Routing for Traffic Networks With Mixed Autonomy

Daniel A. Lazar, Samuel Coogan, *Member, IEEE*, and Ramtin Pedarsani, *Member, IEEE*

***Abstract*—In this article, we propose a macroscopic model for studying routing on networks shared between human-driven and autonomous vehicles that captures the effects of autonomous vehicles forming platoons. We use this to study inefficiency due to selfish routing and bound the price of anarchy (PoA), the maximum ratio between total delay experienced by selfish users and the minimum possible total delay. To do so, we establish two road capacity models, each corresponding to an assumption regarding the platooning capabilities of autonomous vehicles. Using these, we develop a class of road delay functions, parameterized by the road capacity, that are polynomial with respect to vehicle flow. We then bound the PoA and the bicriteria, another measure of the inefficiency due to selfish routing, for general networks with multiple source-destination pairs. We find these bounds depend on: the degree of the polynomial in the road delay function; and the degree of asymmetry, the difference in how human-driven and autonomous traffic affect road delay. We demonstrate that these bounds recover the classical bounds when no asymmetry exists. We show the bounds are tight in certain cases and that the PoA bound is order optimal with respect to the degree of asymmetry.**

***Index Terms*—Game theory, transportation networks.**

I. Introduction

In RECENT years, automobiles are increasingly equipped with autonomous and semiautonomous technology, which has potential to dramatically decrease traffic congestion [1]. Specifically, autonomous technologies enable *platooning*, in which these vehicles automatically maintain short headways between them via adaptive cruise control (ACC) or cooperative ACC (CACC). ACC uses sensing such as radar or LIDAR to maintain a specific distance to the preceding vehicle with faster-than-human reaction time, and CACC augments this with vehicle-to-vehicle communications.

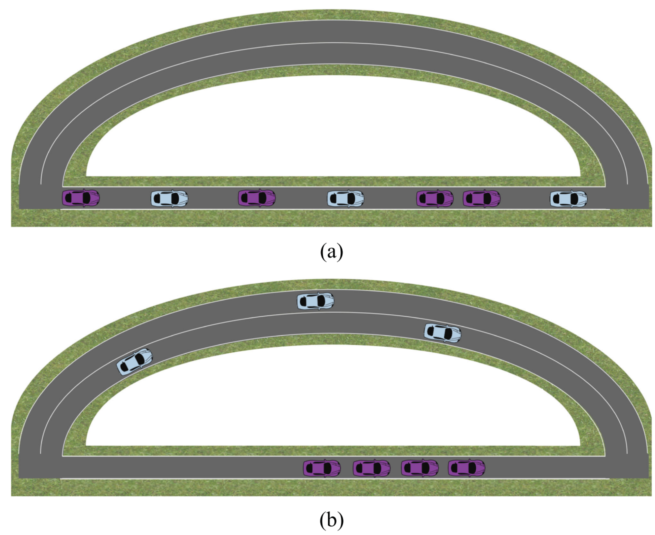

Fig. 1. Social planner can decrease overall travel times by make routing decisions that utilize autonomous vehicles' ability to platoon, and choosing different routes for human-driven vehicles (blue) and autonomous vehicles (purple). (a) When vehicles route selfishly, vehicles pack onto a congested road. (b) In optimal routing, only autonomous vehicles are sent onto the road most amenable to platooning.

When all vehicles are autonomous, the use of platooning has the potential to increase network capacity as much as threefold [2] by enabling synchronous acceleration at green lights [3]. However, the presence of human-driven vehicles—leading to *mixed autonomy*—makes much of these benefits unclear.

Moreover, even in the absence of autonomous capabilities, it is well known that if drivers route selfishly and minimize their individual traffic delays, this does not in general minimize *overall* traffic delay. Understanding the extent of this phenomenon can help city planners—if selfish routing does not adversely affect travel delay too much, then it may not be necessary to try to control vehicle flow using schemes such as tolling. Alternatively, if selfishness can lead to much worse road delay, then a city planner may wish to try to control human routing decisions. See Fig. 1 for an example of selfish routing and optimal routing in mixed autonomy.

The ratio between traffic delay under worst case selfish routing and optimal routing is called the price of anarchy (PoA) and is well understood for networks with only human-driven vehicles [4]–[8]. Many such works also bound the bicriteria, which

Manuscript received December 8, 2019; revised June 25, 2020; accepted July 19, 2020. Date of publication August 28, 2020; date of current version May 27, 2021. This work was supported in part by NSF under Grant 1749357 and in part by UC Office of the President under Grant LFR-18-548175. Recommended by Associate Editor B. De Schutter. *(Corresponding author: Daniel A. Lazar.)*

Daniel A. Lazar and Ramtin Pedarsani are with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, Santa Barbara, CA 93106 USA (e-mail: dlazar@ucsb.edu; ramtin@ucsb.edu).

Samuel Coogan is with the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332 USA (e-mail: sam.coogan@gatech.edu).

Color versions of one or more of the figures in this article are available online at https://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2020.3020059

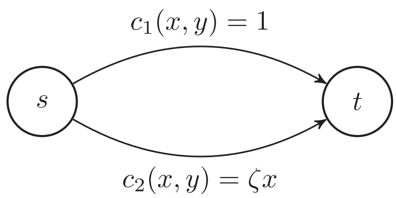

Fig. 2. Road network with PoA and bicriteria that grow unboundedly with ζ when considering $1/\zeta$ units of human-driven flow and 1 unit of autonomous flow demand, with $\zeta \geq 1$. Function arguments x and y, respectively, denote human-driven and autonomous vehicle flow on a road.

quantifies, for any given volume of vehicle flow demand, how much additional flow can be routed optimally to result in the same overall latency as the original volume of traffic routed selfishly. Other studies have bounded the PoA with multiple modes of transportation [9], [10]. However, these prior works require assumptions that do not capture vehicle flow on roads shared between human-driven and autonomous vehicles, leaving open the question of the PoA in mixed autonomy. In fact, *we show that these previous results do not hold, and the PoA for roads with mixed autonomy can in general be unbounded!*

Motivated by this observation, in this article, we provide novel bounds on the PoA and bicriteria that depend on the extent to which platooning affects road delay, as well as the degree of the polynomial describing road delay. To do so, we use two models that describe road capacity as a function of the fraction of vehicles on the road that are autonomous; each model corresponds to a different assumption regarding the technology that enables platooning. We use these capacity models with a known polynomial road delay function, and, for this class of latency functions, we bound the PoA and bicriteria. We develop two mechanisms for bounding the PoA, which yield bounds that are tighter depending on platoon spacing and polynomial degree. In our development, we provide the main elements of our proofs and defer proofs of the lemmas to the appendix.

In our formulation, the benefit due to the presence of autonomous vehicles is limited to platoon formation, and the probability that each vehicle is autonomous is independent of the surrounding vehicles. While we acknowledge that autonomous vehicles yield other benefits, such as smoothing traffic shockwaves, we consider platooning because it is a mature technology that is commercially available. Furthermore, if autonomous vehicles actively rearrange themselves to form platoons, the resulting capacity falls between the two capacity models presented here [11].

Motivating Example: To show that the PoA bounds previously developed for roads with only one type of vehicle (i.e., no autonomous vehicles) do not hold, we present an example of a road network with unbounded PoA (see Fig. 2). Consider a network of two parallel roads, with road latency functions $c_1(x, y) = 1$ and $c_2(x, y) = \zeta x$. On each road, the latency is a function of the human-driven flow (x) and the autonomous flow (y) on that road. Suppose we have $\frac{1}{\zeta}$ units of human-driven vehicle flow and 1 unit of autonomous traffic demand to cross from node s to node t, with $\zeta \geq 1$. Optimal routing puts all human-driven cars on the top road and all autonomous cars on the bottom road; when vehicles route selfishly they all end up on the bottom road. This yields a PoA of $\zeta + 1$. The bicriteria is also $\zeta + 1$, as $\zeta + 1$ times as much traffic, optimally routed, yields the same total cost as the original amount of traffic at Wardrop equilibrium.[1] This examples leads us already to our first proposition, which lays the foundations for the contributions of this article.

Proposition 1: The PoA and bicriteria are in general unbounded in mixed autonomy.

Motivated by this proposition, we develop the notion of the *degree of asymmetry* of a road and use this, in conjunction with the degree of the polynomial cost function, to parameterize the bound on the PoA. To summarize, we

1) show that previous PoA results do not hold for mixed autonomy;
2) develop a realistic class of polynomial cost functions for traffic of mixed autonomy;
3) develop two mechanisms for bounding the PoA using this cost function with both capacity models;
4) bound the PoA and bicriteria and analyze the tightness of our bounds.

Some of these contributions relate to our previous work. In [12], we use similar capacity models and a latency function based on M/M/1 queues to find optimal routing for a network of two parallel roads. In another work, we consider maximizing capacity, using the second capacity model in this article, via a sequence of vehicle reorderings in which autonomous vehicles influence human drivers [11].

This article also relates to [13], which presents a special case of the bounds presented in this article. Lazar *et al.* [13] considered the PoA and bicriteria in mixed autonomy only with *affine* latency functions. The bounds presented there are very loose; in fact, if autonomy increases the capacity of any road in the network by a factor of four or more, the bound does not hold at all. In contrast, the current article uses a new method to derive an entirely new bound, which holds for arbitrarily large capacity increase due to autonomy. Furthermore, in the current article, we base our considered latency functions on *two* capacity models, which are based on different assumptions of the platooning capabilities of autonomous vehicles. Moreover, the bounds are not limited to networks or roads with affine latency functions; the current work considers a class of latency functions that incorporates arbitrary polynomial degree.

II. Related Work

A. Congestion Games and Wardrop Equilibria

Our article is related to the optimal traffic assignment problem, e.g., [14], which studies how to optimally route vehicles on a network when the cost (i.e., delay) on a road link is a function of the flow of vehicles that travel on that link. We are concerned specifically with the relationship between optimal

[1] Although in this case, autonomous vehicles do not affect road delay, other examples in Section V also yield an unbounded PoA with both vehicle types affecting road delay.

traffic assignment and Wardrop Equilibria, which occur when drivers choose their paths selfishly. For a survey on literature on Wardrop Equilibria (see [15]) De Palma and Nesterov[16] described other notions of equilibria. Classic works on Wardrop Equilibria and the associated tools for analyzing them include those presented in [17]–[19].

