A Method for the Determination of the Transfer Function of Electronic Circuits

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Abstract—A general method based on the Laplace expansion for determining the transfer function of a wide variety of linear electronic circuits is discussed. The technique developed requires only the calculation of a number of driving-point resistances to specify the coefficients of the transfer function. Dominant-pole techniques are used and extended, making the procedure useful in both analysis and design. As computation only involves resistance networks, complex arithmetic is not required in determination of the response.

Introduction

DOMINANT-pole techniques [1]–[3] have been used to approximate both the frequency and time-domain responses of linear active systems. Most of these techniques require that the coefficients of the characteristic polynomial be known in order to be applied. This paper describes a method for determining the coefficients of the characteristic polynomial without the need for evaluating the system determinant. In addition, the method allows the circuit designer to relate system performance to specific circuit elements and by means of dominant pole techniques to assess their effect on the circuit.

For convenience, the low-pass case is developed. The results obtained are readily transformed to the high-pass case by duality and frequency translation. The basic approach to the problem is to generate the characteristic polynomial of the form

$$G(s) = \frac{A_0}{1 + a_1 s + a_2 s^2 + \cdots a_n s^n}. \qquad (1)$$

The nth-degree polynomial is considered to arise from a system containing n storage elements. By use of the Laplace expansion of a determinant [4], the coefficients are generated. The calculations involved require only that driving-point functions of purely resistive networks be determined. For a wide variety of electronic circuits, the method significantly reduces the algebra required compared with that required for the evaluation of a determinant.

The n-Capacitance System

Consider an n-port system, shown in Fig. 1, with n capacitances $C_1, C_2, \cdots, C_n$ across ports $1, 2, \cdots, n$, respectively. The circle encloses a linear active network with no energy storage elements. The entire network

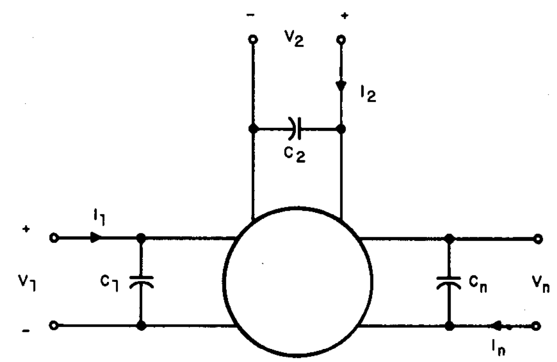

Fig. 1. n-port with n capacitances.

can be represented by a set of node equations with the admittance determinant Δ given by (2), where the C's appear only on the principal diagonal:

$$\Delta = \begin{bmatrix} g_{11} + sC_1 & g_{12} & g_{13} & \cdots & g_{1n} \\ g_{21} & g_{22} + sC_2 & g_{23} & \cdots & g_{2n} \\ g_{31} & g_{32} & g_{33} + sC_3 & \cdots & g_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} + sC_n \end{bmatrix}. \qquad (2)$$

The natural frequencies are determined by $\Delta(s) = \sum_0^n b_i s^i = 0$. In order that $\Delta(s)$ be conveniently compared to the denominator of (1), $\Delta(s)/b_0$ is formed as

$$\frac{\Delta}{b_0} = 1 + \sum_1^n a_i s^i = 1 + a_1 s + a_2 s^2 + \cdots a_n s^n. \qquad (3)$$

To derive the general result it is convenient to define the following:

Δ	determinant, with none of the C's $=0$;
Δ^1	$=\Delta$, when only $C_1 = 0$;
Δ^{12}	$=\Delta$, when only C_1 and $C_2 = 0$;
$\Delta^{12\cdots(n-1)}$	$=\Delta$, when all C's $=0$, except C_n;
$\Delta^{12\cdots n}$	$\equiv \Delta^0 = \Delta$, when all C's $=0$.

The same notation will be followed for all cofactors. Single subscripts will be used for the cofactors, as they are all based on deletion of the same row and column number:

$\Delta_1{}^1$	determinant, when row 1 and column 1 are eliminated with $C_1 = 0$;
$\Delta_{13}{}^{12}$	determinant, when rows and columns 1 and 3 are eliminated with $C_1 = C_2 = 0$.

