, respectively.

The Fourier transform of a waveform $f(t)$ is defined by the integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad \text{(W1-1)}$$

As you would expect from our study of Laplace transforms, there is an inverse procedure which converts a Fourier transform into a time-domain waveform. Inverse Fourier transforms are defined by the integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \qquad \text{(W1-2)}$$

which yields $f(t)$ when given $F(\omega)$. The waveform $f(t)$ and transform $F(\omega)$ comprise a **Fourier transform pair**. We say that $F(\omega)$ is the Fourier transform of $f(t)$, and conversely, $f(t)$ is the inverse transform of $F(\omega)$. The short hand notation $F(\omega) = \mathscr{F}\{f(t)\}$ and $f(t) = \mathscr{F}^{-1}\{F(\omega)\}$ is used to denote these operations.

The limits on these integrals indicate important features of Fourier transforms. The direct transform in Eq. (W1-1) is a time integration across the interval $-\infty < t < \infty$. This points out that Fourier transforms exist for either causal or noncausal waveforms.[1] The inverse transform in Eq. (W1-2) is an integration on the real frequency variable ω across the range $-\infty < \omega < \infty$. This points out that ω is a continuous variable that takes on positive and negative values. Thus, Fourier transforms have spectral content across a continuous range of frequencies, whereas the spectral content of a Fourier series exists only at discrete, harmonic frequencies.

The infinite limits in Eq. (W1-1) should also alert the reader to the question of convergence and existence of $F(\omega)$. Existence is assured by the **Dirichlet conditions,** which require that the waveform $f(t)$ have a finite number of discontinuities and be absolutely

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[1] Recall from Chapter 9 that unique Laplace transforms only exist for causal waveforms for which $f(t) = 0$ for $t < 0$.

integrable, i.e,

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \qquad \text{(W1-3)}$$

A number of useful waveforms meet the Dirichlet conditions, for example, the rectangular pulse $[u(t) - u(t - T)]$ ($|T| < \infty$) and the causal exponential $u(t)e^{-\alpha t}$ ($\alpha > 0$). While the Dirichlet conditions are mathematically sufficient they are not necessary *and* sufficient, there being many useful signals with Fourier transforms that fail to meet them.

Finally, whenever $F(\omega)$ exists there is a unique, one-to-one correspondence between a waveform $f(t)$ and its Fourier transform $F(\omega)$. Uniqueness means that a transform pair can be used to go from waveform to transform or transform to waveform. Once we have developed a basic set of pairs we will summarize our results in a table of Fourier transform pairs similar to our table of Laplace transform pairs.

Example W1-1

Find the Fourier transform of the rectangular pulse in Figure W1-1 and plot its amplitude spectrum.

FIGURE W1-1

Solution:

The specified waveform meets the Dirichlet conditions since it is clearly absolutely integrable and has only two isolated discontinuities. For this waveform integration limits in Eq. (W1-1) are $-T/2$ and $+T/2$ since $f(t)$ is zero everywhere outside this range. As a result $F(\omega)$ is found as

$$F(\omega) = \int_{-T/2}^{T/2} A e^{-j\omega t} dt = -\frac{A}{j\omega} e^{-j\omega t} \Bigg|_{-T/2}^{T/2}$$

$$= \frac{A}{\omega/2} \left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j2} \right]$$

$$= AT \frac{\sin(\omega T/2)}{\omega T/2}$$

Figure W1-2 shows a plot of the amplitude spectrum $|F(\omega)|$ versus ω.[2] The spectrum of a rectangular pulse is a continuous function of a continuous frequency variable. There is some spectral content in every nonzero frequency range no matter how small. The isolated nulls in the spectral plot occur at the discrete frequencies $\omega = 2k\pi/T$ ($k = \pm1, \pm2, \ldots$) where $\sin(\omega T/2) = 0$.

FIGURE W1-2

Example W1-2

Find the Fourier transform of the casual exponential $f(t) = Ae^{-\alpha t}\, u(t)$ and plot its amplitude and phase spectra. Assume $\alpha > 0$.

Solution:

This waveform also meets the Dirichlet conditions since it has only one isolated discontinuity and is absolutely integrable in the sense of Eq. (W1-3). The waveform is causal so the

[2] The transform in this example involves a function usually denoted as $\text{sinc}(x) = \sin x/x$. At first glance sinc($x$) may appear to blow up at $x = 0$. However, a single application of L'Hopital's rule shows that sinc(0) = 1.

integration in Eq. (W1-1) extends from $t = 0$ to $t = +\infty$.

$$F(\omega) = \int_0^\infty A e^{-\alpha t} e^{-j\omega t} dt = A \int_0^\infty e^{-(\alpha + j\omega)t} dt$$

$$= -A \left. \frac{e^{-(\alpha + j\omega)t}}{\alpha + j\omega} \right|_0^\infty$$

For $\alpha > 0$ the integral vanishes at the upper limit and $F(\omega)$ becomes

$$F(\omega) = \frac{A}{\alpha + j\omega} \quad (\alpha > 0)$$

The amplitude and phase spectra of $F(\omega)$ are

$$|F(\omega)| = \frac{|A|}{\sqrt{\alpha^2 + \omega^2}} \quad \text{(Amplitude)}$$

and

$$\phi(\omega) = \angle F(\omega) = -\tan^{-1}(\omega/\alpha) \quad \text{(Phase)}$$

Figure W1-3 show plots of the amplitude and phase spectra. The amplitude spectrum of this aperiodic waveform is a continuous function of a continuous frequency variable. There is some spectral content at every frequency, including negative frequencies.

FIGURE W1-3

Exercise W1-1

Find the Fourier transform of the waveform in Figure W1-4.

FIGURE W1-4

Answer:

$$F(\omega) = -jAT\frac{1 - \cos(\omega T/2)}{\omega T/2}$$

Inverse Fourier Transforms

Unlike the inversion integral for Laplace transforms, the Fourier inversion integral in Eq. (W1-2) is comparatively easy to use. As an example, consider the transform

$$F(\omega) = 2\pi\delta(\omega - \beta) \tag{W1-4}$$

This transform is a frequency-domain impulse located at $\omega = \beta$ with an area of 2π. The corresponding waveform is found by applying the inversion integral in Eq. (W1-2). Because $F(\omega)$ is zero everywhere except at $\omega = \beta$ the integration limits need only span the range from $\omega = \beta-$ to $\omega = \beta+$, just slightly above and below the frequency $\omega = \beta$.

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{\beta-}^{\beta+} 2\pi\delta(\omega - \beta)e^{j\omega t}d\omega \\ &= e^{j\beta t}\int_{\beta-}^{\beta+}\delta(\omega - \beta)d\omega = e^{j\beta t} \end{aligned} \tag{W1-5}$$

Straight forward use of the inversion integral yields $\mathscr{F}^{-1}\{2\pi\delta(\omega - \beta)\} = e^{j\beta t}$. Because the Fourier transformation is unique, we also know that $\mathscr{F}\{e^{j\beta t}\} = 2\pi\delta(\omega - \beta)$.

In particular, if $\beta = 0$, then $f(t) = e^0 = 1$ for all time t and $F(\omega) = 2\pi\delta(\omega)$. In other words, the Fourier transform of a dc waveform with amplitude $A = 1$ is an impulse of area 2π at $\omega = 0$. Figure W1-5 shows the waveform and Fourier transform of a dc signal. Note that the dc waveform is noncausal so it can not be represented by a Laplace transform. Likewise, the dc waveform does not meet the Dirichlet conditions because it is not absolutely integrable.

FIGURE W1-5

Example W1-3

$$F(\omega) = \frac{\pi A}{\beta}[u(\omega + \beta) - u(\omega - \beta)]$$

Use the inversion integral to find the waveform corresponding to the transform

Solution:

For this transform the integration limits on the inversion integral run from $\omega = -\beta$ to $\omega = +\beta$ because $F(\omega)$ is zero everywhere except in this range. Hence, the inverse transform yields

$$f(t) = \frac{1}{2\pi}\int_{-\beta}^{\beta} \pi \frac{A}{\beta} e^{j\omega t} d\omega = \frac{A}{2\beta}\frac{e^{j\omega t}}{jt}\bigg|_{-\beta}^{\beta}$$

$$= \frac{A}{\beta t}\frac{e^{j\beta t} - e^{-j\beta t}}{j2} = A\frac{\sin(\beta t)}{\beta t}$$

Figure W1-6 shows the given amplitude spectrum $|F(\omega)|$ and the resulting inverse transform $f(t)$. The waveform $f(t) = A\sin(\beta t)/(\beta t)$ is another example of the function $\text{sinc}(x) = \sin(x)/x$ which plays an important role in Fourier analysis. Note that $f(t)$ is a noncausal waveform for which there is no Laplace transform.

FIGURE W1-6

Exercise W1-2

Use the inversion integral to find the inverse transform of $F(\omega) = j\omega\pi[u(\omega+1) - u(\omega-1)]$.

Answer:

$$f(t) = \frac{\cos t}{t} - \frac{\sin t}{t^2}$$

W1-2 LAPLACE TRANSFORMS AND FOURIER TRANSFORMS

For certain signals there is a simple relationship between the Laplace and Fourier transforms. In Chapter 9 the integral definition of Laplace transforms is given as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s) \tag{W1-6}$$

where $s = \sigma + j\omega$ is the complex frequency variable. The lower limit of this integration reminds us that $f(t)$ must be causal for a unique transform pair to exist. If $f(t)$ is absolutely integrable in the sense defined in Eq. (W1-3), then the Laplace transform integration converges when $\sigma = 0$, in which case Eq. (W1-6) becomes

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-j\omega t}dt = F(s)\big|_{\sigma = 0} \tag{W1-7}$$

On the other hand, when $f(t)$ is causal and absolutely integrable its Fourier transform exists and is found from Eq. (W1-1) to be

$$\mathcal{F}\{f(t)\} = \int_0^\infty f(t)e^{-j\omega t}dt = F(\omega) \tag{W1-8}$$

Comparing Eqs. (W1-7) and (W1-8) we conclude that

$$F(\omega) = F(s)\big|_{\sigma = 0} \tag{W1-9}$$

provided that $f(t)$ is causal and absolutely integrable.

For a causal $f(t)$ to be absolutely integrable it must have finite duration or decay to zero as $t \to \infty$. A sufficient condition for $f(t)$ to decay to zero is that all of the poles of $F(s)$ lie in the left half plane. For example, the Laplace transform of the causal exponential is $\mathscr{L}\{Ae^{-\alpha t}u(t)\} = A/(s+\alpha)$. The transform has a pole at $s = -\alpha$, which lies in the left half of the s plane for $\alpha > 0$. When $\alpha > 0$ the Fourier transform of the causal exponential is found from its Laplace transform to be

$$F(\omega) = F(s)\big|_{\sigma = 0} = \frac{A}{j\omega + \alpha}$$

The $F(\omega)$ obtained here using Laplace transforms agrees with Example W1-2, where the integral definition of Fourier transforms gave the same result.

We can use Eq. (W1-9) to find $F(\omega)$ provided the poles of $F(s)$ avoid both the right half plane and the j-axis boundary. For example, the Fourier transforms of a causal sinusoid cannot be found from its Laplace transform. The reason is that the transform

$$F(s) = \mathscr{L}\{u(t)\sin(\beta t)\} = \beta/(s^2 + \beta^2)$$

has poles on the j axis at $s = \pm j\beta$.

The Laplace transform method can be used to find the Fourier transform of the unit impulse because $\delta(t)$ is causal and absolutely integrable. Applying Eq. (W1-9) we obtain

$$\mathscr{F}\{\delta(t)\} = \mathscr{L}\{\delta(t)\}\big|_{\sigma = 0} = 1 \qquad \text{(W1-10)}$$

Figure W1-7 shows impulse waveform $\delta(t)$ and its constant amplitude spectrum $|\Delta(\omega)|$. Note the duality between Figure W1-7 and Figure W1-5. Figure W1-5 shows that a constant time-domain waveform leads to an impulse in the frequency domain. The dual result in Figure

W1-7 shows that an impulse in the time-domain leads to a constant in the frequency domain.

FIGURE W1-7

Laplace transform concepts can be used to find inverse Fourier transforms as well. Given a Fourier transform $F(\omega)$, we form a Laplace transform $F(s)$ by replacing $j\omega$ by s, or equivalently, replacing ω by $-js$. If the poles of $F(s)$ all lie in the left half of the s plane, then by the uniqueness property of Fourier and Laplace transforms we have

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \mathscr{L}^{-1}\{F(s)\} \quad (t > 0) \tag{W1-11}$$

Partial fraction expansion can be used to find the inverse transform of $F(s)$ in this equation. However, keep in mind that the Laplace transform method requires either: (1) $f(t)$ to be causal and absolutely integrable, or (2) all of the poles of $F(s)$ to be in the left half of the s plane.

Exercise W1-3

Use Laplace transforms to find the Fourier transforms of the following causal waveforms. Assume $\alpha > 0$.

$$\text{(a) } f(t) = A[e^{-\alpha t}\sin \beta t]u(t)$$

$$\text{(b) } f(t) = A[\alpha t e^{-\alpha t}]u(t)$$

Answers:

$$\text{(a)}\ F(\omega) = \frac{A\beta}{(j\omega + \alpha)^2 + \beta^2}$$

$$\text{(b)}\ F(\omega) = \frac{A\alpha}{(j\omega + \alpha)^2}$$

Example W1-4

Use Laplace transforms to find the inverse Fourier transform of

$$F(\omega) = \frac{10}{(j\omega + 2)(j\omega + 4)}$$

Solution:

Replacing $j\omega$ by s yields $F(s)$ yields

$$F(s) = \frac{10}{(s + 2)(s + 4)}$$

$F(s)$ is a rational function of s with poles at $s = -2$ and $s = -4$. Both poles are in the left-half plane so the inverse transform of $F(s)$ is the inverse transform of $F(\omega)$. Expanding $F(s)$ by partial fractions yields

$$F(s) = \frac{k_1}{s + 2} + \frac{k_2}{s + 4}$$

The poles are simple so the residues k_1 and k_2 are

$$k_1 = (s + 2)F(s)\big]_{s=-2} = 5$$
$$k_2 = (s + 4)F(s)\big]_{s=-4} = -5$$

and the inverse transform of $F(\omega)$ is

$$\mathscr{F}^{-1}\{F(\omega)\} = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{5}{s+2}\right\} + \mathscr{L}^{-1}\left\{\frac{-5}{s+4}\right\}$$

$$= \left[5e^{-2t} - 5e^{-4t}\right]u(t)$$

Exercise W1-4

Use the Laplace transform method to find the causal waveform corresponding to the following Fourier transform:

$$F(\omega) = \frac{5(j\omega - 3)}{(j\omega + 1)(j\omega + 5)}$$

Answer:

$$f(t) = \left[-5e^{-t} + 10e^{-5t}\right]u(t)$$

Exercise W1-5

Explain why Laplace transforms cannot be used to find the Fourier transforms of the following waveforms. Assume $\alpha > 0$.