In an important development, Smith [17] established the widely used *variational inequality* and used it to describe flows at Wardrop equilibrium, in which all users sharing an origin and destination used paths of equal cost and no unused path had a smaller cost. For any feasible flow z and equilibrium flow z^{EQ}, the variational inequality dictates that

$$\langle c(z^{\mathrm{EQ}}), z^{\mathrm{EQ}} - z \rangle \leq 0 \tag{1}$$

where z is a vector describing vehicle flow on each road, $c(z)$ maps a vector of flows to a vector of the delay on each road, and $\langle \cdot, \cdot \rangle$ denotes the inner product of two terms. Note that in the absence of an assumption about the *monotonicity* of c (see the following section for a definition), the variational inequality is a necessary but not sufficient condition for equilibria [16]. The variational inequality is fundamental for establishing our PoA bound.

B. Multiclass Traffic

Some previous works consider traffic assignment and Wardrop Equilibria with multiclass traffic, meaning traffic with multiple vehicle types, and transportation modes that affect and experience road latency differently (e.g., [20]–[22]).

Florian [22] demonstrated how to calculate equilibria for a multimodal system involving personal automobiles and public transportation. They used a relaxation that assumed that public transportation would take the path that would be the shortest in the absence of cars. In the case of mixed autonomy, this is not a fair assumption.

Dafermos [20] assumed that the Jacobian of the cost function was symmetric and positive definite. Similarly, Hearn *et al.* [21] dealt with a monotone cost function, i.e., satisfying the property

$$\langle c(z) - c(q), z - q \rangle \geq 0 \tag{2}$$

for flow vectors z and q.

However, traffic networks with mixed autonomy are in general nonmonotone. To see this, consider a network of two roads with costs $c_1(x, y) = 3x + y + t_1$ and $c_2(x, y) = 3x + 2y + t_2$, where t_1 and t_2 are constants denoting the free-flow latency on roads 1 and 2. This corresponds to a road in which autonomous vehicles can platoon closely and another road on which they cannot platoon as closely. The Jacobian of the cost function is as follows:

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

which is not symmetric, and the vector $z = [-1 \quad 2 \quad 0 \quad 0]^T$ demonstrates that it is also not positive semidefinite. Monotonicity is closely related to the positive (semi-) definiteness of the Jacobian of the cost function. To show that the monotonicity condition is violated as well, consider that there are 2 units of human-driven flow demand and 3 units autonomous flow demand. With one routing in which all human-driven flow is on the first road and all autonomous flow is on the second and another routing with these reversed, we find that the monotonicity condition is violated.

Similarly, Faroukhi *et al.* [23] proved that in heterogeneous routing games with cost functions that were continuously differentiable, nonnegative for feasible flows, and nondecreasing in each of their arguments, then at least one equilibrium was guaranteed to exist. These mild conditions are satisfied in our setting. For heterogeneous games with two types, they further prove a necessary and sufficient condition for a potential function (and therefore unique equilibrium) to exist. However, the condition required can be considered a relaxation of the condition that the Jacobian of the cost function be symmetric. While broader than strict symmetry, this condition is still not satisfied in mixed autonomy. Notably, they describe tolls that, when applied, yield a cost function that satisfies this condition.

As described above, these previous works in multiclass traffic require restrictive assumptions and therefore do not apply to the case of mixed autonomy. In fact, in the case of mixed autonomy, the routing game is not formally a proper congestion game, as it cannot be described with a potential function. Nonetheless, in this article, we adapt tools developed for such games to derive results for mixed autonomous traffic.

C. Price of Anarchy (PoA)

There is an abundance of research into the PoA in nonatomic congestion games, codified in [4]–[8]. In [8], Correa *et al.* developed a general tool for analyzing PoA in nonatomic congestion games. Although their development is specific to monotone cost functions, in this article, we broaden it to cost functions that are not necessarily monotone. Also relatedly, we find that in the case of no asymmetry, our PoA bound for polynomial cost functions simplifies to the classic bound in [5] and [6].

The previously mentioned works consider primarily single-type traffic. Perakis [9] considered PoA in multiclass traffic using nonseparable, asymmetric, nonlinear cost functions with inelastic demand. However, they restrict their analysis to the case that the Jacobian matrix of the cost function is positive semidefinite. Similarly, Chau and Sim [10] considered the PoA for multiclass traffic with *elastic* demand with symmetric cost functions and positive semidefinite Jacobian of the cost function. As demonstrated earlier, these assumptions are violated in the case of mixed autonomy.

D. Autonomy

In one line of research, autonomous vehicles are controlled to *locally* improve traffic by smoothing out stop-and-go shockwaves in congested traffic [24]–[32], optimally sending platooned vehicles through highway bottlenecks [33], [34], and simultaneously accelerating platooned vehicles at signalized intersections [2], [3]. Other papers investigate fuel savings attained using autonomous vehicles [35]–[38] or jointly controlling vehicles on a highway to localize and eliminate traffic disturbances [39]. Some works consider optimally routing and rebalancing a fleet of autonomous vehicles [40], although these generally consider a simpler model for road latency, in which capacitated roads have constant latency for flows below their capacity, and all roads are considered to be in this regime.

Some previous works have related models for road capacity and throughput under mixed autonomy, in particular [3], [12], [13], and [41]. Chen *et al.* [42] provided a capacity model which assumed that all autonomous vehicles are platooned in periodic platoons, each with the same number of vehicles and the same number of human-driven vehicles between platoons. In contrast, we consider two capacity models: one in which autonomous vehicles can maintain a short headway behind any vehicle they follow; and one in which autonomous vehicles are placed randomly as the result of a Bernoulli process and only platoon opportunistically. Another work shows that autonomy can *increase* the total delay experienced by users [43].

In our previous work [13], we found the PoA for affine latency functions, incorporating the first capacity model. This is a special case of the results in this article, which goes beyond the previous work by considering polynomial cost functions and incorporating both capacity models, resulting in a much broader class of functions. Furthermore, we introduce a novel mechanism for finding the PoA in mixed autonomy, leading to a tighter bound than the one previously found.

III. Network Model

Consider a congestion game on a network of N roads, with nonatomic drivers (meaning each control an infinitesimally small unit of vehicle flow) traveling across m origin–destination pairs, each pair associated with β_i units of human-driven vehicle flow demand and γ_i units of autonomous vehicle flow demand. We use $[N] = \{1, 2, \ldots, N\}$ to denote the set of roads. We fully describe driver behavior on a network by using a vector of vehicle flows, which describes the volume of vehicles of each type that travels on each road. This vector has size equal to twice the number of roads and uses alternating entries to denote human-driven and autonomous vehicle flow on a road. We use x_i and y_i to refer to human-driven and autonomous flow on road i, respectively. Then, the flow vector z is as follows:

$$z = [x_1\ y_1\ x_2\ y_2\ \ldots\ x_N\ y_N]^T \in \mathbb{R}^{2N}_{\geq 0}\ .$$

We refer to this flow vector as a *routing* or a *strategy*. We use $\mathcal{X} \subseteq \mathbb{R}^{2N}_{\geq 0}$ to denote the set of feasible routings, meaning routings that route all flow demand from their origin nodes to their destination nodes while respecting conservation of flow in the network.

When needing to distinguish between two vectors, we use v and w in place of x and y and q in place of z. We assume that human-driven and autonomous vehicles experience road delay identically. To capture the differing effects of each type of flow on a road's latency, we construct cost function $c(z) : \mathbb{R}^{2N}_{\geq 0} \to \mathbb{R}^{2N}_{\geq 0}$ as follows:

$$c(z) = \begin{bmatrix} c_1(x_1, y_1) \\ c_1(x_1, y_1) \\ c_2(x_2, y_2) \\ c_2(x_2, y_2) \\ \cdots \\ c_N(x_N, y_N) \\ c_N(x_N, y_N) \end{bmatrix}$$

where $c_i(x_i, y_i)$ is the latency on road i when x_i units of human-driven vehicles and y_i units of autonomous vehicles use the road. The social cost, which is the *aggregate delay* experienced by all users of the network, is then $C(z) := \langle c(z), z \rangle$. A social planner then wishes to find the *socially optimal routing*, which is the feasible routing that minimizes the social cost, and therefore solves the following optimization:

$$\min_{z \in \mathcal{X}} C(z)\ .$$

In contrast, selfish users in a Wardrop equilibrium do not try to minimize the social delay. Instead, they selfishly choose routes. This implies that if a route has positive flow on it, all other routes between the same source–destination pair have equal or greater delay. In the following sections, we develop models for road capacity and road delay in order to construct the cost functions.

A. Capacity Models

We model the capacity of a road under two assumptions: first, autonomous vehicles can platoon (follow closely) behind any vehicle; and second, autonomous vehicles can only platoon behind other autonomous vehicles. Let d_i denote road length times the road's nominal velocity and let $\bar{h}_i$ and h_i denote the nominal space taken up by a platooned and nonplatooned vehicle, respectively. The capacity will be a function of the *autonomy level* of the road, denoted by $\alpha(x_i, y_i) = y_i/(x_i + y_i)$. We define the capacity of a road as the number of vehicles that can travel on a road at the road's nominal velocity. This is calculated by dividing the length of the road by the average space taken up by a car on the road, which is a function of autonomy level, and multiplying it by the free-flow velocity of the road. We formalize this in the following proposition.[2]

Proposition 2: Assume that vehicles are placed on a road as the result of a Bernoulli process. If autonomous vehicles can platoon behind any vehicle, therefore occupying road length $\bar{h}_i$ when traveling at nominal velocity, and human driven vehicles do not platoon (therefore occupying road length h_i at nominal velocity), then the capacity is

$$m_i(x_i, y_i) = \frac{d_i}{\alpha(x_i, y_i)\bar{h}_i + (1 - \alpha(x_i, y_i))h_i}\ . \qquad (3)$$

If autonomous vehicles only platoon behind other autonomous vehicles and human driven vehicles cannot platoon, then the

[2]Found in [12], contemporaneously in [41].

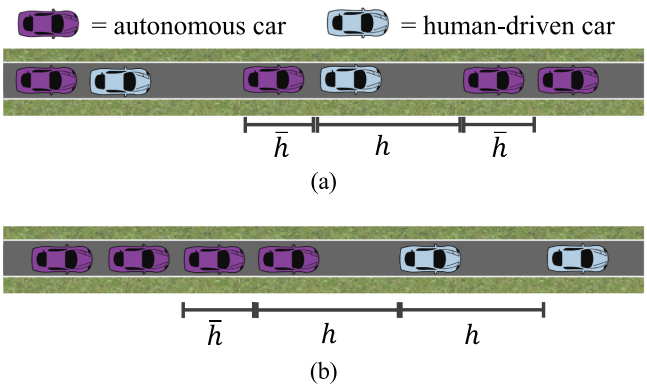

Fig. 3. Capacity models 1 and 2. In capacity model 1 (left), autonomous cars can platoon behind any vehicle, and therefore take up length $\bar{h}$ when traveling at the free-flow velocity. In capacity model 2 (right), autonomous vehicles can only platoon behind other autonomous vehicles; in that case, they take up length $\bar{h}$, but if following a human-driven vehicle, they take up length h. Human-driven vehicles always take up length h.

capacity is

$$m_i(x_i, y_i) = \frac{d_i}{\alpha^2(x_i, y_i)\bar{h}_i + (1 - \alpha^2(x_i, y_i))h_i} \,. \tag{4}$$

Proof: We first justify the proposition for the first capacity model. Autonomous vehicles follow any vehicle with the same headway (occupying total space $\bar{h}_i$), as do human-driven vehicles with a different headway (occupying h_i). The space taken up by an average vehicle, as the number of vehicles grows large, is a weighted combination of those two spacings that depends on the autonomy level. Note that this capacity model does not depend on the ordering of the vehicles.