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Without loss of generality, the results are derived for $n=4$ and are extended by induction. The procedure is based on the Laplace expansion of $\Delta(s)$ and is as follows:

$$\Delta = \Delta^1 + sC_1\Delta_1{}^1 \quad (4)$$

$$\Delta^1 = \Delta^{12} + sC_2\Delta_2{}^{12} \quad (5)$$

$$\Delta_1{}^1 = \Delta_1{}^{12} + sC_2\Delta_{12}{}^{12}. \quad (6)$$

Combination of (12)–(14) yields

$$\Delta^1 = \Delta^{12} + sC_2\Delta_2{}^{12} + sC_1\Delta_1{}^{12} + s^2C_1C_2\Delta_{12}{}^{12}. \quad (7)$$

Continuation in this manner results in

$$\begin{aligned}\Delta = {} & s^4C_1C_2C_3C_4\Delta_{1234}{}^0 + s^3\{C_1C_2C_3\Delta_{123}{}^0 \\ & + C_1C_2C_4\Delta_{124}{}^0 + C_2C_3C_4\Delta_{234}{}^0\} \\ & + s^2\{C_1C_2\Delta_{12}{}^0 + C_1C_3\Delta_{13}{}^0 + C_1C_4\Delta_{14}{}^0 \\ & + C_2C_3\Delta_{23}{}^0 + C_2C_4\Delta_{24}{}^0 + C_3C_4\Delta_{34}{}^0\} \\ & + s\{C_1\Delta_1{}^0 + C_2\Delta_2{}^0 + C_3\Delta_3{}^0 + C_4\Delta_4{}^0\} + \Delta^0 \quad (8) \\ = {} & b_4s^4 + b_3s^3 + b_2s^2 + b_1s + \Delta^0. \quad (9)\end{aligned}$$

Division of (9) by Δ^0 gives the characteristic polynomial in the form $a_4s^4+a_3s^3+a_2s^2+a_1s+1$, from which the a_1 coefficient is

$$a_1 = \frac{b_1}{\Delta^0} = \frac{C_1\Delta_1{}^0 + C_2\Delta_2{}^0 + C_3\Delta_3{}^0 + C_4\Delta_4{}^0}{\Delta^0} \quad (10)$$

$$= \sum_{i=1}^{4} \frac{C_i\Delta_i{}^0}{\Delta^0}. \quad (11)$$

$\Delta_i{}^0/\Delta^0$ is merely the resistance seen at the ith port with all the C's$=0$. Define this resistance as

$$R_{ii}{}^0 \equiv \frac{\Delta_i{}^0}{\Delta^0} \quad (12)$$

which allows a_1 to be written as

$$a_1 = R_{11}{}^0C_1 + R_{22}{}^0C_2 + R_{33}{}^0C_3 + R_{44}{}^0C_4. \quad (13)$$

It can be seen that for any value of n,

$$a_1 = \sum_{i=1}^{n} R_{ii}{}^0C_i. \quad (14)$$

The a_2 coefficient is given as

$$a_2 = \frac{b_2}{\Delta^0} = \frac{C_1C_2\Delta_{12}{}^0 + C_1C_3\Delta_{13}{}^0 + \cdots + C_3C_4\Delta_{34}{}^0}{\Delta^0} \quad (15)$$

$$= \sum_{i=1}^{3}\sum_{j=i+1}^{4} C_iC_j \frac{\Delta_{ij}{}^0}{\Delta^0}. \quad (16)$$

Evaluation of a_2 in a more useful form is possible by noting that

$$\frac{\Delta_{ij}{}^0}{\Delta^0} = \frac{\Delta_i{}^0}{\Delta^0} \times \frac{\Delta_{ij}{}^0}{\Delta_i{}^0} = R_{ii}{}^0 \times \frac{\Delta_{ij}{}^0}{\Delta_i{}^0}. \quad (17)$$

$\Delta_j{}^0/\Delta^0$ can be expanded as

$$\frac{\Delta_j{}^0}{\Delta^0} = \frac{\Delta_j{}^0|_{g_i=0} + g_i\Delta_{ji}{}^0}{\Delta^0|_{g_i=0} + g_i\Delta_i{}^0}. \quad (18)$$