(a) $f(t) = A\alpha t u(t)$

(b) $f(t) = Ae^{-\alpha|t|}$

(c) $f(t) = A \sin \alpha t$.

Answers:

(a) $f(t)$ is not absolutely integrable or equivalently, $F(s)$ has a double pole at $s = 0$.

(b) & (c) $f(t)$ is not causal.

Exercise W1-6

Explain why Laplace transforms cannot be used to find the inverse transforms of the following functions. Assume $\alpha > 0$.

$$\text{(a)} \quad F(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$\text{(b)} \quad F(\omega) = e^{-\alpha|\omega|}$$

Answers:

(a) $F(s)$ has a right half plane pole at $s = +\alpha$.

(b) $F(s)$ is not a rational function.

W1-3 BASIC FOURIER TRANSFORM PROPERTIES AND PAIRS

This section introduces eight properties of Fourier transforms and uses these properties to derive additional transform pairs.

Linearity

Both Laplace and Fourier transforms are linear transformations which means that the transform of a sum of waveforms is the sum of their transforms. Stated formally the linearity property is

$$\mathscr{F}\{Af_1(t) + Bf_2(t)\} = AF_1(\omega) + BF_2(\omega) \tag{W1-12}$$

where A and B are constants. Proof of this property follows directly from the integral definition of Fourier transforms.

$$\begin{aligned}\mathscr{F}\{Af_1(t) + Bf_2(t)\} &= \int_{-\infty}^{\infty}\left[Af_1(t) + Bf_2(t)\right]e^{-j\omega t}d\omega \\ &= A\int_{-\infty}^{\infty}f_1(t)e^{-j\omega t}d\omega + B\int_{-\infty}^{\infty}f_2(t)e^{-j\omega t}d\omega \\ &= AF_1(\omega) \qquad + \qquad BF_2(\omega)\end{aligned}$$

The following example is an application of linearity.

Example W1-5

Find the Fourier transforms of the eternal sinusoids cos βt and sin βt.

Solution:

Laplace transforms method doesn't work here because these waveforms are not causal. We previously found the Fourier transform of complex exponential $e^{j\beta t}$ to be

$$\mathscr{F}\{e^{j\beta t}\} = 2\pi\delta(\omega - \beta)$$

Using Euler's relationship for cos βt and the linearity property we can write

$$\begin{aligned}\mathscr{F}\{\cos\beta t\} &= \mathscr{F}\left\{\frac{e^{j\beta t} + e^{-j\beta t}}{2}\right\} \\ &= \frac{1}{2}\mathscr{F}\left\{e^{j\beta t}\right\} + \frac{1}{2}\mathscr{F}\left\{e^{-j\beta t}\right\} \\ &= \pi\left[\delta(\omega - \beta) + \delta(\omega + \beta)\right]\end{aligned}$$

Similarly for sin βt we have

$$\begin{aligned}\mathscr{F}\{\sin\beta t\} &= \mathscr{F}\left\{\frac{e^{j\beta t} - e^{-j\beta t}}{j2}\right\} \\ &= \frac{1}{j2}\mathscr{F}\left\{e^{j\beta t}\right\} - \frac{1}{j2}\mathscr{F}\left\{e^{-j\beta t}\right\} \\ &= -j\pi\left[\delta(\omega - \beta) - \delta(\omega + \beta)\right]\end{aligned}$$

Here we see that the Fourier transforms of cos βt and sin βt involve impulses located at $\omega = \pm\beta$. In general, the spectral content of a Fourier transform is distributed across a continuous range of frequencies. However, for eternal sinusoids the spectral content is concentrated at discrete frequencies $\omega = \pm\beta$. where the frequency-domain impulses occur.

Time Differentiation and Integration

The **time differentiation** property of Fourier transforms states that differentiating $f(t)$ in the time domain corresponds to multiplying $F(\omega)$ by $j\omega$ in the frequency domain. That is

$$\mathscr{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega) \tag{W1-13}$$

Derivation of this property of Fourier transforms begins with the inversion integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

We first differentiate both sides of this equation with respect to time:

$$\frac{df(t)}{dt} = g(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right]$$

Assuming the order of integration and differentiation on the right side of this equation can be interchanged, we obtain

$$\frac{df(t)}{dt} = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{d}{dt}F(\omega)e^{j\omega t}\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)e^{j\omega t}d\omega$$

The right side of the last line in this equation implies $\mathscr{F}\{g(t)\} = j\omega F(\omega)$. Since $g(t) = df/dt$ we conclude that

$$\mathscr{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

as expected.

Given the form of the differentiation property, it is reasonable to expect that integrating $f(t)$ in the time domain corresponds to dividing $F(\omega)$ by $j\omega$ in the frequency domain. This proves to be correct, except that integrating a waveform may produce a constant offset or dc component. As we have seen, the Fourier transform of a dc waveform is an impulse at $\omega = 0$.

As a result, the statement of the **integration property** is

$$\mathscr{F}\left\{\int_{-\infty}^{t} f(x)dx\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) \qquad \text{(W1-14)}$$

Integrating $f(t)$ in the time domain leads to division of $F(\omega)$ by $j\omega$ plus an additive term $\pi F(0)\delta(\omega)$ to account for the possibility of a dc component that may appear in the integral of $f(t)$.

Example W1-6

Use the integration property to find the Fourier transform of the step function $u(t)$.

Solution:

The Fourier transform of an impulse was previously found to be $\mathscr{F}\{\delta(t)\} = \Delta(\omega) = 1$. Since the step function is the integral of an impulse the integration property yields the Fourier transform of the unit step function.

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\int_{-\infty}^{t}\delta(x)dx\right\} = \frac{\Delta(\omega)}{j\omega} + \pi\Delta(0)\delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\delta(\omega)$$

Reversal

The **reversal** property states that

$$\text{If } \mathcal{F}\{f(t)\} = F(\omega) \text{ then } \mathcal{F}\{f(-t)\} = F(-\omega) \qquad \text{(W1-15)}$$

Simply stated, reversing $f(t)$ in the time domain reverses $F(\omega)$ in the frequency domain.

Deriving this property begins with the integral definition of Fourier transforms.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad \text{(W1-16)}$$

Defining $\mathcal{F}\{f(-t)\} = G(\omega)$ we can write

$$G(\omega) = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt$$

Changing the dummy variable of integration to $\tau = -t$ produces

$$G(\omega) = -\int_{\infty}^{-\infty} f(\tau)e^{j\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)e^{-j(-\omega)\tau}d\tau = F(-\omega)$$

We conclude that $\mathscr{F}\{f(-t)\} = G(\omega) = F(-\omega)$ as stated in Eq. (W1-15).

The reversal property is used to derive the Fourier transforms of waveforms of the type in Figure W1-8. The mathematical expressions for these two waveforms are

$$\begin{aligned} \text{sgn}(t) &= u(t) - u(-t) \\ e^{-\alpha|t|} &= u(t)e^{-\alpha t} + u(-t)e^{-\alpha(-t)} \end{aligned} \tag{W1-17}$$

The first waveform is called the **signum function**, which is defined as plus one for $t > 0$, minus one for $t < 0$ and zero at $t = 0$. As a result sgn(t) can be expressed as the difference between a step $u(t)$ to a reversed step $u(-t)$ as indicated in Eq. (W1-17).

The second waveform in Figure W1-8 is called a **double-sided** exponential which is defined as the sum of a causal exponential and a reversed causal exponential. This waveform is one at $t = 0$ and decays exponentially to zero in both directions along the time axis.

FIGURE W1-8

The signum and double-sided exponential are examples of noncausal waveforms that can be written as $f(t) = g(t) \pm h(-t)$, where $g(t)$ and $h(t)$ are causal. If the Fourier transforms of $g(t)$ and $h(t)$ are known, then the reversal property yields $F(\omega) = G(\omega) \pm H(-\omega)$.

Example W1-7

Use the reversal property to find the Fourier transform of $f(t) = \text{sgn}(t)$

Solution:

The signum function is expressed as $\text{sgn}(t) = u(t) - u(-t)$. In Example W1-6 we found that

$\mathscr{F}\{u(t)\} = 1/j\omega + \pi\delta(\omega)$. Using the reversal property yields

$$\begin{aligned} \mathscr{F}\{\text{sgn}(t)\} &= \mathscr{F}\{u(t)\} - \mathscr{F}\{u(-t)\} \\ &= \left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \left[\frac{1}{j(-\omega)} + \pi\delta(-\omega)\right] \\ &= \frac{2}{j\omega} \end{aligned}$$

In the last line of this equation we have used the fact that $\delta(\omega) = \delta(-\omega)$ since both impulses occur at $\omega = 0$.

Exercise W1-7

Use the reversal property to find the Fourier transform of $f(t) = e^{-\alpha|t|}$. Assume $\alpha > 0$.

Answer:

$$F(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}$$

Duality

The **duality property** states that

$$\text{If } \mathscr{F}\{f(t)\} = F(\omega) \text{ then } \mathscr{F}\{F(t)\} = 2\pi f(-\omega) \tag{W1-18}$$

Proof of this property begins with the inversion integral written in the form

$$2\pi f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Replacing t by $-t$ yields

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega$$

Now interchanging t and ω produces

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}dt = \mathscr{F}\{F(t)\}$$

Thus, $\mathscr{F}\{F(t)\} = 2\pi f(-\omega)$ as stated in the duality property.

Figure W1-9 illustrates the waveforms/transforms duality for a rectangular functions. The upper pair comes from Example W1-1 where we found that a rectangular waveform has a wiggly Fourier transform. The lower pair comes from Example W1-3 where we found that a rectangular transform produces a wiggly waveform. In sum, a rectangular function in one domain leads to an oscillatory sinc(x) function in the other domain.

FIGURE W1-9

Example W1-8

Given that $\mathscr{F}\{e^{-\alpha|t|}\} = 2\alpha/(\alpha^2 + \omega^2)$, use the duality property to find the Fourier transform of $g(t) = \alpha^2/(\alpha^2 + t^2)$.

Solution:

If we define $f(t) = e^{-\alpha|t|}$ then $F(\omega) = 2\alpha/(\alpha^2 + \omega^2)$. According to duality

$$\mathscr{F}\{F(t)\} = 2\pi f(-\omega) = 2\pi e^{-\alpha|-\omega|} = 2\pi e^{-\alpha|\omega|}$$

But since

$$F(t) = \frac{2\alpha}{\alpha^2 + t^2} = \frac{2}{\alpha}\left[\frac{\alpha^2}{\alpha^2 + t^2}\right] = \frac{2}{\alpha}g(t)$$

it follows from linearity that

$$\mathscr{F}\{g(t)\} = \frac{\alpha}{2}\mathscr{F}\{F(t)\} = \pi\alpha e^{-\alpha|\omega|}$$

Time and Frequency Shifting

The two shifting properties of the Fourier transforms are

$$\begin{aligned}&(1)\,\text{Time Shift: If } \mathscr{F}\{f(t)\} = F(\omega) \text{ then } \mathscr{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)\\&(2)\,\text{Freq. Shift: If } \mathscr{F}^{-1}\{F(\omega)\} = f(t) \text{ then } \mathscr{F}^{-1}\{F(\omega-\beta)\} = e^{j\beta t}f(t)\end{aligned} \qquad \text{(W1-19)}$$

Proof of these properties follow directly from the definitions in Eq. (W1-1) and (W1-2).

The first statement in Eq. (W1-19) says that time shifting a waveform by an amount T

is equivalent to multiplying its transform by $e^{-j\omega T}$. As a result we find that delaying a waveform $f(t)$ does not distort its amplitude spectrum since

$$|\mathcal{F}\{f(t - T)\}| = |e^{-j\omega T}F(\omega)| = |e^{-j\omega T}||F(\omega)| = |F(\omega)|$$

The second statement In Eq. (W1-19) says that shifting the frequency of a transform by an amount β is equivalent to multiplying its waveform by $e^{j\beta t}$. The frequency shifting property is the basis for a signal processing operation called modulation in which the spectrum of a signal is shifted from one frequency range to another.

Scaling

The **scaling property** of the Fourier transformation is

$$\text{If } \mathcal{F}\{f(t)\} = F(\omega) \text{ then } \mathcal{F}\{f(at)\} = \frac{1}{|a|}F(\omega/a) \qquad \text{(W1-20)}$$

where a is a real constant.

Figure W1-10 shows the Fourier transforms of two rectangular pulses with different pulse durations. The figure points out that shortening the pulse duration ($a < 1$) causes the transform to spread out in the frequency domain, and vice versa. For this reason the scaling property is sometimes called the *reciprocal spreading* property. That is, compressing a signal in one domain causes it to spread out in the other domain. An important implication is that reducing pulse duration in the time domain increases the bandwidth requirement since the signal spectrum expands in the frequency domain.

FIGURE W1-10

Summary Tables

Table W1-1 lists a basic set of Fourier transform pairs. Because of the uniqueness property this table can be used in either direction. Additional pairs are easy to derive using the pairs in Table W1-1 together with the transform properties summarized in Table W1-2.

SIGNAL	**WAVEFORM $f(t)$**	**TRANSFORM $F(\omega)$**
Impulse	$\delta(t)$	1
Constant (dc)	1	$2\pi\delta(\omega)$
Step function	$u(t)$	$1/j\omega + \pi\delta(\omega)$
Signum	$\text{sgn}(t)$	$2/j\omega$
Causal Exponential	$[e^{-\alpha t}]u(t)$	$1/(\alpha + j\omega)$
Two-Sided Exponential	$e^{-\alpha\|t\|}$	$2\alpha/(\alpha^2 + \omega^2)$
Complex Exponential	$e^{j\beta t}$	$2\pi\delta(\omega - \beta)$
Cosine (ac)	$\cos(\beta t)$	$\pi[\delta(\omega - \beta) + \delta(\omega + \beta)]$
Sine (ac)	$\sin(\beta t)$	$-j\pi[\delta(\omega - \beta) - \delta(\omega + \beta)]$
Sinc	$\sin(\beta t)/(\pi t)$	$u(\omega - \beta) - u(\omega + \beta)$

TABLE W1-1 Basic Fourier Transform Pairs

PROPERTY	TIME DOMAIN	FREQUENCY DOMAIN
Linearity	$Af_1(t) + Bf_2(t)$	$AF_1(\omega) + BF_2(\omega)$
Differentiation	$\dfrac{df(t)}{dt}$	$j\omega F(\omega)$
Integration	$\int_{-\infty}^{t} f(x)dx$	$\dfrac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
Reversal	$f(-t)$	$F(-\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Time shift	$f(t - T)$	$e^{-j\omega T}F(\omega)$
Frequency shift	$e^{j\beta t}f(t)$	$F(\omega - \beta)$
Scaling	$\|a\|f(at)$	$F(\omega/a)$

TABLE W1-2 Basic Fourier Transform Properties

Exercise W1-8

Use Tables W1-1 and W1-2 to find the Fourier transforms of

(a) $f_1(t) = \text{sgn}(t) + 2$

(b) $f_2(t) = 2\cos(2t) + 1$

Answers:

$$\text{(a)}\quad F_1(\omega) = \frac{2}{j\omega} + 4\pi\delta(\omega)$$

$$\text{(b)}\quad F_2(\omega) = 2\pi[\delta(\omega - 2) + \delta(\omega) + \delta(\omega + 2)]$$

Exercise W1-9

Use Tables W1-1 and W1-2 to find the inverse Fourier transforms of

$$\text{(a)}\quad F_1(\omega) = \frac{1}{j\omega} + 3\pi\delta(\omega)$$

$$\text{(b)}\quad F_2(\omega) = \frac{2}{4 + j\omega} - 2 + 4\pi\delta(\omega + 2)$$

Answers:

(a) $f_1(t) = u(t) + 1$

(b) $f_2(t) = 2\,e^{-4t} - 2\delta(t) + 2\,e^{-j2t}$

W1-4 CIRCUIT ANALYSIS USING FOURIER TRANSFORMS

Because of our previous study of Laplace transforms it should come as no surprise that Fourier transforms can be used in linear circuit analysis. To see how the Fourier transforms apply to circuit analysis, we must examine how they effect connection and device constraints.