For the second capacity model, we assume that the vehicles are placed as the result of a Bernoulli process with parameter α_i. Consider M vehicles, each with length L, with s_m denoting the headway of vehicle m. Note that the front vehicle will have $s_m = 0$. The expected total space taken up is, due to linearity of expectation

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^{M} L + s_m\right] &= ML + \sum_{m=1}^{M-1} \mathbb{E}[s_m] \\ &= (M-1)(\alpha^2(x_i, y_i)\bar{h}_i + (1 - \alpha^2(x_i, y_i))h) + L \,.\end{aligned}$$

Then, as the number of vehicles grows, the average space occupied by a vehicle approaches $\alpha^2(x_i, y_i)\bar{h}_i + (1 - \alpha^2(x_i, y_i))h$, yielding the above expression for capacity model 2. ∎

Fig. 3 provides an illustration of the technology assumptions. To make the meaning of nominal vehicle spacing more concrete, we offer one way of calculating spacing: let L denote vehicle length, and $\tau_{h,i}$ and $\tau_{a,i}$ denote the reaction speeds of human-driven and autonomous vehicles, respectively. Let v_i be the nominal speed on road i, which is likely the road's speed limit. Then, we consider $\bar{h}_i = L + v_i\tau_{a,i}$ and $h_i = L + v_i\tau_{h,i}$.[3]

[3]In general, we consider $h_i \geq \bar{h}_i$, but we do not formally make this assumption. Our theoretical results hold even in the case that $h_i \geq \bar{h}_i$ on some roads and $h_j < \bar{h}_j$ on others.

B. Fundamental Diagram of Traffic (FDT)-Based Delay Model

An FDT-based models of vehicle flow dictate a relationship between vehicle flow and density in which flow increases with density until the *critical density* is reached (uncongested regime), after which the flow decreases as density increases (congested regime) [44]. We consider a *triangular* FD with respect to the total flow, where we model the critical density as the capacity functions in (3) and (4). This leads to a flow-density relationship as follows [45]:

$$Q_i(\phi_i^{\mathrm{h}}, \phi_i^{\mathrm{a}}) := \begin{cases} \bar{v}_i \cdot (\phi_i^{\mathrm{h}} + \phi_i^{\mathrm{a}}), & \text{if } \phi_i^{\mathrm{h}} + \phi_i^{\mathrm{a}} \leq m_i(\phi_i^{\mathrm{h}}, \phi_i^{\mathrm{a}}) \\ \frac{\bar{v}_i \cdot m_i(\phi_i^{\mathrm{h}}, \phi_i^{\mathrm{a}}) \cdot (\bar{\phi}_i - (\phi_i^{\mathrm{h}} + \phi_i^{\mathrm{a}}))}{\bar{\phi}_i - m_i(\phi_i^{\mathrm{h}}, \phi_i^{\mathrm{a}})}, & \text{if } m_i(\phi_i^{\mathrm{h}}, \phi_i^{\mathrm{a}}) \leq \phi_i^{\mathrm{h}} + \phi_i^{\mathrm{a}} \leq \bar{\phi}_i \\ 0, & \text{otherwise.} \end{cases}$$

where $\bar{\phi}_i$ and $\bar{v}_i$, respectively, denote the jam density and free-flow velocity on road i, and ϕ_i^{h} and ϕ_i^{a} denote the human-driven density and autonomous vehicle density, respectively. Then, using the relationship that vehicle flow is equal to the product of the density and velocity, we find a relationship between road delay and vehicle flow, where s_i is a binary argument that indicates whether the road is congested

$$c_i(x_i, y_i, s_i) = \begin{cases} \frac{d_i}{\bar{v}_i} & \text{if } s_i = 0 \\ d_i\left(\frac{\bar{\phi}_i}{x_i + y_i} + \frac{m_i(x_i, y_i) - \bar{\phi}_i}{\bar{v}_i \cdot m_i(x_i, y_i)}\right) & \text{if } s_i = 1. \end{cases} \tag{5}$$

This leads to the following proposition [45].

Proposition 3: Using the FDT-based model for road latency in mixed autonomy in (5), the PoA is infinite.

Proof: Consider a single road with a fixed autonomy level. The worst case cost has the road in a congested state, and the best case cost has it in an uncongested state. The ratio of these costs is $c_i(x_i, y_i, 1)/c_i(x_i, y_i, 0)$. If we consider a flow demand that approaches zero, this quantity grows unboundedly.[4] ∎

C. Bureau of Public Roads (BPR)-Based Delay Model

We now propose a model, similar to BPR model [22], [46], [47], for the road delay incurred by mixed traffic resulting from the capacity models derived above.

Assumptions 1: In the remainder, we assume the following relationship between the flow of vehicles on a road and the delay on the road:

$$c_i(x_i, y_i) = t_i\left(1 + \rho_i\left(\frac{x_i + y_i}{m_i(x_i, y_i)}\right)^{\sigma_i}\right). \tag{6}$$

where t_i denotes the free-flow delay on road i, and ρ_i and σ_i are model parameters. Typical values for ρ_i and σ_i are 0.15 and 4, respectively [47]. However, our solution methodology is valid for any parameters such that $t_i \geq 0$, $\rho_i \geq 0$, and $\sigma_i \geq 1$.

Remark 1: This model of delay function assumes that the density of vehicles on a road remains low enough that the vehicle flow does not enter the congested regime, in which delay

[4]A similar proof applies for a more general FDT that is not necessarily triangular, as well as in the case of only a single vehicle type.

increases as flow *decreases*. In the absence of this assumption, the PoA is trivially infinite, as shown in Proposition 3.

Remark 2: This choice of cost functions implies that road delay is separable, meaning that the vehicles on one road do not affect those on another. In the conference version of this paper [13], we bound the PoA and bicriteria for some nonseparable affine cost functions.

The class of cost functions we consider are not monotone, meaning they do not necessarily satisfy (2), but are elementwise monotone, defined in the following.

Definition 1: A cost function $c : R^{2N}_{\geq 0} \to R^{2N}_{\geq 0}$ is *elementwise monotone* if it is nondecreasing in each of its arguments, i.e., $\frac{\mathrm{d}c_i(z)}{\mathrm{d}z_j} \geq 0 \; \forall i, j \in [2N]$.

IV. Bounding the PoA

In this section, we bound the PoA and bicriteria of traffic networks with mixed autonomy. As established in Section I, the PoA is in general unbounded in traffic networks with mixed autonomy. However, we can establish a bound for the PoA by parameterizing it as described in the following.

Definition 2: The *degree of asymmetry* on a road is the maximum ratio of road space utilized by a car of one type to a car of another type on the same road while traveling at nominal velocity. The *maximum degree of asymmetry* k is the maximum of the above quantity over all roads in the network. Formally, $k := \max_{i\in[N]} \max(h_i/\bar{h}_i, \bar{h}_i/h_i)$.

Note that we do not assume that one vehicle type affects delay more than another type on all roads. For example, autonomous vehicles may require shorter headways than human-driven vehicles on highways but longer headways on neighborhood roads to maintain safety for pedestrians.

Definition 3: The *maximum polynomial degree*, denoted by σ, for a road network with cost functions in the form (6) is the maximum degree of a polynomial denoting the cost on all roads in the network: $\sigma = \max_{i\in[N]} \sigma_i$.

We use $\mathcal{C}_{k,\sigma}$ to denote the class of cost functions of the form (6), with maximum degree of asymmetry k and maximum polynomial degree σ, with cost functions using m_i from either capacity mode 1 in (3) or capacity model 2 in (4). Let

$$\xi(\sigma) := \sigma(\sigma+1)^{-\frac{\sigma+1}{\sigma}} \; . \tag{7}$$

Note that for $\sigma \geq 1$, $\xi(\sigma) < 1$. With this, we present our first bound.

Theorem 1: Consider a class of nonatomic congestion games with cost functions drawn from $\mathcal{C}_{k,\sigma}$, under Assumption 1. Let z^{EQ} be an equilibrium and z^{OPT} be a social optimum for this game. Then

$$C(z^{EQ}) \leq \frac{k^\sigma}{1-\xi(\sigma)} C(z^{OPT}) \; .$$

Proof: Given any road cost function c (and social cost C) and equilibrium z^{EQ}, we define an aggregate cost function c^{AGG} (and social cost C^{AGG}) with corresponding equilibrium flow f^{EQ}, both parameterized by z^{EQ}. This allows us to combine human-driven and autonomous flow into one flow type in the aggregate function so we can bound the PoA for the aggregate cost function. We then find the relationship between the optimal routing for the aggregate cost function to that of the original cost function. Formally, the steps of the proof are as follows:

$$C(z^{\mathrm{EQ}}) = C^{\mathrm{AGG}}(f^{\mathrm{EQ}}) \tag{8}$$

$$\leq \frac{1}{1-\xi(\sigma)} C^{\mathrm{AGG}}(f^{\mathrm{OPT}}) \tag{9}$$

$$\leq \frac{1}{1-\xi(\sigma)} k^\sigma C(z^{\mathrm{OPT}}) \; . \tag{10}$$

We begin by introducing the tool with which we bound the PoA in (9). We then define c^{AGG} and f^{EQ} such that (8) holds and show that f^{EQ} is an equilibrium for c^{AGG}. We discuss the structure of the tool used to bound the PoA and provide an intuitive explanation of how the chosen structure of c^{AGG} leads to a tighter PoA bound than an alternative choice. We then provide lemmas corresponding to inequality (9), which bounds the PoA of this new cost function, and (10), which relates the social cost of optimal routing under c^{AGG} to that of the original cost function c. We defer proofs of the lemmas to the appendix.

We first introduce a general tool that we use for our results by extending the framework established by Correa *et al.* [8], which relies on the variational inequality to bound the PoA. We use the following parameters:

$$\beta(c,q) := \max_{z\in\mathbb{R}^{2N}_{\geq 0}} \frac{\langle c(q)-c(z), z\rangle}{\langle c(q), q\rangle}$$

$$\beta(\mathcal{C}) := \sup_{c\in\mathcal{C}, q\in\mathcal{X}} \beta(c,q) \tag{11}$$

where $0/0 = 0$ by definition, and $\mathcal{C}$ is the class of network cost functions being considered. Then, the following lemma is considered.