As g_i approaches infinity, the right-hand side of (18) approaches $\Delta_{ji}{}^0/\Delta_i{}^0$, but $\Delta_{ji}{}^0=\Delta_{ij}{}^0$ and $\Delta_j{}^0/\Delta^0=R_{jj}{}^0$ so that (17) becomes

$$\frac{\Delta_{ij}{}^0}{\Delta^0} = R_{ii}{}^0 \times R_{jj}{}^0|_{g_i=\infty}. \quad (19)$$

The second factor of (19) is merely the resistance seen at the jth port when the ith port is shorted, or $C_i=\infty$ with all other C's$=0$. For convenience we shall define $\Delta_{ij}{}^0/\Delta_i{}^0$ as

$$\frac{\Delta_{ij}{}^0}{\Delta^0} = R_{jj}{}^0|_{g_i=\infty} = R_{jj}{}^i \quad (20)$$

$$\begin{aligned}a_2 = {} & R_{11}{}^0R_{22}{}^1C_1C_2 + R_{11}{}^0R_{33}{}^1C_1C_3 + R_{11}{}^0R_{44}{}^1C_1C_4 \\ & + R_{22}{}^0R_{33}{}^2C_2C_3 + R_{22}{}^0R_{44}{}^2C_2C_4 \\ & + R_{33}{}^0R_{44}{}^3C_3C_4. \quad (21)\end{aligned}$$

It can be seen that for any value of n,

$$a_2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} C_iC_jR_{ii}{}^0R_{jj}{}^i. \quad (22)$$

The coefficient a_3 is obtained as

$$a_3 = \frac{b_3}{\Delta^0} = \sum_{i=1}^{2}\sum_{j=i+1}^{2}\sum_{k=j+1}^{4} C_iC_jC_k \frac{\Delta_{ijk}{}^0}{\Delta^0}. \quad (23)$$

Since

$$\frac{\Delta_{ijk}{}^0}{\Delta^0} = \frac{\Delta_i{}^0}{\Delta^0} \times \frac{\Delta_{ij}{}^0}{\Delta_i{}^0} \times \frac{\Delta_{ijk}}{\Delta_{ij}{}^0} = R_{ii}{}^0R_{jj}{}^i \times \frac{\Delta_{ijk}{}^0}{\Delta_{ij}{}^0} \quad (24)$$

it follows that the procedure used for evaluating $\Delta_{ij}{}^0/\Delta_i{}^0$ in (18)–(20) can be used to evaluate $\Delta_{ijk}{}^0/\Delta_{ij}{}^0$. That is,

$$\frac{\Delta_{ijk}{}^0}{\Delta_{ij}{}^0} = R_{kk}{}^0\Big|_{g_i=g_j=\infty} = R_{kk}{}^{ij} \quad (25)$$

which gives a_3 as

$$\begin{aligned}a_3 = {} & C_1C_2C_3R_{11}{}^0R_{22}{}^1R_{33}{}^{12} + C_1C_2C_4R_{11}{}^0R_{22}{}^1R_{44}{}^{12} \\ & + C_1C_3C_4R_{11}{}^0R_{33}{}^1R_{44}{}^{13} + C_2C_3C_4R_{22}{}^0R_{33}{}^2R_{44}{}^{23}. \quad (26)\end{aligned}$$

Thus for any value of n,

$$a_3 = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} C_iC_jC_kR_{ii}{}^0R_{jj}{}^iR_{kk}{}^{ij}. \quad (27)$$

The coefficient a_4 is obtained as

$$a_4 = \frac{b_4}{\Delta^0} = C_1C_2C_3C_4\frac{\Delta_{1234}{}^0}{\Delta^0} = C_iC_jC_kC_l\frac{\Delta_{ijkl}{}^0}{\Delta_{ijk}{}^0} \quad (28)$$

but

$$\frac{\Delta_{ijkl}{}^0}{\Delta^0} = R_{ii}{}^0R_{jj}{}^iR_{kk}{}^{ij}R_l{}^{ijk} \quad (29)$$

which gives a_4 as

$$a_4 = C_1C_2C_3C_4R_{11}{}^0R_{22}{}^1R_{33}{}^{12}R_{44}{}^{123}. \quad (30)$$