In the time domain a typical KVL constraint might be

$$v_1(t) + v_2(t) - v_3(t) = 0$$

Because of the linearity property the Fourier transform of this equation is

$$V_1(\omega) + V_2(\omega) - V_3(\omega) = 0$$

This example obviously generalizes to any KVL or KCL constraint. The Fourier transformation changes waveforms into transforms but leaves the form of the connection constraints unchanged.

The frequency-domain element constraints are found by transforming the time domain *i-v* characteristics of the passive elements. Using the differentiation and linearity properties of Fourier transforms we write the time-domain and frequency-domain element constraints as

	Time Domain	Frequency Domain
Resistor:	$v(t) = Ri(t)$	$V(\omega) = RI(\omega)$
Inductor:	$v(t) = L\dfrac{di}{dt}$	$V(\omega) = j\omega LI(\omega)$
Capacitor:	$i(t) = C\dfrac{dv}{dt}$	$I(\omega) = j\omega CV(\omega)$

As might be expected, in the frequency domain the element constraints are algebraic equations similar in form to Ohm's law. The proportionality factors relating the voltage and current transforms are the ac impedance of the passive elements, namely

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$$

The form of the frequency domain connection and element constraints have a familiar ring. The key points are: (1) The Fourier transformation does not change the form of the connection constraints and, (2) The element constraints are similar to Ohm's law. We have seen analogous conclusion before in our study of sinusoidal steady-state circuit analysis and again in *s*-domain circuit analysis. As a result, we know that our repertoire of algebraic circuit analysis techniques can be applied using Fourier transforms.

To use Fourier methods in circuit analysis we represent currents and voltages as transforms and passive elements as ac impedances. We then use analysis techniques such as voltage division, current division, equivalence, or even node/mesh analysis to solve for the unknown current or voltage transforms. Inverse Fourier transforms are then used to obtain the response waveform in the time domain.

The discussion above closely parallels the steps in *s*-domain circuit analysis, except for one thing--there is no mention of initial conditions. The reason for this omission is that the lower limit on the defining integral for Fourier transforms is $t = -\infty$. In other words, Fourier transforms account for the entire history of a circuit beginning at $t = -\infty$. With Laplace transforms the lower limit on the defining integral is $t = 0$, and the circuit's history for $t < 0$ is accounted for by the initial conditions at $t = 0$.

In summary, the Fourier transform method of circuit analysis explicitly takes into account all inputs including those that happened prior to $t = 0$. Fourier transforms exist for noncausal waveforms and the method yields responses that are valid for $-\infty < t < \infty$. As a result, Fourier transform methods cannot handle circuit analysis problems in which initial conditions are used to account for inputs prior to $t = 0$.

Example W1-9

Use Fourier transforms to find $v_2(t)$ in the circuit in Figure W1-11 for $v_1(t) = u(t)$.

FIGURE W1-11

Solution:

This example asks us to find the step response of a first-order RC circuit. In Chapter 7 we solved this problem using classical differential equation methods. The problem was revisited in Chapters 9 and 11 using Laplace transforms. From this previous experience we already know that the answer is $v_2(t) = [1 - e^{-t/RC}]u(t)$. Our purpose here is simply to show how Fourier transforms produce the same result.

The circuit in Figure W1-11 is shown in the frequency domain so we use voltage division to relate the input and output transforms.

$$V_2(\omega) = \left[\frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}}\right] V_1(\omega) = \left[\frac{1/RC}{j\omega + 1/RC}\right] V_1(\omega)$$

For a unit step function input $V_1(\omega) = \mathscr{F}\{u(t)\} = 1/j\omega + \pi\delta(\omega)$ and the output transform is

$$V_2(\omega) = \left[\frac{1/RC}{j\omega + 1/RC}\right]\frac{1}{j\omega} + \left[\frac{1/RC}{j\omega + 1/RC}\right]\pi\delta(\omega)$$

The first term on the right side can be expanded by partial fractions as

$$\frac{1/RC}{(j\omega + 1/RC)j\omega} = \frac{1}{j\omega} - \frac{1}{j\omega + 1/RC}$$

The second term on the right side reduces to

$$\left[\frac{1/RC}{j\omega + 1/RC}\right]\pi\delta(\omega) = \left[\frac{1/RC}{1/RC}\right]\pi\delta(\omega) = \pi\delta(\omega)$$

That is, in general $F(\omega)\delta(\omega) = F(0)\delta(\omega)$ because the impulse $\delta(\omega)$ is zero everywhere except at $\omega = 0$. Combining these results the output transform can be arranged as

$$V_2(\omega) = \underbrace{\frac{1}{j\omega} + \pi\delta(\omega)}_{\text{step function}} - \underbrace{\frac{1}{j\omega + 1/RC}}_{\text{causal exponential}}$$

Each term in this expansion is listed in Table W1-1 leading to

$$\begin{aligned} v_2(t) &= \mathscr{F}^{-1}\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} - \mathscr{F}^{-1}\left\{\frac{1}{j\omega + 1/RC}\right\} \\ &= u(t) - e^{-t/RC}u(t) \end{aligned}$$

As expected the step response waveform is $v_2(t) = [1 - e^{-t/RC}]u(t)$. This example illustrates a rather subtle point. For causal inputs Fourier transforms inherently yield only the zero-state response, whereas Laplace transforms yield both the zero-state and zero-input responses.

Example W1-10

Use Fourier transforms to find $v_2(t)$ in the circuit in Figure W1-11 for $v_1(t) = \text{sgn}(t)$.

Solution:

In this example we find the output of a first-order RC circuit for a noncausal input. From Table W1-1 the input transform is $V_1(\omega) = 2/j\omega$. Using the voltage division result from Example W1-9 we write

$$V_2(\omega) = \left[\frac{1/RC}{j\omega + 1/RC}\right]\frac{2}{j\omega}$$

Expanding by partial fractions:

$$V_2(\omega) = \frac{2/RC}{j\omega(j\omega + 1/RC)} = \underbrace{\frac{2}{j\omega}}_{\text{signum}} - \underbrace{\frac{2}{j\omega + 1/RC}}_{\text{exponential}}$$

Each term in this expansion is listed in Table W1-1. The inverse transform produces

$$\begin{aligned} v_2(t) &= \mathscr{F}^{-1}\left\{\frac{2}{j\omega}\right\} - \mathscr{F}^{-1}\left\{\frac{2}{j\omega + 1/RC}\right\} \\ &= \text{sgn}(t) - 2e^{-t/RC}u(t) \end{aligned}$$

Both a noncausal signum waveform and a causal exponential are required to describe $v_2(t)$ on the interval $-\infty < t < +\infty$. Figure W1-12 shows how these waveforms combine to produce $v_2(t)$. A physical interpretation of this response is that signum waveform applies an input $v_1(t) = -1$ V at $t = -\infty$ (a long time ago). After about five time constants ($t = -\infty + 5RC$, still a long time ago) the circuit output reaches a steady-state condition of $v_2(t) = -1$ V. Thus, for all practical purposes $v_2(t) = -1$ V for all $t < 0$. At $t = 0$ the signum input jumps from $v_1(t) = -1$ to +1 V. This drives the output from $v_2(t) = -1$ V at $t = 0$ to $v_2(t) = +1$ V for $t > 5RC$.

FIGURE W1-12

Exercise W1-10

Use Fourier transforms to find $v_2(t)$ in the circuit in Figure W1-11 for $v_1(t) = u(t) - 1$.

Answer:

$v_2(t) = -u(-t) - e^{-t/RC}\, u(t)$

W1-5 IMPULSE RESPONSE AND CONVOLUTION

The previous section shows that Fourier transforms can be used to analyze linear circuits. The process is straight forward once we accept the idea of noncausal waveforms. However, the important applications of Fourier transforms are found in system analysis and signal processing rather than traditional circuit analysis.

Fourier transforms find applications in these areas because they can represent idealized models of signal processors. These ideal models often involve noncausal responses that cannot be represented by Laplace transforms. Of course real physical systems must have causal responses so these models can only be approximated by engineering hardware. Nevertheless, ideal models are useful in preliminary design studies and as "gold standards" for evaluating the performance of real systems.

Signal processors are often described in the frequency domain by the Fourier transform equivalent of the *s*-domain transfer function. To develop the Fourier equivalent we begin with the convolution integral from Chapter 11:

$$y(t) = \int_{0}^{t} h(t - \tau)\, x(\tau)\, d\tau$$

In this expression $y(t)$ is the output of a system whose impulse response is $h(t)$ when the input is $x(t)$. The limits of integration are $\tau = 0$ to $\tau = t$ because Chapter 11 assumes that $h(t)$ and $x(t)$ are causal waveforms. To remove this limitation the integration limits are extended backward to $\tau = -\infty$ to accommodate a noncausal input $x(\tau)$ and forward to $t = +\infty$ to accommodate a noncausal $h(t)$. The result is

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)\, x(\tau)\, d\tau \qquad \text{(W1-21)}$$

This equation is a general form of the convolution integral that applies to causal and noncausal waveforms alike.

We now take the Fourier transformation on both sides of Eq. (W1-21).

$$\mathscr{F}\{y(t)\} = Y(\omega) = \mathscr{F}\left\{ \int_{\tau=-\infty}^{\infty} h(t - \tau) x(\tau) d\tau \right\}$$

$$= \int_{t=-\infty}^{\infty} \left[\int_{\tau=-\infty}^{\infty} h(t - \tau) x(\tau) d\tau \right] e^{-j\omega t} dt$$

Next we interchange the order of integration and factor out $x(\tau)$, since it does not depend on t.

$$Y(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau) \underbrace{\left[\int_{t=-\infty}^{\infty} h(t - \tau) e^{-j\omega t} dt \right]}_{H(\omega) e^{-j\omega\tau}} d\tau$$

Using the time-shifting property the quantity within the bracket is $\mathscr{F}\{h(t - \tau)\} = H(\omega)e^{-j\omega\tau}$, and this equation reduces to

$$Y(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau)H(\omega)e^{-j\omega\tau}\,d\tau$$

$$= H(\omega)\underbrace{\left[\int_{\tau=-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau\right]}_{X(\omega)}$$

By definition the remaining integral inside the bracket is $\mathscr{F}\{x(\tau)\} = X(\omega)$. We conclude that

$$Y(\omega) = H(\omega)X(\omega) \qquad \text{(W1-22)}$$

Thus, in the frequency domain convolution is accomplished by multiplying the Fourier transform of the system impulse response by the Fourier transform of the input.

Equation (W1-22) looks suspiciously like the s-domain relationship $Y(s) = T(s)X(s)$, where $T(s)$ is the s-domain transfer function. Since the two input-output relationships have the same form we call $H(\omega)$ the **Fourier-domain transfer function**. Key differences between $T(s) = \mathscr{L}\{h(t)\}$ and $H(\omega) = \mathscr{F}\{h(t)\}$ are:

(1) $H(\omega)$ exists for noncausal impulse responses for which $T(s)$ does not exist.

(2) $T(s)$ exists for unstable impulse responses for which $H(\omega)$ does not exist.

Example W1-11

The impulse response of a system is $h(t) = e^{-|t|}$. Use Fourier transform convolution to find the output $y(t)$ when $x(t) = \text{sgn}(t)$.

Solution:

Using transform pairs in Table W1-1 we have

$$H(\omega) = \frac{2}{1 + \omega^2} \quad \text{and} \quad X(\omega) = \frac{2}{j\omega}$$

Using Eq. (W1-22) the output transform is found as

$$Y(\omega) = H(\omega)X(\omega) = \frac{4}{(1 + \omega^2)j\omega}$$

Expanding $Y(\omega)$ by partial fractions

$$Y(\omega) = \frac{4}{(1 + j\omega)(1 - j\omega)j\omega} = \frac{-2}{1 + j\omega} + \frac{2}{1 - j\omega} + \frac{4}{j\omega}$$

Using Table W1-1 the inverse transform of the first term in the expansion is

$$\mathscr{F}^{-1}\left\{\frac{-2}{1 + j\omega}\right\} = -2e^{-t}u(t)$$

According to the reversal property $\mathscr{F}^{-1}\{F(-\omega)\} = f(-t)$, hence the inverse transform of the second term is

$$\mathscr{F}^{-1}\left\{\frac{2}{1 + j(-\omega)}\right\} = 2e^{t}u(-t)$$

The inverse transform of the third term is $\mathscr{F}^{-1}\{4/j\omega\} = 2\ \mathrm{sgn}(t)$. So finally the output waveform is

$$y(t) = -2e^{-t}u(t) + 2e^{t}u(-t) + 2\,\mathrm{sgn}(t)$$

Notice that we can not use Laplace transforms here because the input, output, and impulse response are all noncausal waveforms.

Exercise W1-11

The impulse response of a system is $h(t) = 2\,e^{-4t}u(t)$. Find the output $y(t)$ when $x(t) = -2 + 4u(t)$.

Answer:

$$y(t) = \mathrm{sgn}(t) - 2e^{-4t}u(t)$$

The Fourier domain transfer function $H(\omega)$ represents the system frequency response, including noncausal systems. For example, consider the $H(\omega)$ shown in Figure W1-13. In the filter terminology the gain $|H(\omega)|$ is an *ideal low-pass filter* with a passband gain of one, a stopband gain of zero, and a bandwidth of β. In Example W1-3 we found that the waveform corresponding to a rectangular transform is the noncausal $h(t)$ shown in Figure W1-13. The noncausal $h(t)$ means that an ideal low-pass filter responds *before* the impulse is applied at $t = 0$. Such anticipatory behavior is physically impossible, which means that we can't actually building an ideal filter using physical hardware. Nevertheless, the ideal low-pass filter model is used in conceptual design work because it is easily implemented in the system simulation software and provides a "gold standard" for evaluating the performance of the real systems.