Lemma 1: Let z^{EQ} be an equilibrium of a nonatomic congestion game with cost functions drawn from a class $\mathcal{C}$ of elementwise monotone cost functions.

a) If z^{OPT} is a social optimum for this game and $\beta(\mathcal{C}) < 1$, then

$$C(z^{EQ}) \leq (1-\beta(\mathcal{C}))^{-1} C(z^{OPT}) \; .$$

b) If q^{OPT} is a social optimum for the same game with $1+\beta(\mathcal{C})$ times as much flow demand of each type, then

$$C(z^{EQ}) \leq C(q^{OPT}) \; .$$

The lemma and proof are nearly identical to that of Correa *et al.* [8], extended to encompass nonmonotone, yet elementwise monotone, cost functions.

We now explain our choice of c^{AGG} and f^{OPT} that yields (8) then provide an intuitive explanation for this choice. Recall that we define

$$z^{\mathrm{EQ}} = \begin{bmatrix} x_1^{\mathrm{EQ}} & y_1^{\mathrm{EQ}} & x_2^{\mathrm{EQ}} & y_2^{\mathrm{EQ}} & \ldots & x_N^{\mathrm{EQ}} & y_N^{\mathrm{EQ}} \end{bmatrix}^T \; .$$

We define a new flow vector that aggregates the regular and autonomous flows: $f^{\mathrm{EQ}} = x^{\mathrm{EQ}} + y^{\mathrm{EQ}}$, where $z^{\mathrm{EQ}} \in \mathbb{R}^{2N}_{\geq 0}$ and $x^{\mathrm{EQ}}, y^{\mathrm{EQ}}, f^{\mathrm{EQ}} \in \mathbb{R}^{N}_{\geq 0}$. We define a new cost function c^{AGG} that is a mapping from flow vector (with one flow for each road) to road latencies, i.e., $c^{\mathrm{AGG}} : \mathbb{R}^{N}_{\geq 0} \to \mathbb{R}^{N}_{\geq 0}$. We define c^{AGG} so

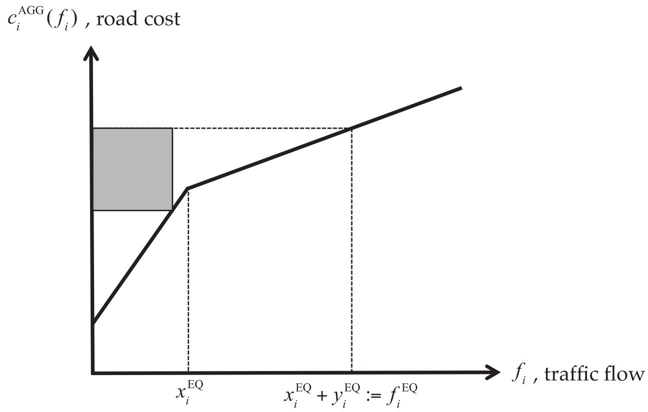

Fig. 4. Illustration of the geometric interpretation of the parameter $\beta(\mathcal{C}^{\mathrm{AGG}})$ where $\mathcal{C}^{\mathrm{AGG}}$ represents the class of aggregate cost functions. Parameter $\beta(\mathcal{C}^{\mathrm{AGG}})$ is an upper bound on the ratio between the size of the shaded rectangle and the dashed rectangle. This is an upper bound over all choices of $c_i^{\mathrm{AGG}} \in \mathcal{C}^{\mathrm{AGG}}$ and x^{EQ} and $y^{\mathrm{EQ}} \geq 0$.

that it has the same road costs with flow f^{EQ} as c does with flow z^{EQ}. Note, however, that c is a mapping from flows, with two flow types per road, to road latencies, again with each road represented twice ($c : \mathbb{R}^{2N}_{\geq 0} \to \mathbb{R}^{2N}_{\geq 0}$). However, c^{AGG} represents each road once.

We formally define c^{AGG}, which depends on the equilibrium flow being considered, i.e., z^{EQ}. This cost function is defined below for both capacity models. In general terms, c^{AGG} adds the "costly" type of vehicle flow first, then adds the "less costly" vehicle flow. If $\bar{h}_i \leq h_i$

$$c_{i,1}^{\mathrm{AGG}}(f_i) \tag{12}$$

$$:= \begin{cases} t_i \left(1 + \rho_i \left(\frac{h_i f_i}{d_i}\right)^{\sigma_i}\right) & f_i \leq x_i^{\mathrm{EQ}} \\ t_i \left(1 + \rho_i \left(\frac{\bar{h}_i f_i + (h_i - \bar{h}_i) x_i^{\mathrm{EQ}}}{d_i}\right)^{\sigma_i}\right) & f_i > x_i^{\mathrm{EQ}} \end{cases}$$

$$c_{i,2}^{\mathrm{AGG}}(f_i)$$

$$:= \begin{cases} t_i \left(1 + \rho_i \left(\frac{h_i f_i}{d_i}\right)^{\sigma_i}\right) & f_i \leq x_i^{\mathrm{EQ}} \\ t_i \left(1 + \rho_i \left(\frac{h_i f_i^2 - (h_i - \bar{h}_i)(f_i - x_i^{\mathrm{EQ}})^2}{d_i f_i}\right)^{\sigma_i}\right) & f_i > x_i^{\mathrm{EQ}} \end{cases}. \tag{13}$$

If $\bar{h}_i > h_i$, then swap $\bar{h}_i$ and h_i above, and replace x_i^{EQ} with y_i^{EQ}. In all cases, $c_i^{\mathrm{AGG}}(f_i^{\mathrm{EQ}}) = c_j(x_j^{\mathrm{EQ}}, y_j^{\mathrm{EQ}})$, where $j \in [2N]$ and $i = \lceil j/2 \rceil \in [N]$. Since the road latencies under $c^{\mathrm{AGG}}(f^{\mathrm{EQ}})$ are the same as under $c(z^{\mathrm{EQ}})$, f^{EQ} is an equilibrium for c^{AGG}.

To provide some intuition as to why we add the "costly" vehicle type first, consider the affine case with the first capacity model. Correa *et al.* give a geometric interpretation of the parameter $\beta(\mathcal{C}^{\mathrm{AGG}})$ when cost are separable, meaning road latency only depends on one element of the flow vector. They show that for any cost function drawn from $\mathcal{C}^{\mathrm{AGG}}$, $\beta(\mathcal{C}^{\mathrm{AGG}})$ provides an upper bound on the ratio of the area of a rectangle above the cost function curve to the area of a rectangle enclosing it, where the enclosing rectangle has one corner at the origin. See Fig. 4 for an illustration.

This interpretation provides the intuition that the more convex a function can be, the greater $\beta(\mathcal{C}^{\mathrm{AGG}})$ can grow. Thus, to make our bound as tight as possible in our case, we add the costly vehicle type first. In the affine case with the first capacity model, this makes the class of cost functions concave. Then, the element of this class that maximizes the size of the interior rectangle relative to the exterior rectangle minimizes the concavity of the function by setting $x^{\mathrm{EQ}} = 0$ or $y^{\mathrm{EQ}} = 0$. Thus, the PoA bound does not depend on the degree of asymmetry. Although this exact interpretation does not apply for $\sigma > 1$ or for the second capacity model, the intuition is nonetheless useful.

With this intuition, we now present inequalities (9) and (10) as lemmas, which we prove in the appendix.

Lemma 2: Consider a nonatomic congestion game with road cost functions of the form (12) or (13), with maximum polynomial degree σ. Then

$$C^{\mathrm{AGG}}(f^{\mathrm{EQ}}) \leq \frac{1}{1 - \xi(\sigma)} C^{\mathrm{AGG}}(f^{\mathrm{OPT}})$$

where $\xi(\sigma) = \sigma(\sigma + 1)^{-\frac{\sigma+1}{\sigma}}$.

Lemma 3: Let c be a cost function composed of road costs of the form (6) with maximum degree of asymmetry k and maximum polynomial degree σ. Let c^{AGG} be an aggregate cost function of c, as defined in (12) and (13). Let the flow vector z^{OPT} be a minimizer of C and f^{OPT} be a minimizer of C^{AGG}, with $\sum_{i \in [2N]} z_i^{\mathrm{OPT}} = \sum_{i \in [N]} f_i^{\mathrm{OPT}}$. Then

$$C^{\mathrm{AGG}}(f^{\mathrm{OPT}}) \leq k^{\sigma} C(z^{\mathrm{OPT}}) .$$

We prove Lemma 2 by bounding $\beta(\mathcal{C})$ for the class of aggregate cost functions and applying Lemma 1. We analyze the structures of c and c^{AGG} to prove Lemma 3. With these lemmas, the theorem is proved. ■

Note that for $k = 1$ (i.e., no asymmetry), the PoA bound simplifies to those in [5] and [6]. If the cost functions are affine and there is no asymmetry, this reduces to the classic $\frac{4}{3}$ bound [4]. We characterize the tightness of this bound in the following corollary.

Corollary 1: Given a maximum polynomial degree σ, the PoA bound is order optimal with respect to the maximum degree of asymmetry k.

We provide an example proving the corollary in Section V. When considering road networks with low asymmetry, we can establish another bound.

Theorem 2: Consider a class of nonatomic congestion games with cost functions drawn from $\mathcal{C}_{k,\sigma}$, under Assumption 1. Let z^{EQ} be an equilibrium and z^{OPT} a social optimum for this game. If $k\xi(\sigma) < 1$, then

$$C(z^{EQ}) \leq \frac{1}{1 - k\xi(\sigma)} C(z^{OPT}) .$$

Proof: To prove this theorem, instead of going through an aggregate cost function, we directly find $\beta(\mathcal{C})$ for our class of cost functions and apply Lemma 1. We do this in two lemmas: we first find a relationship between the parameter $\beta(c, v)$ and the road capacity model $m_i(x_i, y_i)$, then we bound the resulting expression.

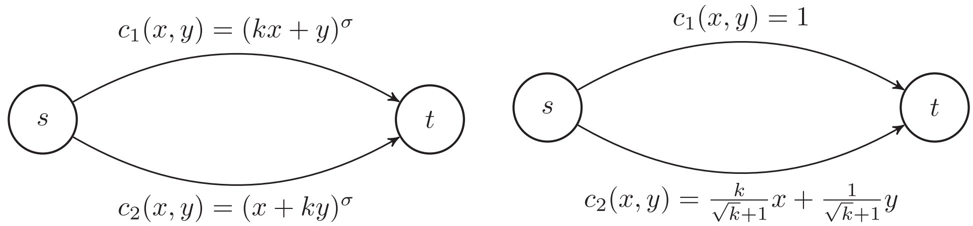

Fig. 5. From left to right: (a) Network from Example 1 with two-sided asymmetry. One unit of human-driven and one unit of autonomous flow cross from node s to t. (b) Network from Example 2 with two-sided asymmetry. $\frac{1}{\sqrt{k}}$ units of human-driven and one unit of autonomous flow cross from node s to t.