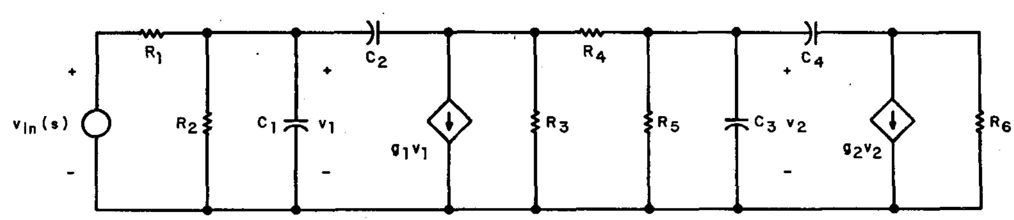

Fig. 2. Equivalent circuit a for two-stage transistor amplifier. $R_1=R_2=R_6=1$ kΩ; $R_3=20$ kΩ; $R_4=200$; $R_5=800$; $C_1=C_3=200$ pF; $C_2=C_4=5$ pF; $g_1=g_2=0.021$.

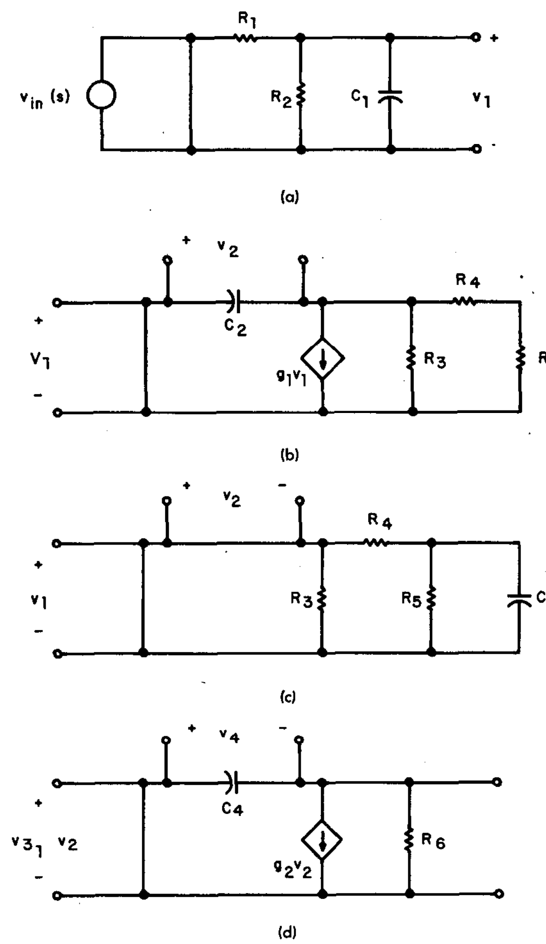

Fig. 3. Circuits for finding the four resistances for a_4. (a) Finding R_{11}^0. (b) Finding R_{22}^1. (c) Finding R_{33}^{12}. (d) Finding R_{44}^{123}.