FIGURE W1-13

Fourier transforms highlight the relationship between impulse response and frequency response. Digital filter design involves defining an impulse response $h(t)$ whose Fourier transform $H(\omega)$ has a desired frequency response. The signal processing action is actually carried out using software that convolves $h(t)$ and $x(t)$ in the time domain. Fast algorithms for performing time-domain convolution are readily available in commercial software. Thus, Fourier methods play a key role in digital signal processing because they provide a computationally efficient way to implement signal filtering.

Example W1-12

Describe the frequency response of a system whose impulse response is

$$h(t) = \frac{\alpha}{2} e^{-\alpha |t|}$$

Solution:

The given $h(t)$ is a two-sided exponential whose transform from Table W1-1 is

$$H(\omega) = \frac{\alpha}{2} \mathcal{F}\{e^{-\alpha|t|}\} = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

Figure W1-14 shows the plot of gain $|H(\omega)|$ versus normalized frequency. The system acts like a low-pass filter with a dc gain of $H(0) = 1$ and the high-frequency gain rolls off α^2/ω^2, or -40 dB/decade, so to speak. The passband cutoff frequency occurs when $|H(\omega_C)| = H(0)/\sqrt{2}$, which requires

$$H(\omega_C) = \frac{\alpha^2}{\alpha^2 + \omega_C^2} = \frac{1}{\sqrt{2}}$$

yielding

$$\omega_C = \pm\alpha\sqrt{\sqrt{2} - 1} = \pm 0.644\alpha$$

FIGURE W1-14

Exercise W1-12

(a) Find the transfer function of a system whose impulse response is $h(t) = \delta(t) - \alpha e^{-\alpha|t|}$.

(b) Describe the frequency response of the system.

Answer:

(a) $H(\omega) = (\omega^2 - \alpha^2)/(\omega^2 + \alpha^2)$.

(b) System has a bandstop characteristic with a notch at $\omega = \pm\alpha$.

W1-6 PARSEVAL'S THEOREM

Parseval's theorem relates the energy carried by a waveform to the amplitude spectrum of its Fourier transform. The total energy carried by any waveform is defined to be

$$W_T = \int_{-\infty}^{+\infty} p(t)\,dt$$

where $p(t)$ is the power the waveform delivers to a specified load. For a resistive load $p(t) = v^2(t)/R = i^2(t)R$ so the energy delivered to a 1-Ω resistive load can be written in the form

$$W_{1\Omega} = \int_{-\infty}^{+\infty} f^2(t)\,dt \qquad \text{(W1-23)}$$

where $f(t)$ can be either a voltage or a current waveform. Equation (W1-23) is often used as the definition of the energy carried by a waveform, although an implied 1-Ω resistance is required for the result to have the dimensions of energy.

Parseval's theorem states that the total energy carried by a signal can be calculated in either the time domain or the frequency domain.

$$W_{1\Omega} = \int_{-\infty}^{+\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \qquad \text{(W1-24)}$$

That is, the total 1-Ω energy can be found from either the waveform $f(t)$ or its transform $F(\omega)$.

Parseval's theorem assumes that the integrals in Eq. (W1-24) converge to a finite value. Signals for which $W_{1\Omega}$ is finite are called **energy signals**. Examples of energy signals are the causal exponential, the two-sided exponential, the rectangular pulse, and the damped sine. Finite energy is a stronger requirement than the absolutely integrable. As a result not all signals that have Fourier transforms are energy signals. For example, the impulse, step, signum, and eternal sinusoid all have Fourier transforms but are not energy signals.

The derivation of Parseval's theorem begins with energy in the time domain. Assuming the Fourier transform of $f(t)$ exists, this energy can be written in the form

$$W_{1\Omega} = \int_{-\infty}^{+\infty} [f(t)][f(t)]dt = \int_{-\infty}^{+\infty} [f(t)] \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega \right]}_{f(t) \,=\, \mathscr{F}^{-1}\{F(\omega)\}} dt$$

The integration within the bracket does not involve time, so $f(t)$ can be moved inside the second integral and the $1/2\pi$ moved outside the first integral to produce

$$W_{1\Omega} = \int_{-\infty}^{+\infty} [f(t)][f(t)]dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) F(\omega) e^{j\omega t} d\omega dt$$

Reversing the order of integration and factoring out $F(\omega)$ produces

$$W_{1\Omega} = \int_{-\infty}^{+\infty} [f(t)][f(t)]dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \underbrace{\left[\int_{-\infty}^{+\infty} f(t) e^{-j(-\omega)t} dt \right]}_{F(-\omega)} d\omega$$

By definition the function inside the bracket is $F(-\omega)$. But $F(-\omega)$ is formed by replacing $j\omega$ by $j(-\omega) = -j\omega$, which means that $F(-\omega) = F(\omega)^*$, the conjugate of $F(\omega)$. Thus, the product

$F(\omega)F(-\omega)$ is the square of the magnitude of $F(\omega)$ and this equation reduces to

$$W_{1\Omega} = \int_{-\infty}^{+\infty} f(t)^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\, d\omega$$

as stated in Parseval's theorem.

Since $|F(\omega)|^2$ is an even function the integral from $-\infty$ to 0 is the same as the integral from 0 to $+\infty$. Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2\, d\omega = \frac{1}{\pi}\int_{0}^{+\infty} |F(\omega)|^2\, d\omega$$

and Parseval's theorem can also be written as

$$W_{1\Omega} = \int_{-\infty}^{+\infty} f(t)^2\, dt = \frac{1}{\pi}\int_{0}^{+\infty} |F(\omega)|^2\, d\omega \qquad \text{(W1-25)}$$

Example W1-13

Derive an expression for the 1-Ω energy in the causal exponential $v(t) = V_A\, e^{-\alpha t}\, u(t)$

(a) in the time domain and (b) in the frequency domain.

Solution:

(a) The given waveform is causal so the time integration extends from $t = 0$ to $t = +\infty$.

$$W_{1\Omega} = \int_0^{+\infty} \left(V_A e^{-\alpha t}\right)^2 dt = \frac{V_A^2}{-2\alpha} e^{-2\alpha t} \Big|_0^{+\infty} = \frac{V_A^2}{2\alpha}$$

This result applies if $\alpha > 0$ since for $\alpha < 0$ the waveform is unbounded and the integral does not converge. In other words, an unbounded exponential is not an energy signal.

(b) The Fourier transform of the signal is $V(\omega) = V_A/(j\omega + \alpha)$ provided $\alpha > 0$. In the frequency domain the 1-Ω energy is found to be

$$W_{1\Omega} = \frac{1}{\pi} \int_0^{+\infty} \frac{V_A^2}{\alpha^2 + \omega^2} d\omega = \frac{V_A^2}{\pi\alpha} \tan^{-1}(\omega/\alpha) \Big|_0^{+\infty}$$

$$= \frac{V_A^2}{\pi\alpha}\left[\frac{\pi}{2} - 0\right] = \frac{V_A^2}{2\alpha}$$

which is the same as the result in (a).

Applications of Parseval's Theorem

The square amplitude spectrum $|F(\omega)|^2$ is called the **energy spectral density** because it describes how the total energy carried by $f(t)$ is distributed among the frequencies in its spectrum. In this sense we think of the integral

$$W_{12} = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega$$

as the amount of energy carried by frequencies in the band $\omega_1 < \omega < \omega_2$. Notice that the energy distribution for a Fourier transform is the fundamentally different than for a Fourier series. With a Fourier transform the energy carried by $f(t)$ is distributed across all of the frequencies between ω_1 and ω_2. With a Fourier series the energy is only located at the discrete harmonic frequencies.

Example W1-14

(a) Find the percentage of the 1-Ω energy in $v(t) = V_A e^{-\alpha t} u(t)$ that is carried in the frequency band $|\omega| < \alpha$.

(b) What frequency band required to carry 90% of the 1-Ω energy?

Solution:

(a) The energy in the band $|\omega| < \alpha$ is

$$W_\alpha = \frac{1}{\pi}\int_0^\alpha \frac{V_A^2}{\alpha^2 + \omega^2} d\omega = \frac{V_A^2}{\pi\alpha}\tan^{-1}(\omega/\alpha)\Big|_0^\alpha$$

$$= \frac{V_A^2}{\pi\alpha}\left(\frac{\pi}{4} - 0\right) = \frac{V_A^2}{4\alpha}$$

In Example W1-13 we found that the total energy is $W_{1\Omega} = (V_A)^2/2\alpha$, hence energy ratio is

$$\frac{W_\alpha}{W_{1\Omega}} = \frac{V_A^2/4\alpha}{V_A^2/2\alpha} = \frac{1}{2}$$

That is, 50% of the total energy is carried in the frequency band $|\omega| < \alpha$.

(b) Let the frequency band $|\omega| < \beta$ carry 90% of the total energy. The energy in this band is

$$W_\beta = \frac{1}{\pi}\int_0^\beta \frac{V_A^2}{\alpha^2 + \omega^2} d\omega = \frac{V_A^2}{\pi\alpha}\tan^{-1}(\omega/\alpha)\Big|_0^\beta$$
$$= \frac{V_A^2}{\pi\alpha}\tan^{-1}(\beta/\alpha) = 0.9\frac{V_A^2}{2\alpha}$$

The equality sign in the last line of this equation requires

$$\tan^{-1}(\beta/\alpha) = 0.9\frac{\pi}{2} \quad \text{or} \quad \beta = \alpha\tan(9\pi/20) = 6.31\alpha$$

Hence , the band $|\omega| < 6.31\alpha$ carries 90% of the total energy .

The energy approach offers a way to quantify the selectivity of filters. Since the output of a filter is $Y(\omega) = H(\omega)X(\omega)$, the energy spectral density of the output signal is

$$|Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2 \tag{W1-26}$$

Combining Eqs. (W1-25) and (W1-26) yields the energy in the output signal as

$$W_{1\Omega} = \frac{1}{\pi}\int_0^{+\infty}|Y(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{+\infty}|H(\omega)|^2|X(\omega)|^2 d\omega \tag{W1-27}$$

We can view filtering as a process that shapes the output energy spectral density by accepting input energy at frequencies in the passband and rejecting input energy at frequencies in the stopband. Note that we calculate the energy processing from the Fourier transforms rather than the signal waveforms.

Example W1-15

The signal $x(t) = 10\,e^{-20t}\,u(t)$ is input to an ideal low-pass filter with $H(\omega) = u(\omega + 100) - u(\omega - 100)$. Find the percentage of the input energy in the output signal.

Solution:

The 1-Ω energy in the input signal is

$$W_{1\Omega,\mathrm{IN}} = \int_0^{+\infty} 100e^{-40t}dt = -\left.\frac{100e^{-40t}}{40}\right|_0^{\infty} = 2.5 \text{ J}$$

The Fourier transform of the input signal is $10/(20 + j\omega)$. The frequency response of the ideal low-pass filter is $|H(\omega)| = 1$ for $|\omega| < 100$ and $|H(\omega)| = 0$ elsewhere. The 1-Ω energy in the output signal is

$$W_{1\Omega,\mathrm{out}} = \frac{1}{\pi}\int_0^{100} 1\times|X(\omega)|^2 d\omega$$

$$= \frac{1}{\pi}\int_0^{100} \frac{10^2}{20^2 + \omega^2} d\omega = \left.\frac{100}{20\pi}\tan^{-1}(\omega/20)\right|_0^{100}$$

$$= \frac{5}{\pi}\tan^{-1}(5) = 2.19 \text{ J}$$

The fraction of the input energy in the output signal is 2.19/2.5, or about 87.6%.

Example W1-16

The transfer function of a first-order low-pass filter is $H(\omega) = 100/(100 + j\omega)$. The input signal to the filter is $x(t) = 10\,e^{-20t}\,u(t)$.

(a) Find the percentage of the input energy in the filter output signal.

(b) Find the percentage of the output energy that lies within the filter passband.

Solution:

(a) The input signal is the same as in Example W1-15 where we found $W_{1\Omega,\mathrm{IN}} = 2.5$ J. The energy in the output signal is

$$W_{1\Omega,\mathrm{OUT}} = \frac{1}{\pi}\int_0^\infty |H(\omega)|^2\,|X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^\infty \frac{100^2}{100^2 + \omega^2}\,\frac{10^2}{20^2 + \omega^2} d\omega$$

The integrand in this expression can be expanded by partial fractions as

$$\frac{10^6}{(100^2 + \omega^2)(20^2 + \omega^2)} = \frac{k_1}{100^2 + \omega^2} + \frac{k_2}{20^2 + \omega^2}$$

The constants in this expansion are

$$k_1 = \left.\frac{10^6}{20^2 + \omega^2}\right|_{\omega^2 = -100^2} = -\frac{625}{6}$$

$$k_2 = \left.\frac{10^6}{10^4 + \omega^2}\right|_{\omega^2 = -20^2} = -k_1$$

The output energy is

$$W_{1\Omega,\mathrm{OUT}} = \frac{625}{6\pi}\left[-\int_0^\infty \frac{d\omega}{100^2 + \omega^2} + \int_0^\infty \frac{d\omega}{20^2 + \omega^2}\right] = \frac{625}{6\pi}\left[\frac{\pi}{40} - \frac{\pi}{200}\right] = 2.08 \text{ J}$$

The fraction of the input energy in the output signal is 2.08/2.5, or about 83.2%.

(b) The cutoff frequency of the low-pass filter is 100 rad/s. The output energy in the passband is

$$W_{1\Omega,\text{OUT}} = \frac{625}{6\pi}\left[-\int_0^{100}\frac{d\omega}{100^2+\omega^2} + \int_0^{100}\frac{d\omega}{20^2+\omega^2}\right]$$

$$= \frac{625}{6\pi}\left[-\frac{\tan^{-1}100/100}{100} + \frac{\tan^{-1}100/20}{20}\right] = 2.02 \text{ J}$$

The fraction of the output energy in the filter pass band is 2.02/2.08, or about 97.1%.

Exercise W1-13

(a) Find the total 1-Ω energy carried a double-sided exponential $f(t) = A\, e^{-\alpha|t|}$.

(b) Find the fraction of the total energy is in the frequencies band $|\omega| < \alpha$.

Answers:

(a) $(A)^2/\alpha$

(b) $0.5 + 1/\pi = 0.818$

Exercise W1-14

Find the total 1-Ω energy carried by $f(t) = 10\, e^{-t} \sin(2t)\, u(t)$.

Answer: 2 J

Exercise W1-15

The current in a 5-kΩ resistor is $i(t) = 12\, e^{-200\,t}\, u(t)$ mA. Find the total energy delivered to the resistor.