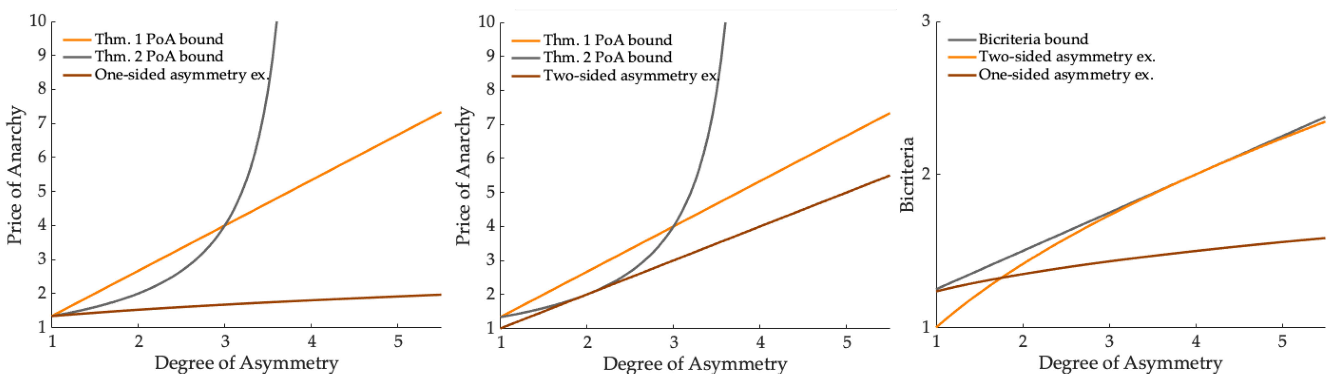

Fig. 6. From left to right: (a) Comparison of the PoA of Example 1 with the upper bound, with $\sigma = 1$. (b) Comparison of the PoA of Example 2 with the upper bound, with $\sigma = 1$. (c) Bicriteria of Examples 1 and 2 (with $\sigma = 1$), compared with the bicriteria bound.

Lemma 4: For cost functions of the form (6), the parameter $\beta(\mathcal{C})$ is bounded by

$$\beta(\mathcal{C}) \leq \max_{i\in[N],q_i,z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i + y_i}{v_i + w_i}\left(1 - \left(\frac{m_i(v_i, w_i)(x_i + y_i)}{m_i(x_i, y_i)(v_i + w_i)}\right)^\sigma\right).$$

Lemma 5: For capacities of the forms (3) or (4)

$$\max_{i\in[N],q_i,z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i + y_i}{v_i + w_i}\left(1 - \left(\frac{m_i(v_i, w_i)(x_i + y_i)}{m_i(x_i, y_i)(v_i + w_i)}\right)^\sigma\right) \leq k\xi(\sigma).$$

These lemmas, together with Lemma 1, prove the theorem as well as Theorem 3 below. ∎

Note that this bound may not necessarily be tighter in all regimes so our new PoA bound is $\min(\frac{k^\sigma}{1-\xi(\sigma)}, \frac{1}{1-k\xi(\sigma)})$. Though we cannot in closed form determine the region for which it is tighter, we can do so numerically. For example, for affine cost functions with $k = 2$, $\frac{k^\sigma}{1-\xi(\sigma)} = \frac{8}{3}$ and $\frac{1}{1-k\xi(\sigma)} = 2$. In Section V, we show via example that the bound in Theorem 2 is tight in this case.

The method used for establishing Theorem 2 also gives a bound on the bicriteria stated in the following theorem.

Theorem 3: Consider a class of nonatomic congestion games with cost functions drawn from $\mathcal{C}_{k,\sigma}$, under Assumption 1. Let z^{EQ} be an equilibrium for this game. If q^{OPT} is a social optimum for the same game with $1 + k\xi(\sigma)$ times as much flow demand of each type, then

$$C(z^{\text{EQ}}) \leq C(q^{\text{OPT}}) .$$

As an example, if road delays are described by polynomials of degree 4, and the maximum asymmetry between the spacing of platooned and nonplatooned vehicles is 3, then the cost of selfishly routing vehicles will be less than optimally routing $1 + 3\,\xi(4) \approx 2.61$ times as much vehicle flow of each type.

V. Establishing Lower Bounds by Example

In this section, we provide examples that give a lower bound on the PoA for this class of networks and serve to illustrate the tightness of the bounds. The examples are shown in Fig. 5 and the comparison of the PoA and bicriteria are shown in Fig. 6. We discuss notions of one-sided and two-sided asymmetries: a network has one-sided asymmetry if $\bar{h}_i \leq h_i\,\tilde{}\,\forall i \in [N]$ (human-driven cars always contribute more to road delay than autonomous cars) or $\bar{h}_i \geq h_i\,\tilde{}\,\forall i \in [N]$ (human-driven cars always contribute less to road delay than autonomous cars); otherwise the network has two-sided asymmetry. We provide two example networks (see Fig. 5), one with two-sided asymmetry and one with one-sided asymmetry. We compare the PoA and bicriteria in those networks to the upper bounds established earlier.

Through the first example, we prove Corollary 1. In this example, we consider two roads, one of which is well-suited for autonomous vehicles (such as a highway) and the other is well-suited for human-driven vehicles (such as an urban road).

Example 1: Consider the network of parallel roads in Fig. 5(a), where one unit of human-driven and one unit of autonomous flow wish to cross from node s to t. The roads have costs $c_1(x,y) = (kx+y)^\sigma$ and $c_2(x,y) = (x+ky)^\sigma$, where $k \geq 1$. In worst case equilibrium, all human-driven cars are on the top road and all autonomous cars are on the bottom road. In the best case, these routing are reversed. This yields a PoA of k^σ. To find the bicriteria, we calculate how much traffic, optimally routed, yields a cost equal to $2k^\sigma$, the cost of the original traffic volume at worst case equilibrium. We find that $k^{\frac{\sigma}{\sigma+1}}$ as much traffic, optimally routed, yields this same cost.

We now analyze the setting in which autonomous vehicles *always* increase the capacity of a road. In this case, the tightness of our bound (which holds for two-sided asymmetry as well) remains open.

Example 2: Consider the network of parallel roads in Fig. 5(b), where $\frac{1}{\sqrt{k}}$ unit of human-driven and one unit of autonomous flow wish to cross from node s to t. The roads have costs $c_1(x,y) = 1$ and $c_2(x,y) = \frac{k}{\sqrt{k}+1}x + \frac{1}{\sqrt{k}+1}y$. At equilibrium, all vehicles take the bottom road; optimally routed, human-driven vehicles takes the top road and autonomous vehicles take the bottom. This yields a PoA of $1 + \frac{k}{2\sqrt{k}+1}$. Calculations similar to that in Example 1 yield a bicriteria of $\frac{(-1+\sqrt{1+4\sqrt{k}})(1+\sqrt{k})}{2\sqrt{k}}$.

For affine cost functions, $\sigma = 1$ so $\xi = 1/4$. The PoA bound is then $\min(\frac{4}{4-k}, \frac{4}{3}k)$ and the bicriteria bound is $1 + k/4$. With $\sigma = 1$, the first example has PoA k and bicriteria $\sqrt{k}$, and the second example has PoA of order $\sqrt{k}$ and bicriteria of order $k^{1/4}$. Accordingly, the first example shows that the PoA bound is tight for $k = 2$ and the bicriteria bound is tight for $k = 4$. Note that a realistic range for k is between 1 and 4.

Furthermore, for affine cost functions, the bound in Theorem 2 is tighter than that of Theorem 1 when the degree of asymmetry is low. However, the bound in Theorem 1 scales much better for high degrees of asymmetry. This effect is accentuated for cost functions that have higher order polynomials—the regime for which the bound in Theorem 2 is tighter shrinks as the maximum polynomial degree grows.

As stated in Corollary 1, our bound is order optimal with respect to the maximum degree of asymmetry, i.e., k. Comparing the bound ($\frac{k^\sigma}{1-\xi(\sigma)}$) with the PoA in Example 1 (k^σ) shows that for a fixed σ, the PoA upper bound is within a constant factor of the lower bound, implying that the upper bound is order optimal in k.

It is also worth noting that under the construction used in Theorem 2, the bicriteria is related to the PoA through the quantity $\beta(\mathcal{C})$ [8]. Observe that Example 1 provides a bicriteria of 2 for $k = 4$, implying $\beta(\mathcal{C}_4) \geq 1$. Since the PoA is greater than or equal to $\frac{1}{1-\beta(\mathcal{C})}$, this mechanism cannot bound the PoA for $k \geq 4$. This leads us to rely on the mechanism developed for Theorem 1 for networks with large asymmetry.

VI. Conclusion

In this article, we present a framework, similar to a congestion game, for considering traffic networks with mixed autonomy. To do so, we present two models for the capacity of roads with mixed autonomy, each corresponding to an assumption about the technological capabilities of autonomous vehicles, and we define a class of road latency functions that incorporates these capacity models. Using this framework, we develop two methods of bounding the PoA and show that these bounds depend on the degree of the polynomial describing latency and the difference in the degree to which platooned and nonplatooned vehicles occupy space on a road. In addition, we present a bound on the bicriteria, another measure of inefficiency due to selfish routing. We present that examples showing these bounds are tight in some cases and recover classical bounds when human-driven and autonomous vehicles affect congestion the same way. Moreover, we show that our PoA bound is order optimal with respect to the degree to which vehicle types differently affect latency. Moreover, we show the limitations of the PoA framework when considering the congested regime of vehicle flow.

Some directions for future work are as follows. The capacity models presented assume that vehicle types are determined as a result of a Bernoulli process; a more general capacity model could incorporate autonomous vehicles that actively rearrange themselves as to form platoons. Furthermore, autonomous vehicles can affect vehicle flow in ways not limited to platooning. In addition, our proposed latency function considers only the effect of a vehicle on the road upon which it travels; a more general latency function would consider interaction between roads. Finally, the PoA bound is not shown to be tight but is order optimal in the degree of asymmetry k, and a future work could aim to close this gap. Nonetheless, this article presents a framework that can be used in the future for studying traffic networks in mixed autonomy.