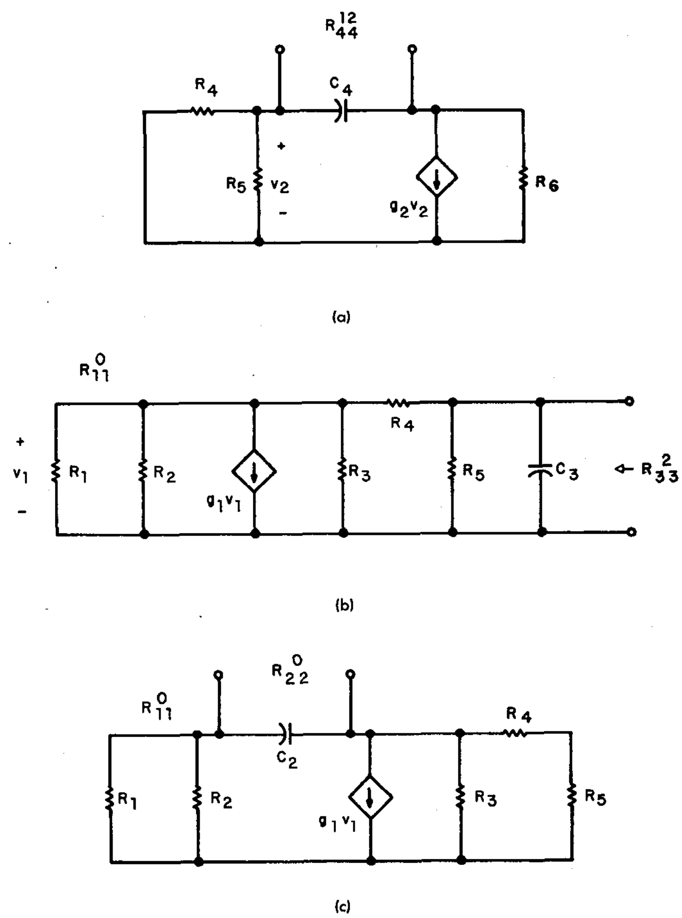

Fig. 4. Circuits for finding three resistances for a_3 in Fig. 2. (a) Finding R_{44}^{12}. (b) Finding R_{33}^2. (c) Finding R_{22}^0.

Thus for any value of n,

$$a_4 = \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} C_iC_jC_kC_lR_{ii}^0R_{jj}^iR_{kk}^{ij}R_{ll}^{ijk}. \quad (31)$$

These results can be extended to find a_k, the coefficient of s^k where $k \leq n$, which becomes

$$a_k = \sum_{i=1}^{n-(k-1)} \sum_{j=i+1}^{n-(k-2)} \cdots \sum^{n} C_iC_j \cdots \cdot C_nR_{ii}^0R_{jj}^i \cdots R_{nn}^{ij} \cdots (n-1). \quad (32)$$

The results expressed in (14), (22), (27), (29), and (32) indicate that simple resistance calculations suffice in determining the coefficients of the transfer function. The example in the next section illustrates the technique.

Example

Consider the two-stage amplifier at high frequencies as shown in Fig. 2. The capacitors C_1–C_4 correspond to the port capacitors in Fig. 1, for $n=4$.

To find the driving-point resistances required to determine a_4, the simplified circuits of Fig. 3 may be used. The results are given as follows:

$$R_{11}^0 = \frac{R_1R_2}{R_1+R_2}$$

$$R_{22}^1 = \frac{R_3(R_4+R_5)}{R_3+R_4+R_5}$$

$$R_{33}^{12} = \frac{R_4R_5}{R_4+R_5}$$

$$R_{44}^{123} = R_6. \quad (33)$$

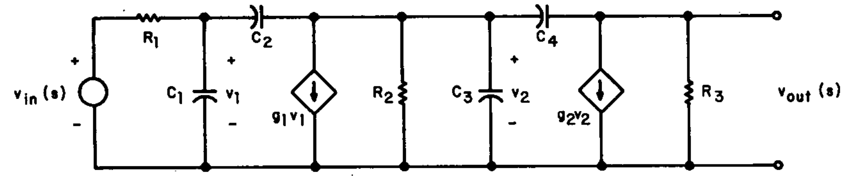

Fig. 5. Equivalent circuit of an amplifier with a capacitor loop made up of C_1, C_2, and C_3.

Substitution of the component values and the use of (33) in conjunction with (30) gives

$$a_4 = \frac{C_1C_2C_3C_4R_1R_2R_3R_4R_5}{(R_1 + R_2)(R_3 + R_4 + R_5)} = 7.6 \times 10^{-32}. \quad (34)$$

Inspection of (26) indicates many of the resistance values are known from a_4. The remaining values may be found by inspection of the circuit that yields

$$R_{33}^{1} = R_5(R_3 + R_4)/(R_3 + R_4 + R_5)$$
$$R_{44}^{13} = R_{44}^{23} = R_6. \quad (35)$$

The remaining resistances R_{44}^{12}, R_{33}^{2}, and R_{22}^{0} can be found using the circuits of Fig. 4.

Fig. 4(a) is the result of taking $C_1 = C_2 = \infty$, but $C_3 = C_4 = 0$. R_{44}^{12} is determined as

$$R_{44}^{12} = \frac{R_4R_5}{R_4 + R_5}(1 + g_2R_6) + R_6 = R_{33}^{12}(1 + g_2R_6) + R_6. \quad (36)$$

Similarly,

$$R_{33}^{2} = R_5 \| R_4 + R_3 \| \frac{R_{11}^{0}}{R_{11}^{0}g_1 + 1}$$
$$R_{22}^{0} = R_{11}^{0}\left(1 + g_1\frac{R_3(R_4 + R_5)}{R_3 + R_4 + R_5}\right) + \frac{R_3(R_4 + R_5)}{R_3 + R_4 + R_5} = R_{11}^{0}(1 + g_1R_2^{1}) + R_{22}^{1}. \quad (37)$$

By use of numerical values of the circuit parameters and the above results, a_3 becomes $a_3 = 1.14 \times 10^{-22}$.