Answer: 1.8 mJ

Exercise W1-16

What percentage of the total energy in $f(t) = 5\, e^{-10\,t}\, u(t)$ is in the frequency band $|\omega| > 5$ rad/s.

Answer: 88.1%

SUMMARY

- The Fourier transformation applies to aperiodic waveforms that may be causal or noncausal. The direct transformation is defined by the integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

The inverse transformation is defined by the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

- Sufficient conditions for the existence of $F(\omega)$ are that $f(t)$ be absolutely integrable and have a finite number of discontinuities. Signals not meeting these conditions may still have a Fourier transform. When it exists, a Fourier transform is unique.
- If $f(t)$ is causal and absolutely integrable, then its Fourier transform can be obtained from its

Laplace transform $F(s)$ by replacing s by $j\omega$. A causal waveform $f(t)$ is absolutely integrable if all of the poles of $F(s)$ lie in the left half of the s plane.

- The basic Fourier transform pairs and properties are listed in Tables W1-1 and W1-2.
- Circuit analysis using Fourier transforms involves finding input transform, transforming the circuit into the frequency domain, solving for the unknown response transform, and performing the inverse transform to obtain the response waveform.
- The frequency domain transfer function $H(\omega)$ is the Fourier transform of the system impulse response. The system output transform is $Y(\omega) = H(\omega)X(\omega)$, where $X(\omega)$ is the Fourier transform of the input.
- Parseval's theorem relates the total 1-Ω energy carried by a waveform $f(t)$ to its Fourier transform $F(\omega)$ as

$$W_{1\Omega} = \int_{-\infty}^{+\infty} f^2(t)dt = \frac{1}{\pi}\int_{0}^{+\infty} |F(\omega)|^2 d\omega$$

The 1-Ω energy can be calculated in either the time domain or the frequency domain.

- Parseval's theorem allows us to find the percentage of the total input energy in the output of a filter and to find the percentage of the output energy carried in specified frequency bands.

PROBLEMS

ERO W1-1 Fourier Transforms (Sects. W1-1, W1-2, W1-3)

(a) Find the Fourier transform of a given waveform using transform properties and pairs, or using the integral definition.

(b) Find the inverse Fourier transform of a given function using transform properties and pairs, or using the integral definition.

(c) Derive Fourier transform properties using the defining integrals or other properties.

See Examples W1-1, W1-2, W1-3, W1-4, W1-5, W1-6, W1-7, W1-8 and

Exercises W1-1, W1-2, W1-3, W1-4.

W1-1 Use the defining integral to find the Fourier transform of $f(t) = A[u(t) - u(t-1)]$.

W1-2 Use the defining integral to find the Fourier transform of $f(t) = At[u(t) - u(t-1)]$.

W1-3 Use the defining integral to find the Fourier transform of the following waveform.

$$f(t) = A\cos(\pi t/2)\,[u(t+1) - u(t-1)]$$

W1-4 Use the inversion integral to find the inverse transform of the following function.

$$F(\omega) = 10\pi[u(\omega+1) - u(\omega-1)]$$

W1-5 Use the inversion integral to find the inverse transform of the following function.

$$F(\omega) = j\pi 10[-u(\omega+1) + 2u(\omega) - u(\omega-1)]$$

W1-6 Use the inversion integral to find the inverse transform of the following function.

$$F(\omega) = \cos(\pi\omega/2)[u(\omega+1) - u(\omega-1)]$$

W1-7 Find the inverse transforms of the following functions.

(a) $F_1(\omega) = \dfrac{400}{(j\omega+20)(j\omega+40)}$ (b) $F_2(\omega) = \dfrac{j\omega}{(j\omega+20)(j\omega+40)}$

W1-8 Find the inverse transforms of the following functions.

$$\text{(a) } F_1(\omega) = \frac{400}{j\omega(j\omega + 20)(j\omega + 40)} \qquad \text{(b) } F_2(\omega) = \frac{-\omega^2}{(j\omega + 20)(j\omega + 40)}$$

W1-9 Find the inverse transforms of the following functions.

$$\text{(a) } F_1(\omega) = \frac{5000}{j\omega(-j\omega + 50)(j\omega + 50)} \qquad \text{(b) } F_2(\omega) = \frac{500j\omega}{(-j\omega + 50)(j\omega + 50)}$$

W1-10 Find the Fourier transforms of the following waveforms.

(a) $f_1(t) = 2u(t) - 2$ (b) $f_2(t) = 2\, sgn(t) - 2u(t)$ (c) $f_3(t) = sgn(t) - 1$

W1-11 Find the Fourier transforms of the following waveforms.

(a) $f_1(t) = 2\, e^{-2t}u(t) + 2\,\text{sgn}(t)$ (b) $f_2(t) = 2\, e^{-2t}\, u(t) + 2u(t)$

W1-12 Find the Fourier transforms of the following waveforms.

(a) $f_1(t) = 2\sin(t) + \cos(t)$ (b) $f_2(t) = (2/t)\sin(t) + \cos(t)$

W1-13 Find the Fourier transforms of the following waveforms.

(a) $f_1(t) = 2\, e^{-2t}\cos(4t)\, u(t)$ (b) $f_2(t) = [2e^{-2t} - e^{-4t}]u(t)$

W1-14 Find the Fourier transforms of the following waveforms.

(a) $f_2(t) = 3\cos(2\pi t)$ (b) $f_1(t) = 3\, e^{j2\pi t}$

W1-15 Find the inverse transforms of the following functions.

(a) $F_1(\omega) = 4\pi\delta(\omega) + 4\pi\delta(\omega - 2) + 4\pi\delta(\omega + 2)$

(b) $F_2(\omega) = 4\pi\delta(\omega) - j2/\omega$

(c) $F_3(\omega) = 2\pi\delta(\omega) - j2/\omega$

W1-16 Use the duality property to find the inverse transforms of the following functions.

(a) $F_1(\omega) = 4\cos(2\omega)$ (b) $F_2(\omega) = 4\,u(\omega) - 2$ (c) $F_3(\omega) = 4\,e^{-|2\omega|}$

W1-17 Use the time-shifting property to find the inverse transforms of the following functions.

(a) $F_1(\omega) = [4\pi\delta(\omega) - j2/\omega]\,e^{-j2\omega}$ (b) $F_2(\omega) = 2\,e^{-j2\omega}/(j\omega + 2)$ (c) $F_3(\omega) = 4\cos(2\omega)/j\omega$

W1-18 Show that $f_1(t) = 2\,u(-t)$ and $f_2(t) = 1 - sgn(t)$ have the same Fourier transform. Does this contradict the uniqueness property of Fourier Transforms? Explain.

W1-19 Given that the Fourier transform of $f(t)$ is

$$F(\omega) = \frac{2}{(j\omega + 2)(j\omega + 4)}$$

Use the integration property to find the waveform $g(t) = \int_{-\infty}^{t} f(x)\,dx$

W1-20 Use the reversal property to show that

$$\mathcal{F}\{A e^{-\alpha|t|}\,\mathrm{sgn(t)}\} = \frac{-2Aj\omega}{\omega^2 + \alpha^2}$$

W1-21 Use the frequency shifting property to prove the modulation property.

$$\mathcal{F}\{f(t)\sin(\beta t)\} = \frac{F(\omega - \beta)}{2j} - \frac{F(\omega + \beta)}{2j}$$

W1-22 Use the frequency shifting property to show that

$$\mathcal{F}\{\cos(\beta t)\, u(t)\} = \frac{j\omega}{\beta^2 - \omega^2} + \frac{\pi}{2}[\delta(\omega - \beta) + \delta(\omega + \beta)]$$

Explain why replacing s by $j\omega$ in $\mathcal{L}\{\cos(\beta t)\, u(t)\} = s/(s^2 + \beta^2)$ does not yield this result.

W1-23 Use the integral definition of Fourier transforms to show that

$$\frac{dF(\omega)}{d\omega} = \mathcal{F}\{-jtf(t)\}$$

W1-24 Use the integral definition of Fourier transforms to show that

$$F(0) = \int_{-\infty}^{+\infty} f(t)\, dt$$

W1-25 Use the integral definition of Fourier transforms to derive the time-shifting property

$$\text{If } \mathcal{F}\{f(t)\} = F(\omega) \text{ then } \mathcal{F}\{f(t - T)\} = e^{-j\omega T}F(\omega)$$

ERO W1-2 Circuit Analysis Using Fourier Transforms (Sect. W1-4)

Given a linear circuit and a Fourier transformable input waveform, use Fourier transforms to find response waveforms.

See Examples W1-9, W1-10 and Exercise W1-10.

W1-26 The input in Fig. PW1-26 is $v_1(t) = 10\, sgn(t)$ V. Use Fourier transforms to find $v_2(t)$.

FIGURE PW1-26

W1-27 The input in Fig. PW1-27 is $v_1(t) = 20\, e^{5t}\, u(-t)$ V. Use Fourier transforms to find $v_2(t)$.

FIGURE PW1-27

W1-28 The input in Fig. PW1-27 is $v_1(t) = 20\ \mathrm{sgn}(t)$ V. Use Fourier transforms to find $v_2(t)$.

W1-29 The input in Fig. PW1-29 is $v_1(t) = 10\ u(-t)$ V. Use Fourier transforms to find $v_2(t)$.

FIGURE PW1-29

W1-30 The input in Fig. PW1-29 is $v_1(t) = 20e^{-5|t|}$ V. Use Fourier transforms to find $v_2(t)$.

ERO W1-3 Impulse Response and Convolution (Sect. W1-5)

(a) Given the impulse response of a linear system, find its transfer function and characterize the system's frequency response.

(b) Given the frequency response of a linear system, find its impulse response.

(c) Given the impulse or frequency response of a linear system, find the response for a specified input.

See Examples W1-11, W1-12 and Exercises W1-11, W1-12

W1-31 The impulse response of a linear system is $h(t) = e^{-2t}u(t)$. Find the output for an input $x(t) = u(-t)$.

W1-32 The impulse response of a linear system is $h(t) = e^{-2|t|}$. Find the output for an input $x(t) = u(t)$.

W1-33 The impulse response of a linear system is $h(t) = \delta(t) - 2u(t)e^{-t}$. Find the output for an input $x(t) = \mathrm{sgn}(t)$.

W1-34 The impulse response of a linear system is $h(t) = A\left[\delta(t) - \alpha u(t)e^{-\alpha t}\right]$, with $\alpha > 0$. Make a sketch of $|H(\omega)|$ and describe the system frequency response.

W1-35 The impulse response of a linear system is $h(t) = A\left[\delta(t) - \sin(\beta t)/\pi t\right]$. Make a sketch of $|H(\omega)|$ and describe the system frequency response.

W1-36 The impulse response of a linear system is $h(t) = -Au(t)e^{-\alpha t} + Au(-t)e^{-\alpha(-t)}$, with $\alpha > 0$. Make a sketch of $|H(\omega)|$ and describe the system frequency response.

W1-37 The frequency response of a linear system is shown in Fig. PW1-37. Find the system impulse response $h(t)$.

FIGURE PW1-37

W1-38 The frequency response of a linear system is shown in Fig. PW1-38. Find the system impulse response $h(t)$.

FIGURE PW1-38

W1-39 The impulse response of a linear system is $h(t) = 2\sin(2\beta t)/\pi t$. Find the output $y(t)$ when the input is $x(t) = A\sin(\beta t)/\beta t$

ERO W1-4 Parseval's Theorem (Sect. W1-6)

Given a signal waveform or transform: Find the total 1-Ω energy carried by the signal and the percentage of the total energy in specified frequency bands.

See Examples W1-13, W1-14, W1-15, W1-16 and Exercises W1-13, W1-14, W1-15, W1-16.

W1-40 Find the 1-Ω energy carried by the signal $F(\omega) = 2/(\omega^2 + 1)$.

W1-41 Find the 1-Ω energy carried by the signal $f(t) = \mathrm{A}\, e^{+\alpha t}\, u(-t)$ with $\alpha > 0$.

W1-42 Find the 1-Ω energy carried by the signal $f(t) = A[\sin(\beta t)/(\pi\beta t)]$.

Then find the percentage of the 1-Ω energy carried in the frequency band $|\omega| \leq \beta$.

W1-43 Find the 1-Ω energy carried by the signal

$$F(\omega) = \frac{j\omega A}{\omega^2 + \alpha^2}$$

Then find the percentage of the 1-Ω energy carried in the frequency band $|\omega| \leq \alpha$.

W1-44 The transfer function of an ideal high-pass filter is $H(\omega) = 1$ for $|\omega| \geq 2$ krad/s. The filter input signal is $x(t) = 10\, e^{-2000\,t}\, u(t)$. Find the 1-Ω energy carried by the input signal and . the percentage of the input energy that appears in the output.

W1-45 The impulse response of a filter is $h(t) = 10\, e^{-20\,t}\, u(t)$. Find the 1-Ω energy in the output signal when the input is $x(t) = 20\, e^{-5\,t}\, u(t)$.

W1-46 The current in a 500-Ω resistor is $i(t) = -5\, u(t + 1) + 10\, u(t) - 5\, u(t - 1)$. Find the total energy delivered to the resistor.

W1-47 The transfer function of an ideal bandpass filter is $H(\omega) = 1$ for $1000 \leq |\omega| \leq 1100$ rad/s. Find the 1-Ω energy carried by the output signal when the input is $x(t) = 10\, e^{-500\,t}\, u(t)$. What percentage of the input signal energy appears in the output? Is the filter narrow band or broad band?

INTEGRATING PROBLEMS

W1-48 Bode Plots and Parseval's Theorem **(A)**

The purpose of this problem is to explore using a straight-line approximations to estimate the area under the energy density function. Consider the waveform $v(t) = V_A\, e^{-\alpha t}\, u(t)$ V:

(a) Find the low- and high-frequency asymptotes of the energy density function $|V(\omega)|^2$. Show that these intercepts intersect at $|\omega| = \alpha$. Sketch the asymptotic approximation of $|V(\omega)|^2$.

(b) Use the asymptotic approximation in part (a) to estimate the 1-Ω energy carried by the signal.

(c) Use Parseval's theorem to calculate the actual 1-Ω energy carried by the signal. By how much does the straight-line approximation overestimate the 1-Ω energy?

W1-49 Impulse Generator **(A)**

Theoretically an impulse has an amplitude spectrum that is constant at all frequencies. In practice a constant spectrum across an infinite bandwidth can not be achieved, nor is it really necessary. What is required is an amplitude spectrum that is "essentially" constant over the frequency range of interest. Under this concept an impulse generator is a signal source that produces a pulse waveform whose amplitude spectrum does not vary more than a prescribed amount over a specified frequency range.

Consider a rectangular pulse

$$f(t) = A[u(t + T/2) - u(t - T/2)]$$

Find $F(\omega)$ and sketch its amplitude spectrum $|F(\omega)|$. Select the pulse duration T such that the amplitude spectrum does not change by more than 10% over a frequency range from 1 Mhz to 10 MHz.