Appendix

A. Proof of Lemma 1

We prove Lemma 1, which offers a useful tool for bounding the PoA. To prove part (a)

$$\begin{aligned}\langle c(q^{\mathrm{EQ}}), z\rangle &= \langle c(z), z\rangle + \langle c(q^{\mathrm{EQ}}) - c(z), z\rangle \\ &\leq \langle c(z), z\rangle + \beta(c, q^{\mathrm{EQ}})\langle c(q^{\mathrm{EQ}}), q^{\mathrm{EQ}}\rangle \\ &\leq C(z) + \beta(\mathcal{C})C(q^{\mathrm{EQ}}) \qquad (14)\end{aligned}$$

and by the variational inequality, $C(q^{\mathrm{EQ}}) \leq \langle c(q^{\mathrm{EQ}}), z\rangle$ for any feasible routing z. Completing the proof requires that $\beta(\mathcal{C}) \leq 1$, then replace the generic z with z^{OPT}.

To prove part (b), elementwise monotonicity implies the feasibility of $(1+\beta(\mathcal{C}))^{-1}q^{\mathrm{OPT}}$, which routes the same volume of traffic as z^{EQ}. Using (1)

$$\langle c(z^{\mathrm{EQ}}), z^{\mathrm{EQ}}\rangle \leq \langle c(z^{\mathrm{EQ}}), (1+\beta(\mathcal{C}))^{-1}q^{\mathrm{OPT}}\rangle\,. \qquad (15)$$

Then

$$\begin{aligned}C(z^{\mathrm{EQ}}) &= (1+\beta(\mathcal{C}))\langle c(z^{\mathrm{EQ}}), z^{\mathrm{EQ}}\rangle \\ &\quad - \beta(\mathcal{C})\langle c(z^{\mathrm{EQ}}), z^{\mathrm{EQ}}\rangle \qquad (16)\\ &\leq (1+\beta(\mathcal{C}))\langle c(z^{\mathrm{EQ}}), (1+\beta(\mathcal{C}))^{-1}q^{\mathrm{OPT}}\rangle \\ &\quad - \beta(\mathcal{C})\langle c(z^{\mathrm{EQ}}), z^{\mathrm{EQ}}\rangle \qquad (17)\\ &\leq C(q^{\mathrm{OPT}}) \qquad (18)\end{aligned}$$

where (17) uses (15) and (18) uses (14).

B. Proof of Lemma 2

We prove Lemma 2, an intermediary step for bounding the PoA. We use Lemma 1 and bound $\beta(\mathcal{C}_\sigma^{\text{AGG}})$, where $\mathcal{C}_\sigma^{\text{AGG}}$ denotes the set of aggregate cost functions with maximum polynomial degree σ. First, we will show that

$$\beta(\mathcal{C}_\sigma^{\text{AGG}}) \leq \max_{i\in[N], f_i, g_i \geq 0} \frac{f_i}{g_i}\left(1 - \left(\frac{f_i}{g_i}\right)^{\sigma_i}\right).$$

For the remainder of the proof of the lemma, we drop the aggregate superscript from aggregate cost functions. Then

$$\beta(\mathcal{C}_\sigma^{\text{AGG}}) := \sup_{c\in\mathcal{C}_\sigma^{\text{AGG}}, g\in\mathcal{X}} \max_{f\in\mathbb{R}_{\geq 0}^{[N]}} \frac{\langle c(g) - c(f), f\rangle}{\langle c(g), g\rangle}$$
$$\leq \sup_{c_i\in\mathcal{C}_\sigma^{\text{AGG}}, g_i\geq 0} \max_{f_i \geq 0} \frac{f_i}{g_i}\left(1 - \frac{c_i(f_i)}{c_i(g_i)}\right).$$

Note that this expression equals 0 by definition when $f_i = g_i = 0$; the supremum of this expression is therefore at least 0. Moreover, consider the case that $g_i < f_i$. Since c_i is nondecreasing, $(1 - \frac{c_i(f_i)}{c_i(g_i)}) \leq 0$. Since $g_i \geq 0$ and $f_i \geq 0$, the full expression is then bounded from above by 0, and is equal to 0 when $g_i = 0$. Since we are maximizing this expression, we can rule out $g_i < f_i$ and only consider $g_i \geq f_i$.

We will show that $\beta(\mathcal{C}_\sigma^{\text{AGG}}) \leq \max_{i\in[N], f_i, g_i\geq 0} \frac{f_i}{g_i}(1 - (\frac{f_i}{g_i})^{\sigma_i})$ by showing that $\frac{c_i(f_i)}{c_i(g_i)} \geq (\frac{f_i}{g_i})^{\sigma_i}$ for either capacity model. We will show this for a road in which $\bar{h}_i \leq h_i$, although with the alterations discussed above, the same can be done for a road on which $\bar{h}_i > h_i$.

As mentioned above, we can neglect the case in which $g_i < f_i$. Because of this, to bound the expression, we only need to bound the following six cases:

1) capacity model 1, $g_i \leq x_i^{\text{EQ}}$ and $f_i \leq x_i^{\text{EQ}}$;
2) capacity model 1, $g_i > x_i^{\text{EQ}}$ and $f_i \leq x_i^{\text{EQ}}$;
3) capacity model 1, $g_i > x_i^{\text{EQ}}$ and $f_i > x_i^{\text{EQ}}$;
4) capacity model 2, $g_i \leq x_i^{\text{EQ}}$ and $f_i \leq x_i^{\text{EQ}}$;
5) capacity model 2, $g_i > x_i^{\text{EQ}}$ and $f_i \leq x_i^{\text{EQ}}$;
6) capacity model 2, $g_i > x_i^{\text{EQ}}$ and $f_i > x_i^{\text{EQ}}$.

We will show that in all cases, $\frac{c_i(f_i)}{c_i(g_i)} \geq (\frac{f_i}{g_i})^{\sigma_i}$.

In the first case

$$\frac{c_{i,1}(f_i)}{c_{i,1}(g_i)} = \frac{1 + \rho_i\left(\frac{h_i f_i}{d_i}\right)^{\sigma_i}}{1 + \rho_i(\frac{h_i g_i}{d_i})^{\sigma_i}} \geq \left(\frac{f_i}{g_i}\right)^{\sigma_i}$$

where the inequality follows from $\frac{c_{i,1}(f_i)}{c_{i,1}(g_i)} \leq 1$. In the following cases, we perform the same operation without comment.

In the second case

$$\frac{c_{i,1}(f_i)}{c_{i,1}(g_i)} \geq \left(\frac{h_i f_i}{\bar{h}_i g_i + (h_i - \bar{h}_i)x_i^{\text{EQ}}}\right)^{\sigma_i} \geq \left(\frac{h_i f_i}{\bar{h}_i g_i}\right)^{\sigma_i}$$
$$\geq \left(\frac{f_i}{g_i}\right)^{\sigma_i} \tag{19}$$

where (19) follows from $\bar{h}_i \leq h_i$.

In the third case

$$\frac{c_{i,1}(f_i)}{c_{i,1}(g_i)} \geq \left(\frac{\bar{h}_i f_i + (h_i - \bar{h}_i)x_i^{\text{EQ}}}{\bar{h}_i g_i + (h_i - \bar{h}_i)x_i^{\text{EQ}}}\right)^{\sigma_i} \geq \left(\frac{f_i}{g_i}\right)^{\sigma_i}$$

where the final inequality follows from $h_i \geq \bar{h}_i$ and $f_i \leq g_i$.

The fourth case is equivalent to the first case. In the fifth case

$$\frac{c_{i,2}(f_i)}{c_{i,2}(g_i)} \geq \left(\frac{h_i f_i g_i}{h_i(g_i)^2 - (h_i - \bar{h}_i)(g_i - x_i^{\text{EQ}})^2}\right)^{\sigma_i}$$
$$\geq \left(\frac{h_i f_i g_i}{h_i(g_i)^2}\right)^{\sigma_i} = \left(\frac{f_i}{g_i}\right)^{\sigma_i} \tag{20}$$

where (20) results from $h_i \geq \bar{h}_i$.

In the sixth case

$$\frac{c_{i,2}(f_i)}{c_{i,2}(g_i)} \geq \left(\frac{h_i(f_i)^2 - (h_i - \bar{h}_i)(f_i - x_i^{\text{EQ}})^2}{h_i(g_i)^2 - (h_i - \bar{h}_i)(g_i - x_i^{\text{EQ}})^2} \times \frac{g_i}{f_i}\right)^{\sigma_i}$$
$$\geq \left(\frac{h_i(f_i)^2}{h_i(g_i)^2} \times \frac{g_i}{f_i}\right)^{\sigma_i} = \left(\frac{f_i}{g_i}\right)^{\sigma_i} \tag{21}$$

where (21) results from $(\frac{f_i}{g_i})^2 \geq (\frac{f_i - x_i^{\text{EQ}}}{g_i - x_i^{\text{EQ}}})^2$, as $g_i \geq f_i$.

As we have shown that $\frac{c_i(f_i)}{c_i(g_i)} \geq (\frac{f_i}{g_i})^{\sigma_i}$ in all cases, we now find that

$$\beta(\mathcal{C}_\sigma^{\text{AGG}}) \leq \max_{i\in[N], f_i, g_i\geq 0} \frac{f_i}{g_i} - \left(\frac{f_i}{g_i}\right)^{\sigma_i + 1}.$$

As this expression is concave with respect to f_i, to maximize this with respect to f_i, we set the derivative of this expression with respect to f_i to 0

$$\frac{1}{g_i} - (\sigma + 1)\frac{(f_i^*)^\sigma}{(g_i)^{\sigma+1}} = 0 \Rightarrow f_i^* = (g_i)(\sigma + 1)^{-\frac{1}{\sigma}}.$$

Plugging this in

$$\beta(\mathcal{C}_\sigma^{\text{AGG}}) \leq (\sigma + 1)^{-\frac{1}{\sigma}}\left(1 - \frac{1}{\sigma + 1}\right) = \sigma(\sigma + 1)^{-\frac{\sigma+1}{\sigma}}.$$

This, combined with Lemma 1 and the definition of $\xi(\sigma)$, completes the proof of the lemma.