By following the same processes, the coefficients a_2 and a_1 can be found readily as

$$a_2 = 2.91 \times 10^{-14}$$
$$a_1 = 4.04 \times 10^{-7}.$$

A computer program for this circuit gives the same values for these coefficients, accurate to four decimal places; however, uncertainty about the transistor parameters hardly merits such accuracy. The approximate pole positions are obtained as

$$-p_1 \simeq 1/a_1 = 2.46 \times 10^{6}$$
$$-p_2 \simeq a_1/a_2 = 1.39 \times 10^{7}$$
$$-p_3 \simeq a_2/a_3 = 2.9 \times 10^{8}$$
$$-p_4 \simeq a_3/a_4 = 1.5 \times 10^{9}.$$

The same computer program gives the poles as

$$-p_1 = 3.258 \times 10^{6}$$
$$p_2 = 1.1127 \times 10^{7}$$
$$p_3 = 3.1018 \times 10^{8}$$
$$p_4 = 1.669 \times 10^{9}.$$

The nearly 25-percent deviation between computed values and those using the dominant-pole analysis is expected as the two poles nearest the origin are separated by only a factor of three. However, the use of the technique described permits identification of the RC product closest to the origin and the degree of interaction between the poles.

A Special Case

The previous example was a perfectly general circuit configuration for which the number of coefficients was equal to the number of C's. A special situation arises when there are capacitor loops that occur naturally in some amplifier configurations. Insight gained from the general derivation makes it possible to handle this special case. If C_1, C_2, and C_3 constitute a loop, it can be seen that letting any two of the C's $= \infty$ results in the third capacitance also being shorted. Resistances of the form R_{11}^{23}, R_{22}^{13}, and R_{33}^{12} will be zero wherever they occur. The result is that $a_n = 0$ or the order of the polynomial is reduced. If there is only one loop, the a_{n-1} term will exist, although it may be modified somewhat by the single loop. Two separate loops would result in the a_{n-1} term also going to zero.

An example of a circuit with a capacitor loop is given in Fig. 5. Fig. 5 is the equivalent high-frequency circuit for a two-stage J-FET amplifier. Application of (30) to the circuit of Fig. 5 results in

$$R_{11}^{0} = R_1 \quad R_{22}^{1} = R_2 \quad R_{44}^{123} = R_3 \quad R_{33}^{12} = 0.$$

When $C_1 = C_2 = \infty$, the resistance shunting C_3 is evidently zero. Thus $a_4 = 0$ and the a_3 term will be only

$$a_3 = C_1C_2C_4R_{11}^{0}R_{22}^{1}R_{44}^{12} + C_1C_3C_4R_{11}^{0}R_{33}^{1}R_{44}^{13} + C_2C_3C_4R_{22}^{0}R_{33}^{2}R_{44}^{23} \quad (38)$$

where the $C_1C_2C_3$ combination is missing because of $R_{33}^{12} = 0$. The a_2 and a_1 terms will be unchanged since they do not involve the shorting of two C's.

Low-Pass R–L Circuits

Extension of this procedure to handle R–L networks is relatively straightforward. By duality, replacing C

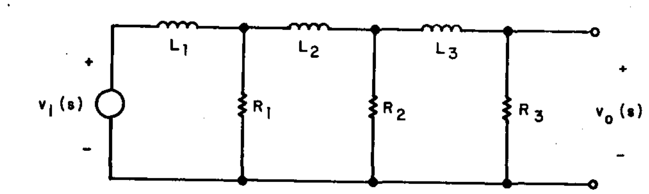

Fig. 6. Network containing three inductors.

by L, the principal diagonal of the mesh matrix contains terms of the form $r_{ii}+sL_i$. By use of the Laplace expansion, the circuit determinant is of the form

$$\Delta(s) = \Delta^0 + s\sum_{i=1}^{n} L_i\Delta_i + \cdots \prod_{i=1}^{n} L_i. \tag{39}$$

Thus (39) is analogous to (8), so that division by Δ^0 permits the identification of appropriate L/R time constants. The interpretation given Δ^0 in (39) is that of the resistive determinant obtained when all $L_i \equiv 0$. By identification with the form of (32), the kth coefficient of Δ/Δ^0 in (39) may be written as

$$a_k = \sum_{i=1}^{n-(k-1)} \sum_{j=i+1}^{n-(k-2)} \cdots \cdot \sum^{n} \frac{L_iL_j \cdots L_n}{R_{ii}{}^{j\cdots n-1}R_{jj}{}^{k\cdots n-1} \cdots R_{nn}{}^0}. \tag{40}$$

The resistances in the denominator of (40) are easily identified as follows.