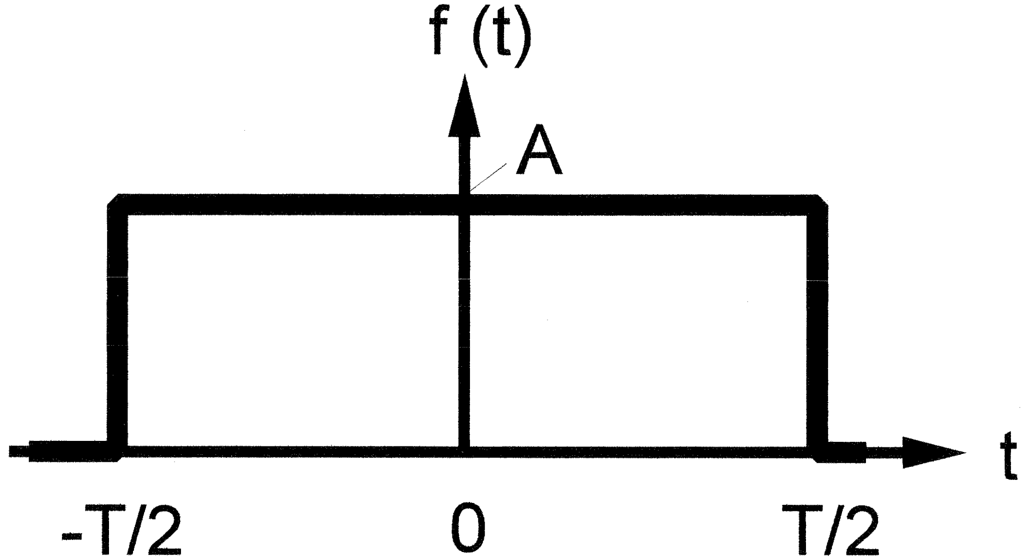


FIGURE W1-1

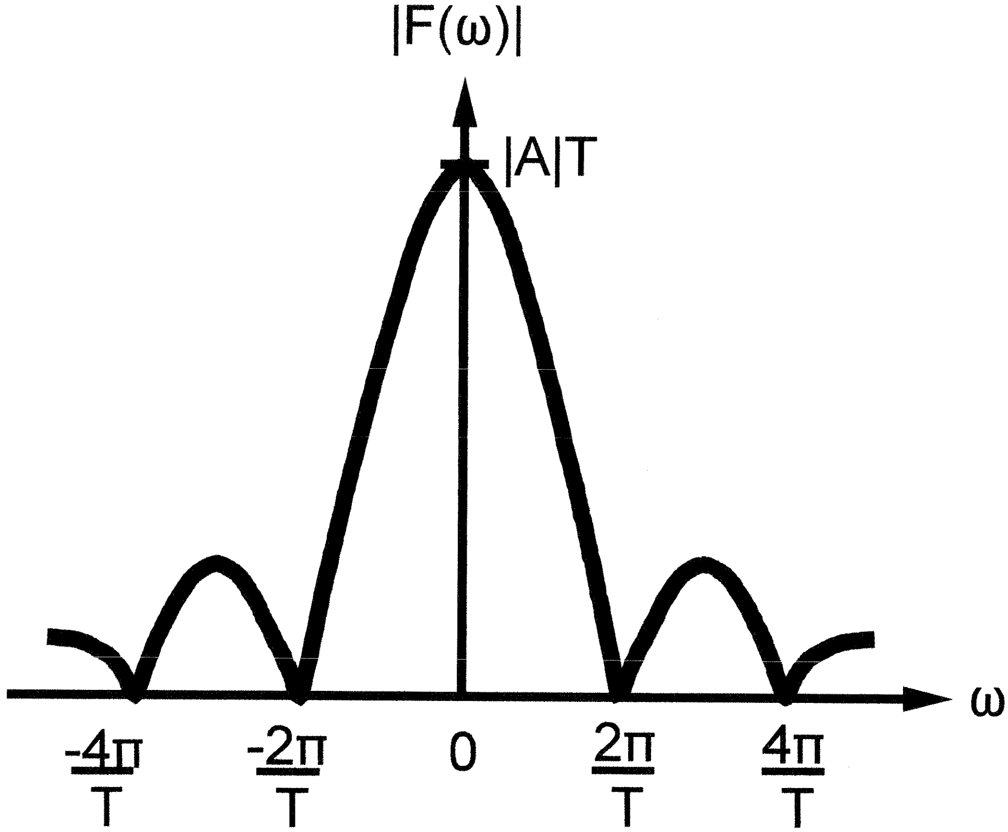


FIGURE W1-2

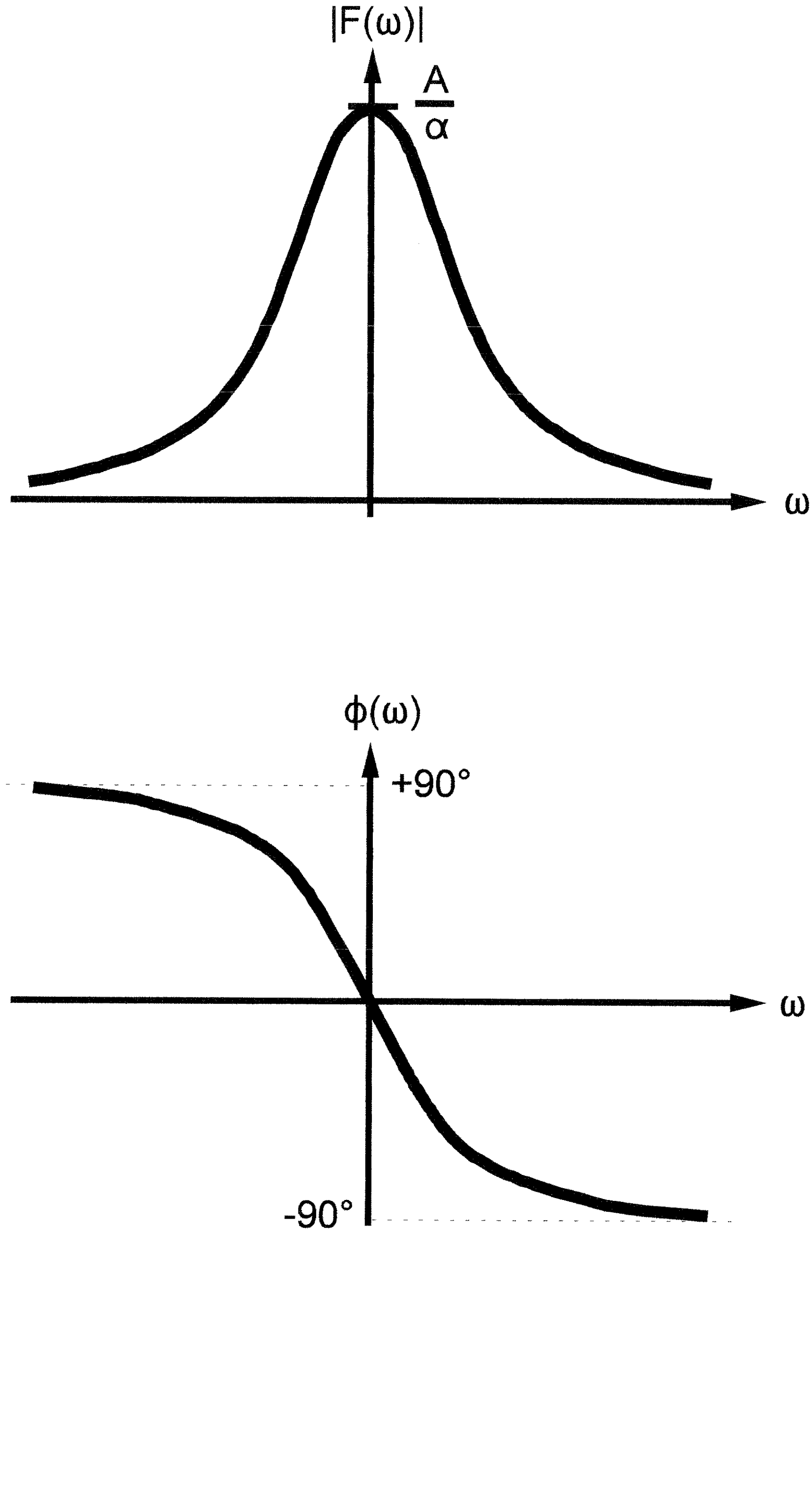


FIGURE W1-3

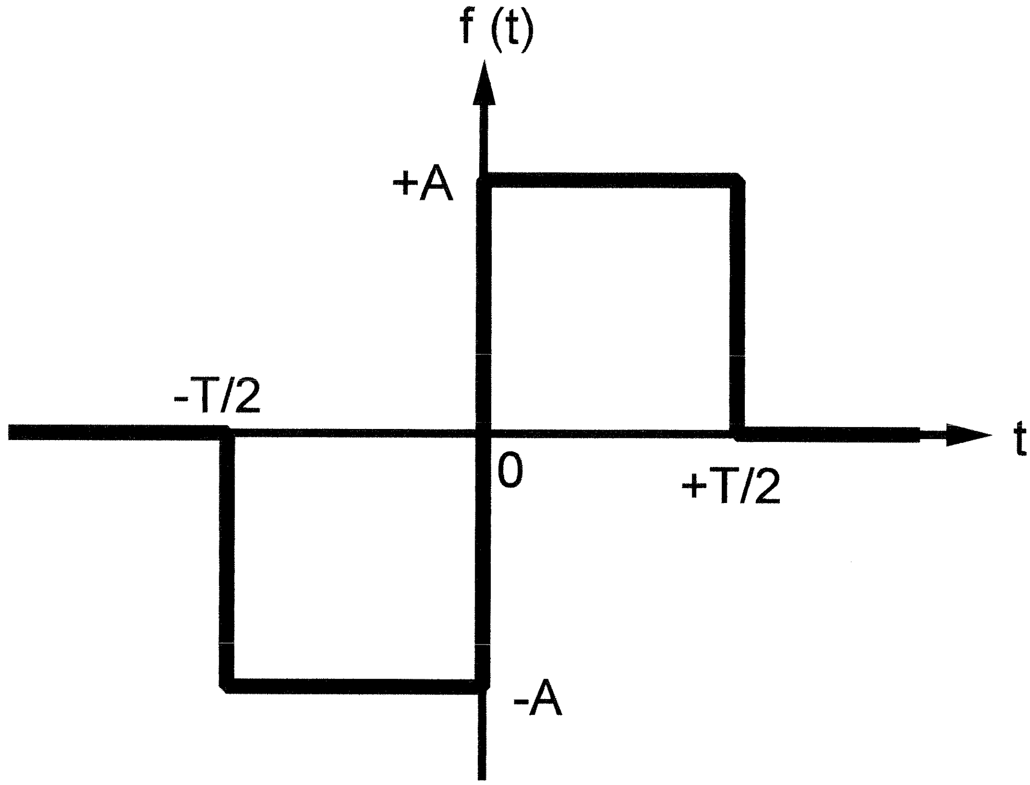


FIGURE W1-4

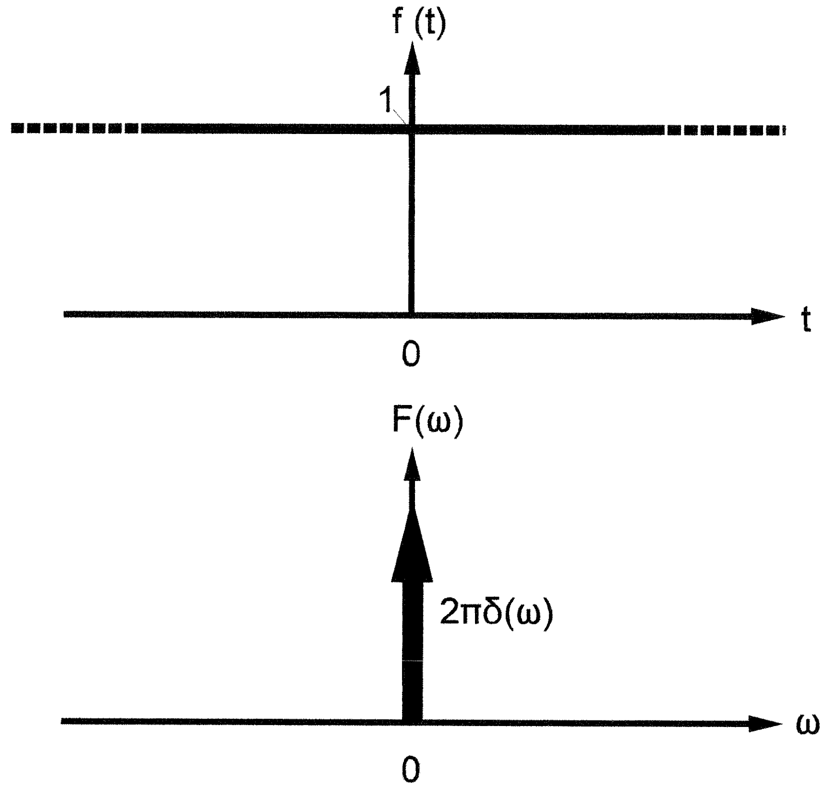


FIGURE W1-5