C. Proof of Lemma 3

We prove the final intermediary step, bounding the difference in social cost between the optimal routing for our intermediary latency functions and the optimal routing for our original latency function. As before, let $z = [x_1 \;\; y_1 \;\; x_2 \;\; y_2 \;\; \ldots \;\; x_N \;\; y_N]^T$. Then

$$k^\sigma c_i(x_i^{\text{OPT}}, y_i^{\text{OPT}})$$
$$\geq \max\left(c_i(x_i^{\text{OPT}} + y_i^{\text{OPT}}, 0), c_i(0, x_i^{\text{OPT}} + y_i^{\text{OPT}})\right)$$
$$\geq c_i^{\text{AGG}}(x_i^{\text{OPT}} + y_i^{\text{OPT}})$$

and by definition of f^{OPT}, $C^{\text{AGG}}(f) \geq C^{\text{AGG}}(f^{\text{OPT}})$ for any feasible vector f with $\sum_{i\in[N]} f_i = \sum_{i\in[N]} f_i^{\text{OPT}} = \sum_{i\in[2N]} z_i$, so

$$k^\sigma C(z^{\text{OPT}}) \geq \sum_{i\in[N]} (x_i^{\text{OPT}} + y_i^{\text{OPT}}) c_i^{\text{AGG}}(x^{\text{OPT}} + y^{\text{OPT}})$$
$$\geq C^{\text{AGG}}(f^{\text{OPT}}).$$

D. Proof of Lemma 4

This lemma and the following one together prove Theorem 2. Using (11)

$$\beta(c,q) = \max_{z\in\mathbb{R}^{2N}_{\geq 0}} \frac{\sum_{i\in[N]} t_i\rho_i \left[\left(\frac{v_i+w_i}{m_i(v_i,w_i)}\right)^{\sigma_i} - \left(\frac{x_i+y_i}{m_i(x_i,y_i)}\right)^{\sigma_i}\right](x_i+y_i)}{\sum_{i\in[N]} t_i\left[1+\rho_i\left(\frac{v_i+w_i}{m_i(v_i,w_i)}\right)^{\sigma_i}\right](v_i+w_i)}$$

$$\leq \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{\rho_i\left[\left(\frac{v_i+w_i}{m_i(v_i,w_i)}\right)^{\sigma_i} - \left(\frac{x_i+y_i}{m_i(x_i,y_i)}\right)^{\sigma_i}\right](x_i+y_i)}{[1+\rho_i(\frac{v_i+w_i}{m_i(v_i,w_i)})^{\sigma_i}](v_i+w_i)} \tag{22}$$

$$\leq \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{\left[\left(\frac{v_i+w_i}{m_i(v_i,w_i)}\right)^{\sigma_i} - \left(\frac{x_i+y_i}{m_i(x_i,y_i)}\right)^{\sigma_i}\right](x_i+y_i)}{\left(\frac{v_i+w_i}{m_i(v_i,w_i)}\right)^{\sigma_i}(v_i+w_i)}$$

$$= \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i}\left(1-\left(\frac{m_i(v_i,w_i)(x_i+y_i)}{m_i(x_i,y_i)(v_i+w_i)}\right)^{\sigma_i}\right)$$

$$\leq \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i}\left(1-\left(\frac{m_i(v_i,w_i)(x_i+y_i)}{m_i(x_i,y_i)(v_i+w_i)}\right)^{\sigma}\right) \tag{23}$$

where the terms of the denominator being nonnegative imply (22), since a term in the summation in the numerator that is negative does not need to be accounted for in the upper bound. Then, $\beta(c,q)\geq 0$ implies (23), allowing us to consider only the maximum allowable degree of polynomial.

E. Proof of Lemma 5

For capacity model 1

$$\beta(c,q) \tag{24}$$

$$\leq \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i} \times\left(1-\left(\frac{\left(h_i-(h_i-\bar{h}_i)\left(\frac{y_i}{x_i+y_i}\right)\right)(x_i+y_i)}{\left(h_i-(h_i-\bar{h}_i)\left(\frac{w_i}{v_i+w_i}\right)\right)(v_i+w_i)}\right)^{\sigma}\right)$$

$$= \max_{i\in[N],z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i}\left(1-\left(\frac{h_ix_i+\bar{h}_iy_i}{h_iv_i+\bar{h}_iw_i}\right)^{\sigma}\right)$$

$$\leq \max_{z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i}\left(1-\left(\frac{kx_i+y_i}{kv_i+w_i}\right)^{\sigma}\right)$$

$$:= \max_{z_i\in\mathbb{R}^2_{\geq 0}} f(x_i,y_i,v_i,w_i). \tag{25}$$

In (25), we use the Definition 2 of the maximum degree of asymmetry. For ease of notation, we drop the subscripts for $f(x,y,v,w)$.

We now investigate this expression more closely, and show that the maximum of this expression with respect to x and y occurs at either $x=0$ or $y=0$. We do this by showing there are no critical points with $x>0$ and $y>0$, and that outside of a finite region, the function is decreasing with respect to both x and y.

First, we show that there exist no critical points, meaning points for which $\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}y}=0$, for $k>1$. We have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{kx+y-(\frac{kx+y}{kv+w})^{\sigma}(kx+y+k\sigma(x+y))}{(v+w)(kx+y)}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} = \frac{kx+y-(\frac{kx+y}{kv+w})^{\sigma}(kx+y+\sigma(x+y))}{(v+w)(kx+y)}.$$

Since $k>1$, we conclude that $\frac{\mathrm{d}f}{\mathrm{d}x}\neq\frac{\mathrm{d}f}{\mathrm{d}y}$ for $x>0$ and $y>0$.

To show the second component, we see that

$$\frac{\mathrm{d}f}{\mathrm{d}x} \leq \frac{1}{v+w}\left(1-\left(\frac{kx+y}{kv+w}\right)^{\sigma}\right) \quad \text{and}$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} \leq \frac{1}{v+w}\left(1-\left(\frac{kx+y}{kv+w}\right)^{\sigma}\right).$$

Therefore, for the region $y>kv+w-kx$, the function is decreasing with x and y.

These two facts together imply that the maximum of f in the first quadrant lies on either the x- or y-axis. Checking these two candidate functions

$$f(x,0,v,w) = \frac{x(1-(\frac{kx}{kv+w})^{\sigma})}{v+w}$$

$$f(0,y,v,w) = \frac{y(1-(\frac{y}{kv+w})^{\sigma})}{v+w}.$$

These functions are concave with respect to x and y, with minima at $x^*=(\frac{1}{\sigma+1})^{1/\sigma}\frac{kv+w}{k}$ and $y^*=(\frac{1}{\sigma+1})^{1/\sigma}(kv+w)$ respectively. When plugging these in, we find that the solution along the y-axis is greater, so

$$\max_{x,y\geq 0} f(x,y,v,w) = \sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma}\frac{kv+w}{v+w}.$$

We then find that when restricted to capacity model 1

$$\beta(\mathcal{C}_{k,\sigma}) \leq \max_{v,w\leq 0} \sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma}\frac{kv+w}{v+w}$$

$$\leq k\sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma}$$

$$= k\xi(\sigma).$$

For capacity model 2

$$\beta(c,q)$$

$$\leq \max_{z_i\in\mathbb{R}^2_{\geq 0}} \frac{x_i+y_i}{v_i+w_i} - \left(\frac{kx_i^2+2kx_iy_i+y_i^2}{kv_i^2+2kv_iw_i+w_i^2}\right)^{\sigma}\left(\frac{v_i+w_i}{x_i+y_i}\right)^{\sigma-1}$$

$$:= g(x_i,y_i,v_i,w_i).$$

We again find that $k>1 \Rightarrow \frac{\mathrm{d}g}{\mathrm{d}x}\neq\frac{\mathrm{d}g}{\mathrm{d}y}$. Furthermore

$$\frac{\mathrm{d}f}{\mathrm{d}x} \leq \frac{1}{v+w}\left(1-\left(\left(\frac{v+w}{x+y}\right)\frac{kx^2+2kxy+y^2}{kv^2+2kvw+w^2}\right)^{\sigma}\right.$$

$$\frac{\mathrm{d}f}{\mathrm{d}y} \leq \frac{1}{v+w}\left(1-\left(\left(\frac{v+w}{x+y}\right)\frac{kx^2+2kxy+y^2}{kv^2+2kvw+w^2}\right)^{\sigma}\right.$$

so

$$\frac{kx^2+2kxy+y^2}{x+y} < \frac{kv^2+2kvw+w^2}{v+w}$$
$$\Rightarrow \frac{\mathrm{d}f}{\mathrm{d}x} < 0 \text{ and } \frac{\mathrm{d}f}{\mathrm{d}y} < 0 \,.$$

Using the same reasoning as above, we now just search the x-axis and y-axis. As above, $g(x,0,v,w)$ is concave with respect to x and $g(0,y,v,w)$ is concave with respect to y, with maxima at $x = (\frac{1}{\sigma+1})^{1/\sigma}\frac{kv^2+2kvw+w^2}{k(v+w)}$ and $y = (\frac{1}{\sigma+1})^{1/\sigma}\frac{kv^2+2kvw+w^2}{v+w}$, respectively. Comparing these, we find the maximum is the latter, so

$$\max_{x,y\geq 0} g(x,y,v,w) = \sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma}\frac{kv^2+2kvw+w^2}{(v+w)^2} \,.$$

Then, for capacity model 2

$$\beta(\mathcal{C}_{k,\sigma}) \leq \max_{v,w\leq 0} \sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma}\frac{kv^2+2kvw+w^2}{(v+w)^2}$$
$$\leq \sigma\left(\frac{1}{\sigma+1}\right)^{1+1/\sigma} k$$
$$= k\xi(\sigma) \,.$$

Together, this shows that regardless of capacity model, $\beta(\mathcal{C}_{k,\sigma}) \leq k\xi(\sigma)$. The application of Lemma 1 completes the proof.

References

[1] *Beyond Traffic 2045: Trends and Choices*. Washington, DC, USA: US Dept. Transp., 2015.

[2] J. Lioris, R. Pedarsani, F. Y. Tascikaraoglu, and P. Varaiya, "Platoons of connected vehicles can double throughput in urban roads," *Transp. Res. C, Emerg. Technol.*, vol. 77, pp. 292–305, Apr. 2017.

[3] A. Askari, D. A. Farias, A. A. Kurzhanskiy, and P. Varaiya, "Modeling and measuring impact of adaptive and cooperative adaptive cruise control on throughput of signalized intersections," 2016, *arXiv:1611.08973*.

[4] T. Roughgarden and É. Tardos, "How bad is selfish routing?" *J. ACM*, vol. 49, pp. 236–259, Mar. 2002.

[5] T. Roughgarden, "The price of anarchy is independent of the network topology," *J. Comput. Syst. Sci.*, vol. 67, pp. 341–364, 2003.

[6] T. Roughgarden, *Selfish Routing and the Price of Anarchy*, vol. 174. Cambridge, MA, USA: MIT Press, 2005.

[7] J. R. Correa, A. S. Schulz, and N. E. Stier-Moses, "Selfish routing in capacitated networks," *Math. Oper. Res.*, vol. 29, pp. 739–978, 2004.

[8] J. R. Correa, A. S. Schulz, and N. E. Stier-Moses, "A geometric approach to the price of anarchy in nonatomic congestion games," *Games Econ. Behav.*, vol. 64, pp. 457–469, 2008.