1) $R_{ii}{}^0$ is the resistance in series with L_i with all other $L_j \equiv 0$.

2) $R_{ii}{}^j$ is the resistance in series with L_1 when $L_j = \infty$ and all other $L_k \equiv 0$.

3) $R_{ii}{}^{jk\cdots q}$ is the resistance in series with L_i when $L_j, L_k, \cdots, L_q \equiv \infty$ and all other $L_M \equiv 0$.

Therefore, a_1 is interpreted as the sum of short-circuit time constants; a_j is the sum of products of one short-circuit time constant with one open-circuit time constant. The interpretation of a_i follows directly. An alternate interpretation of the short- and open-circuit values of L and C is the zero-frequency infinite-frequency calculation described earlier. Consider the network depicted in Fig. 6. This circuit results in a cubic as there are three inductors. By the use of the procedure outlined, the a_1 coefficient is

$$a_1 = \frac{L_1}{R_1\|R_2\|R_3} + \frac{L_2}{R_3\|R_2} + \frac{L_3}{R_3}. \tag{41}$$

The first term is found by taking $L_2 = L_3 = 0$, which parallels all three resistances. When $L_1 = L_3 = 0$, the resistance shunting L_2 is just the parallel combination of R_2 and R_3. With $L_1 = L_2 = 0$, the only resistance shunting L_3 is R_3. The a_2 coefficient will be

$$a_2 = \frac{L_1L_2}{R_{11}{}^2R_{22}{}^0} + \frac{L_1L_3}{R_{11}{}^3R_{33}{}^0} + \frac{L_2L_3}{R_{22}{}^0R_{33}{}^0}. \tag{42}$$

The resistance $R_{11}{}^2$ is the resistance shunting L_1 when $L_2 = \infty$ and $L_3 = 0$, which is just R_1. Thus $R_{11}{}^2R_{22}{}^0 = R_1(R_2\|R_3)$. The resistance shunting L_1 when $L_3 = \infty$ and $L_3 = 0$ is $R_{11}{}^3 = (R_1\|R_2)$, and the resistance shunting L_3 when $L_1 = L_2 = 0$ is $R_{33}{}^0 = R_3$. The resistance shunting L_2 when $L_3 = \infty$ and $L_1 = 0$ is $R_{22}{}^3 = R_2$. Thus in terms of the circuit parameters, the a_3 coefficient becomes

$$a_3 = \frac{L_1L_2}{R_1(R_2\|R_3)} + \frac{L_1L_3}{(R_1\|R_2)R_3} + \frac{L_2L_3}{R_2R_3}. \tag{43}$$

Finally, the a_4 coefficient is found as

$$a_4 = \frac{L_1L_2L_3}{R_{11}{}^{23}R_{22}{}^3R_{33}{}^0} = \frac{L_1L_2L_3}{R_1R_2R_3} \tag{44}$$

where $R_{11}{}^{23}$ is the resistance shunting L_1 when $L_2 = L_3 = \infty$ and $R_{22}{}^3$ and $R_{33}{}^0$ have already been defined.

Conclusions

By use of the Laplace expansion of the system determinant it has been demonstrated that the system function may be determined from a prescribed number of driving-point resistance calculations. The advantages of the method are as follows: 1) simplicity in numerical calculation; 2) avoidance of evaluating the system determinant; 3) the use of real as opposed to complex arithmetic in computation; and 4) the ease with which dominant-pole approximations may be used. In addition, the circuit designer now has available a simple method of evaluating the relative importance on system performance of specific element values and their variation.

These results are readily extended to high-pass circuits (low-frequency performance of electronic circuits) by use of low-pass to high-pass transformation or by the analogy with the classical dominant-pole techniques [5].

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