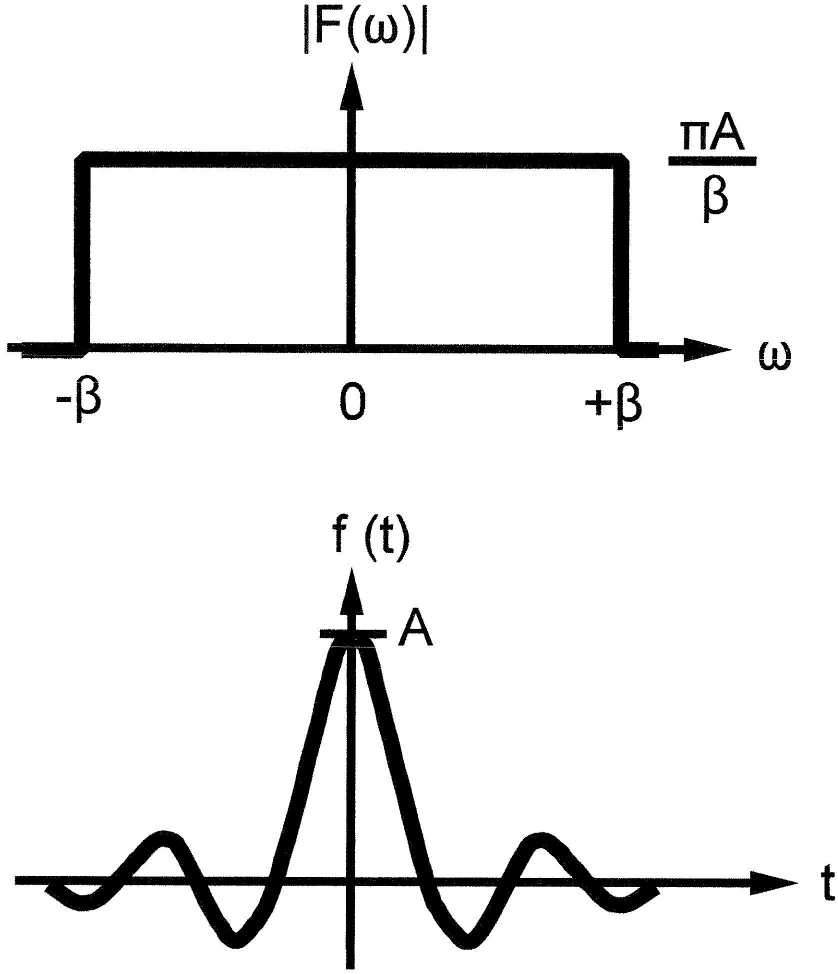


FIGURE W1-6

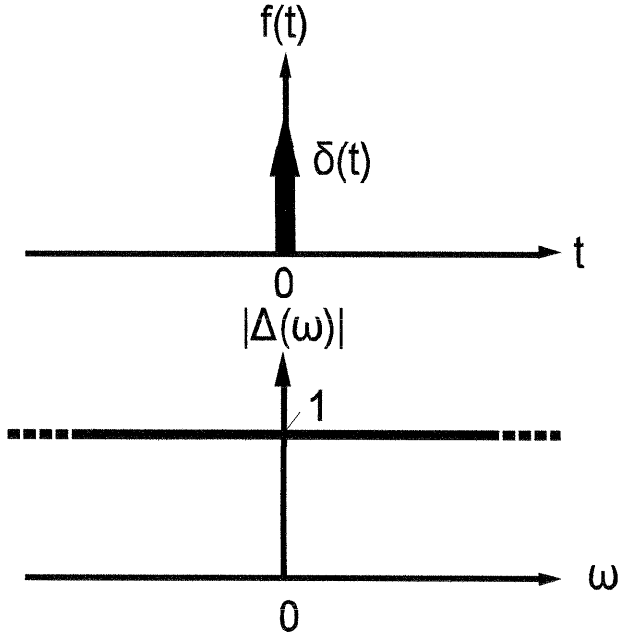


FIGURE W1-7

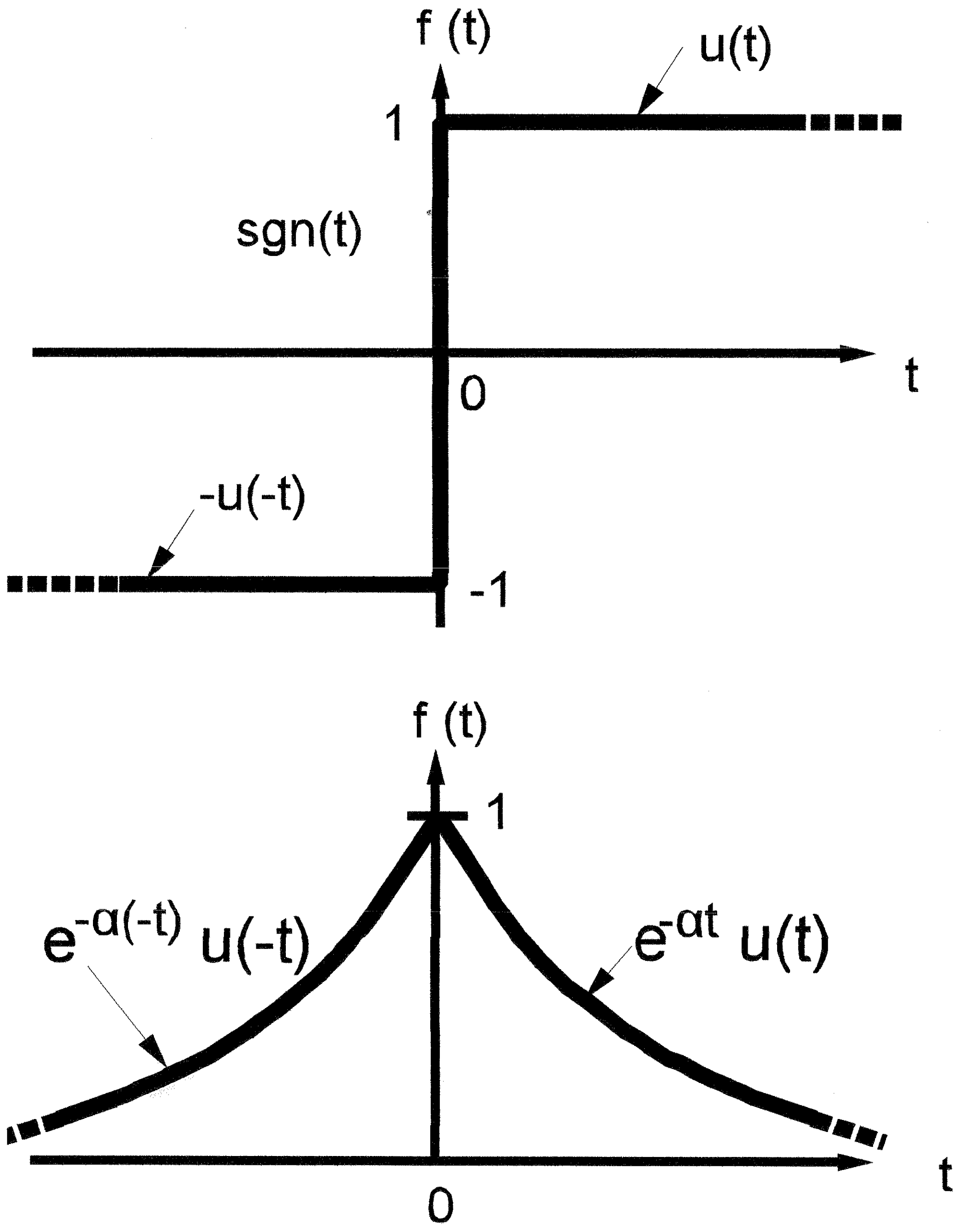
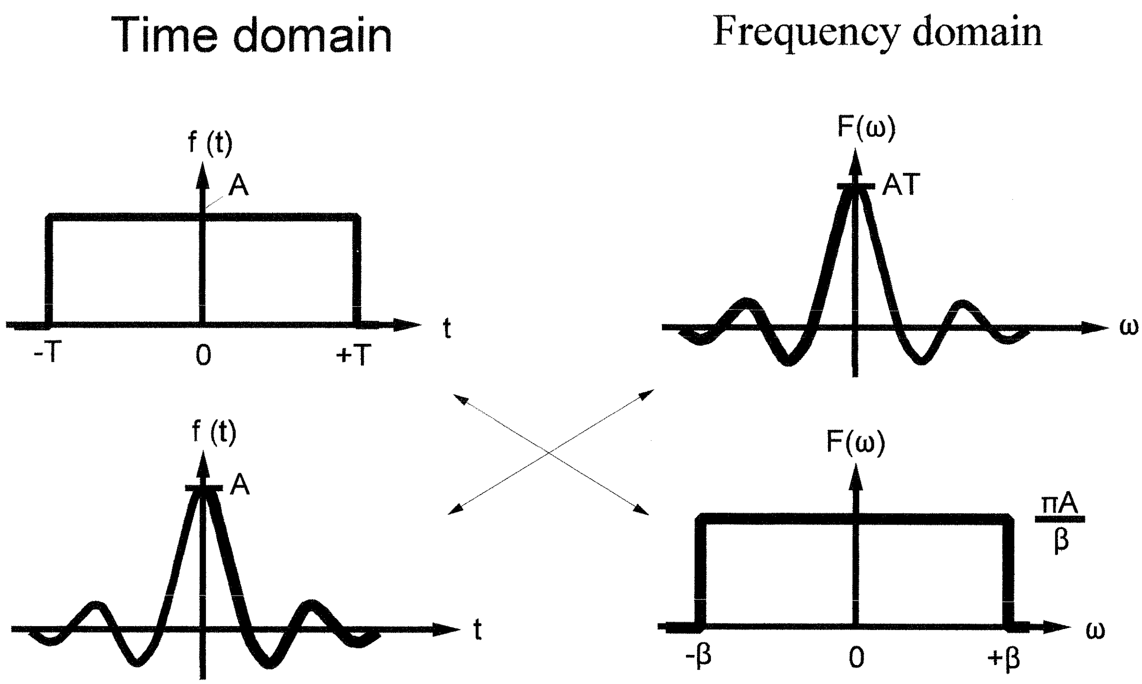


FIGURE W1-8



Time domain
Frequency domain
f (t)
A
t
-T
0
+T
F(ω)
AT
ω
f (t)
A
t
F(ω)
πA
β
ω
-β
0
+β