[9] G. Perakis, "The price of anarchy under nonlinear and asymmetric costs," *Math. Oper. Res.*, vol. 32, pp. 497–768, 2007.

[10] C. K. Chau and K. M. Sim, "The price of anarchy for non-atomic congestion games with symmetric cost maps and elastic demands," *Oper. Rese. Lett.*, vol. 31, pp. 327–334, 2003.

[11] D. A. Lazar, R. Pedarsani, K. Chandrasekher, and D. Sadigh, "Maximizing road capacity using cars that influence people," in *Proc. IEEE Conf. Decis. Control*, 2018.

[12] D. A. Lazar, S. Coogan, and R. Pedarsani, "Capacity modeling and routing for traffic networks with mixed autonomy," in *Proc. IEEE Conf. Decis. Control*, 2017, pp. 5678–5683.

[13] D. A. Lazar, S. Coogan, and R. Pedarsani, "The price of anarchy for transportation networks with mixed autonomy," in *Proc. IEEE Amer. Control Conf.*, 2018, pp. 6359–6365.

[14] S. C. Dafermos and F. T. Sparrow, "The traffic assignment problem for a general network," *J. Res. Nat. Bureau Standards B*, vol. 73B, pp. 91–118, 1969.

[15] J. R. Correa and N. E. Stier-Moses, "Wardrop equilibria, " in, *Wiley Encyclopedia of Operations Research and Management Science*. Hoboken, NJ, USA: Wiley, 2011.

[16] A. De Palma and Y. Nesterov, "Optimization formulations and static equilibrium in congested transportation networks," THEMA (THéorie Economique, Modélisation et Applications), Université de Cergy-Pontoise, Cergy, France, THEMA Working Papers 97-17, 1997.

[17] M. J. Smith, "The existence, uniqueness and stability of traffic equilibria," *Transp. Res. B, Methodological*, vol. 13, pp. 295–304, 1979.

[18] S. Dafermos, "Traffic equilibrium and variational inequalities," *Transp. Sci.*, vol. 14, pp. 1–105, 1980.

[19] M. Florian and D. Hearn, "Network equilibrium models and algorithms," *Handbooks in Operations Research and Management Science*, vol. 7(C), pp. 673–762, 1995.

[20] S. C. Dafermos, "The traffic assignment problem for multiclass-user transportation networks," *Transp. Sci.*, vol. 6, pp. 73–87, 1972.

[21] D. W. Hearn, S. Lawphongpanich, and S. Nguyen, "Convex programming formulations of the asymmetric traffic assignment problem," *Transp. Res. B, Methodological*, vol. 18, pp. 357–365, 1984.

[22] M. Florian, "A traffic equilibrium model of travel by car and public transit modes," *Transp. Sci.*, vol. 11, pp. 107–195, 1977.

[23] F. Farokhi, W. Krichene, M. Alexandre, and K. H. Johansson, "When do potential functions exist in heterogeneous routing games?" KTH Roy. Inst. Technol., Stockholm, Sweden, Tech. Rep. TRITA-EE 2014:009, 2014.

[24] S. E. Shladover, "Longitudinal control of automated guideway transit vehicles within platoons," *J. Dyn. Syst., Meas., Control*, vol. 100, 1978, pp. 302–310.

[25] S. Darbha and K. Rajagopal, "Intelligent cruise control systems and traffic flow stability," *Transp. Res. C, Emerg. Technol.*, vol. 7, no. 6, pp. 329–352, 1999.

[26] J. Yi and R. Horowitz, "Macroscopic traffic flow propagation stability for adaptive cruise controlled vehicles," *Transp. Res. C, Emerg. Technol.*, vol. 14, pp. 81–95, Apr. 2006.

[27] R. Pueboobpaphan and B. van Arem, "Driver and vehicle characteristics and platoon and traffic flow stability: Understanding the relationship for design and assessment of cooperative adaptive cruise control," *Transp. Res. Rec., J. Transp. Res. Board*, vol. 2189, pp. 89–97, 2010.

[28] G. Orosz, "Connected cruise control: Modelling, delay effects, and nonlinear behaviour," *Veh. Syst. Dyn.*, vol. 54, pp. 1147–1176, 2016.

[29] S. Cui, B. Seibold, R. Stern, and D. B. Work, "Stabilizing traffic flow via a single autonomous vehicle: Possibilities and limitations," in *Proc. IEEE Intell. Veh. Symp.*, 2017, pp. 1336–1341.

[30] R. E. Stern *et al.*, "Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments," *Transp. Res. C, Emerg. Technol.*, vol. 89, pp. 205–221, 2018.

[31] C. Wu, A. Kreidieh, E. Vinitsky, and A. M. Bayen, "Emergent behaviors in mixed-autonomy traffic," in *Proc. Conf. Robot Learn.*, 2017, vol. 78, pp. 398–407.

[32] C. Wu, A. M. Bayen, and A. Mehta, "Stabilizing traffic with autonomous vehicles," in *Proc. Int. Conf. Robot. Autom.*, 2018, pp. 6012–6018.

[33] M. Motie and K. Savla, "Throughput analysis of a horizontal traffic queue under safe car following models," in *Proc. IEEE Conf. Decis. Control*, 2016, pp. 6771–6776.

[34] L. Jin, M. Čičić, S. Amin, and K. H. Johansson, "Modeling impact of vehicle platooning on highway congestion: A fluid queuing approach," in *Proc. Int. Conf. Hybrid Syst., Comput. Control*, 2018, pp. 237–246.

[35] K.-Y. Liang, J. Mårtensson, and K. H. Johansson, "Heavy-duty vehicle platoon formation for fuel efficiency," *IEEE Trans. Intell. Transp. Syst.*, vol. 17, no. 4, pp. 1051–1061, Apr. 2016.

[36] A. Adler, D. Miculescu, and S. Karaman, "Optimal policies for platooning and ride sharing in autonomy-enabled transportation," in *Algorithmic Foundations of Robotics XII*. New York, NY, USA: Springer, 2020, pp. 848–863.

[37] V. Turri, B. Besselink, and K. H. Johansson, "Cooperative look-ahead control for fuel-efficient and safe heavy-duty vehicle platooning," *IEEE Trans. Control Syst. Technol.*, vol. 25, no. 1, pp. 12–28, Jan. 2017.

[38] S. van de Hoef, K. H. Johansson, and D. V. Dimarogonas, "Fuel-efficient en route formation of truck platoons," *IEEE Trans. Intell. Transp. Syst.*, vol. 19, no. 1, pp. 102–112, Jan. 2018.

[39] S. Sivaranjani, Y.-S. Wang, V. Gupta, and K. Savla, "Localization of disturbances in transportation systems," in *Proc. IEEE Conf. Decis. Control*, 2015, pp. 3439–3444.

[40] F. Rossi, R. Zhang, Y. Hindy, and M. Pavone, "Routing autonomous vehicles in congested transportation networks: Structural properties and coordination algorithms," *Auton. Robots*, vol. 42, pp. 1427–1442, 2018.
[41] A. Askari, D. A. Farias, A. A. Kurzhanskiy, and P. Varaiya, "Effect of adaptive and cooperative adaptive cruise control on throughput of signalized arterials," in *Proc. IEEE Intell. Veh. Symp.*, 2017, pp. 1287–1292.
[42] D. Chen, S. Ahn, M. Chitturi, and D. A. Noyce, "Towards vehicle automation: Roadway capacity formulation for traffic mixed with regular and automated vehicles," *Transp. Res. B, Methodological*, vol. 100, pp. 196–221, Jun. 2017.
[43] N. Mehr and R. Horowitz, "Can the presence of autonomous vehicles worsen the equilibrium state of traffic networks?" in *Proc. IEEE Conf. Decis. Control*, 2018, pp. 1788–1793.
[44] M. Papageorgiou, J.-M. Blosseville, and H. Hadj-Salem, "Macroscopic modelling of traffic flow on the Boulevard Périphérique in Paris," *Transp. Res. B, Methodological*, vol. 23, pp. 29–47, 1989.
[45] E. Bıyık, D. A. Lazar, R. Pedarsani, and D. Sadigh, "Altruistic autonomy: Beating congestion on shared roads," in *Algorithmic Foundations of Robotics XIII*. New York, NY, USA: Springer, 2018, pp. 887–904.
[46] "Bureau of public roads traffic assignment manual," US Dept. Commerce, Washington, DC, USA, 1964.
[47] Y. Sheffi, *Urban Transportation Networks*, vol. 6. Englewood Cliffs, NJ, USA: Prentice-Hall, 1985.

Daniel A. Lazar received the B.Sc. degree in electrical engineering from Washington University in St. Louis, St. Louis, MO, USA. He is currently working toward the Ph.D. degree with the Department of Electrical and Computer Engineering at The University of California, Santa Barbara, Santa Barbara, CA, USA, in 2014.

After graduating he spent two years working in the field of wireless communications. His research focuses on control of transportation networks.

Sam Coogan (Member, IEEE) received the B.S. degree in electrical engineering from Georgia Institute of Technology, (Georgia Tech), Atlanta, GA, USA, in 2010, and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Berkeley, Berkeley, CA, USA, in 2012 and 2015, respectively.

He is currently an Assistant Professor with Georgia Tech with a joint appointment in the School of Electrical and Computer Engineering and the School of Civil and Environmental Engineering. Prior to joining Georgia Tech in 2017, he was an Assistant Professor with the Electrical Engineering Department, University of California, Los Angeles, Los Angeles, CA, USA, from 2015 to 2017. His research is in the area of dynamical systems and control and focuses on developing scalable tools for verification and control of networked cyber-physical systems with an emphasis on transportation systems.

Dr. Coogan was the recipient of the Outstanding Paper Award from the IEEE Transactions on Control of Network Systems in 2017, a CAREER Award from the National Science Foundation in 2018, a Young Investigator Award from the Air Force Office of Scientific Research in 2018, and the Donald P. Eckman Award from the American Automatic Control Council in 2020.

Ramtin Pedarsani (Member, IEEE) received the B.Sc. degree from the University of Tehran, Tehran, Iran, in 2009, the M.Sc. degree at École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland, in 2011, and the Ph.D. degree in electrical engineering and computer sciences from The University of California, Berkeley, Berkeley, CA, USA, in 2015.

He is currently an Assistant Professor with the Electrical Engineering Department, Santa Barbara, CA, USA. His research interests include machine learning, optimization, information theory, game theory, and transportation systems.

Dr. Pedarsani was the recipient of the Communications Society and Information Theory Society Joint Paper Award in 2020, the Best Paper Award in the IEEE International Conference on Communications in 2014, and the NSF CRII Award in 2017.