FIGURE W1-9

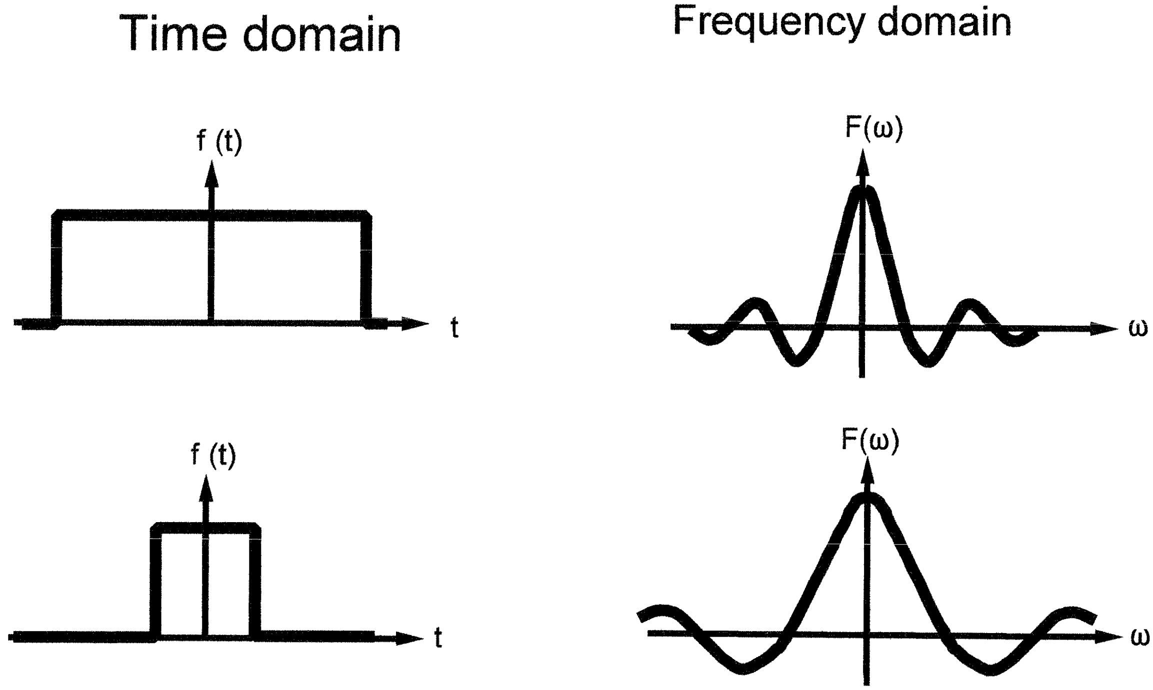


FIGURE W1-10

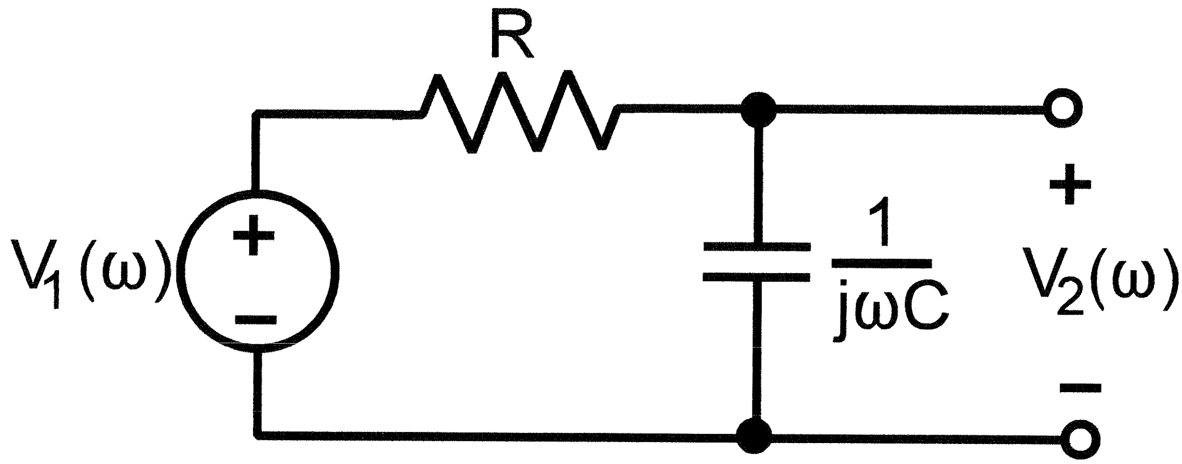


FIGURE W1-11

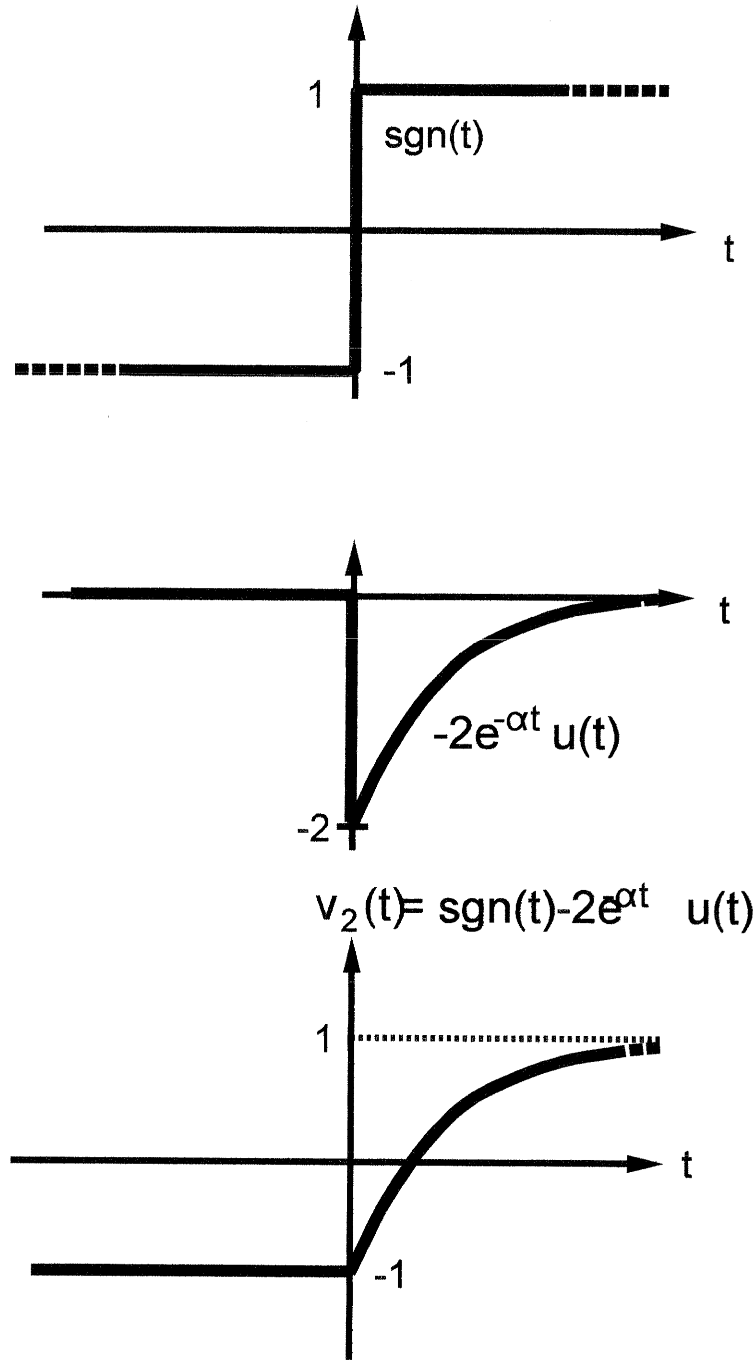


FIGURE W1-12

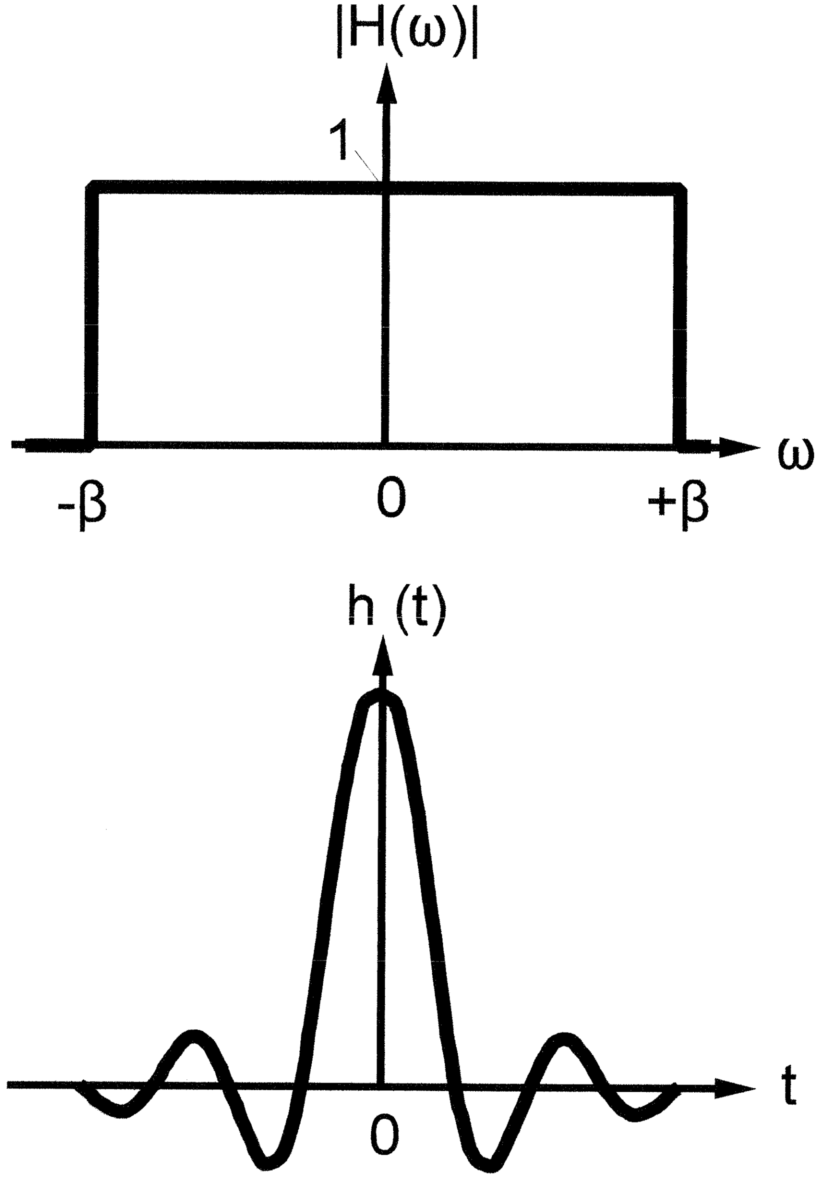


FIGURE W1-13

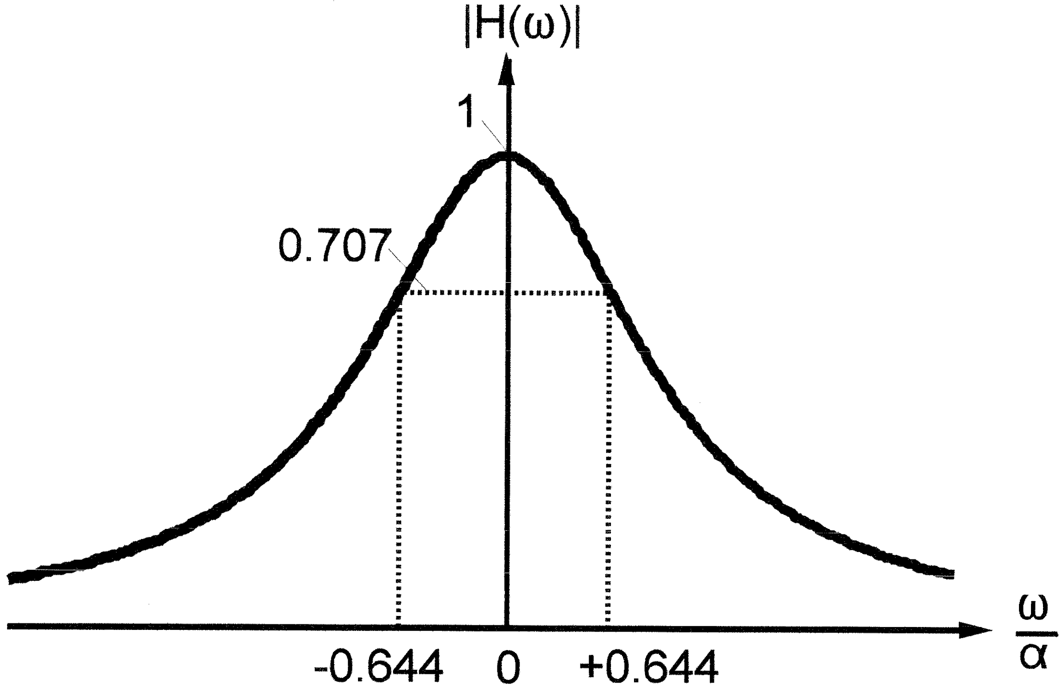


FIGURE W1-14

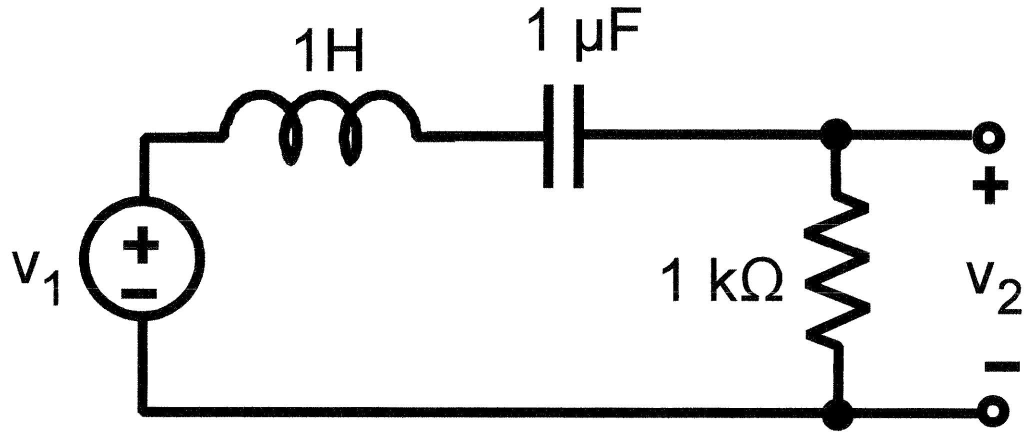


FIGURE PW1-26

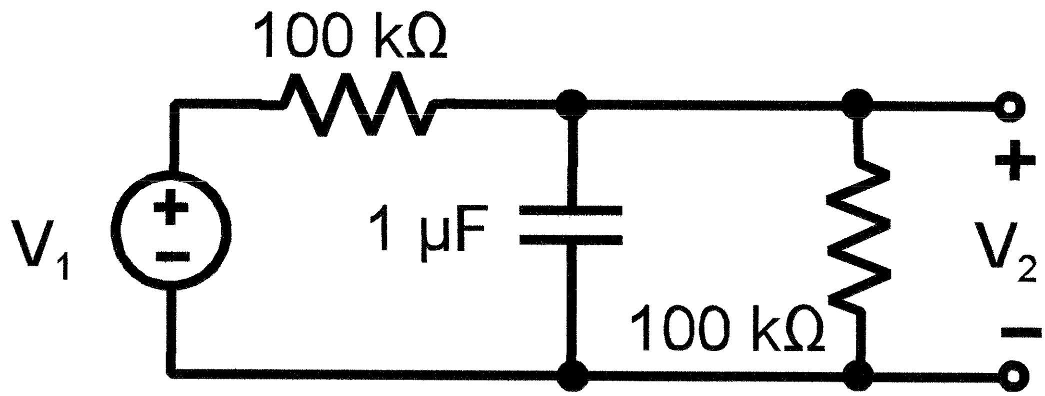


FIGURE PW1-27

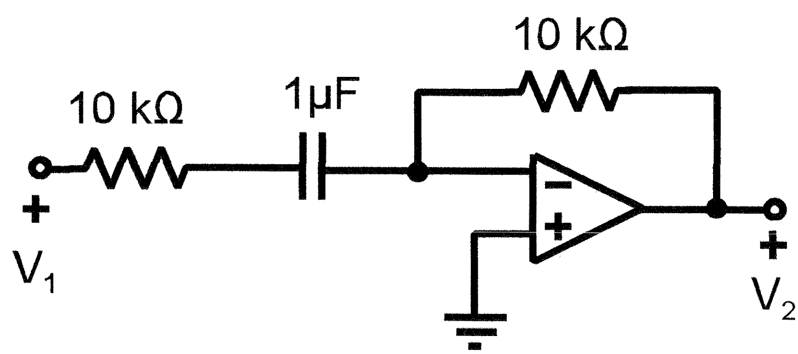


FIGURE PW1-29

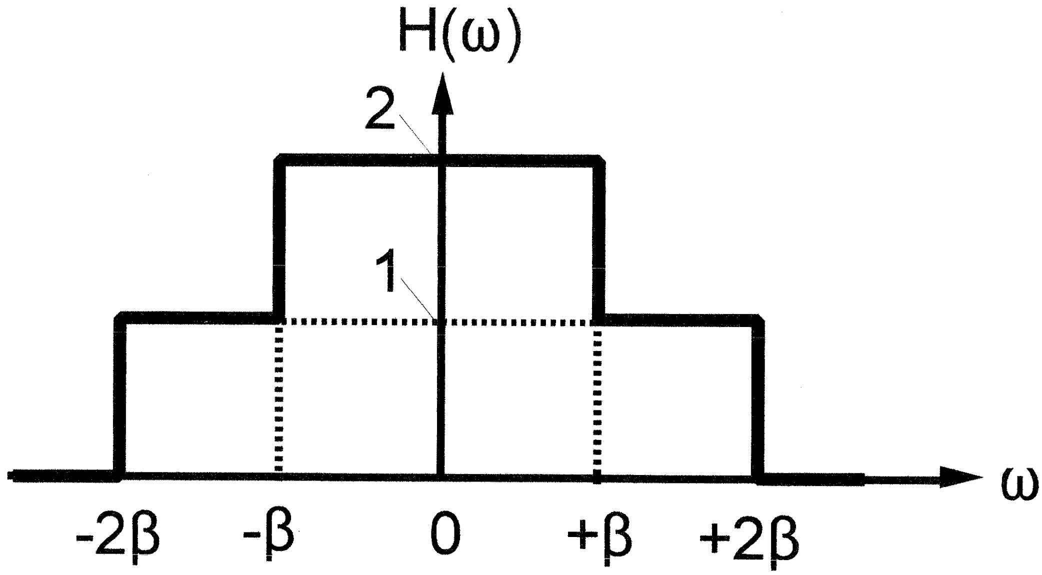


FIGURE PW1-37

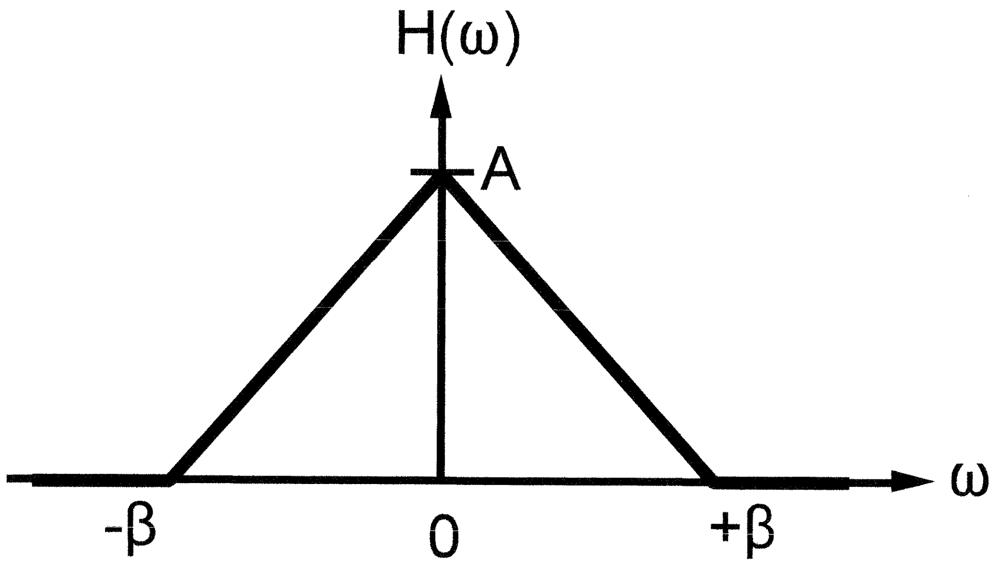


FIGURE PW1-38