A SIMPLE BOUND ON THE BER OF THE MAP DECODER FOR MASSIVE MIMO SYSTEMS

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ABSTRACT

The deployment of massive MIMO systems has revived much of the interest in the study of the large-system performance of multiuser detection systems. In this paper, we prove a non-trivial upper bound on the bit-error rate (BER) of the MAP detector for BPSK signal transmission and equal-power condition. The proof is simple and relies on Gordon’s comparison inequality. Interestingly, we show that under the assumption that Gordon’s inequality is tight, the resulting BER prediction matches that of the replica method when the ratio of receive to transmit antennas exceeds 1.46. Also, we prove that the replica prediction matches the matched filter lower bound (MBF) at high-SNR. We corroborate our results by numerical evidence.

Index Terms— massive mimo, large-system analysis, JO detector, Gaussian process inequalities, replica method.

1. INTRODUCTION

Massive multiple-input multiple-output (MIMO) systems, where the base station is equipped with hundreds of thousands of antennas, promise improved spectral efficiency, coverage and range compared to small-scale systems. As such, they are widely believed to play an important role in 5G wireless communication systems [1]. Their deployment has revived much of the recent interest for the study of multiuser detection schemes in high-dimensions, e.g., [2][3][4][5].

A large host of exact and heuristic detection schemes have been proposed over the years. Decoders such as zero-forcing (ZF) and linear minimum mean square error (LMMSE) have inferior performances [6], and others such as local neighborhood search-based methods [7] and lattice reduction-aided (LRA) decoders [8, 9] are often difficult to precisely characterize. Recently, [10] studied in detail the performance of the box-relaxation optimization (BRO), which is a natural convex relaxation of the maximum a posteriori (MAP) decoder, and which allows one to recover the signal via efficient convex optimization followed by hard thresholding. In particular, [10] precisely quantifies the performance gain of the BRO compared to the ZF and the LMMSE. Despite such gains, it remains unclear the degree of sub-optimality of the convex relaxation compared to the combinatorial MAP detector. The challenge lies in the complexity of analyzing the latter. In particular, known predictions of the performance of the MAP detector are known only via the (non-rigorous) replica method from statistical physics [11][12][13].

In this paper, we derive a simple, yet non-trivial, upper bound on the bit error rate (BER) of the MAP detector. We show (in a precise manner) that our bound is approximately tight at high-SNR, since it is close to the matched filter lower bound (MBF). Our numerical simulations verify our claims and further include comparisons to the replica prediction and to the BER of the BRO. Our proof relies on Gordon’s Gaussian comparison inequality [14]. While Gordon’s inequality is not guaranteed to be tight, we make the following possibly interesting and useful observation. If Gordon’s inequality was asymptotically tight in our setting, then its BER prediction would match the prediction of the replica method (under replica-symmetry) for all values of the SNR and ratio of receive to transmit antennas larger than 1.46.

2. SETTING

We assume a real Gaussian wireless channel, additive Gaussian noise and and uncoded modulation scheme. Also, for concreteness, we focus on the binary-phase-shift-keying (BPSK) transmission; but, the techniques naturally extend to other constellations. Formally, we seek to recover an $n$-dimensional BPSK vector $\mathbf{x}_0 \in \{\pm 1\}^n$ from the noisy MIMO relation $y = A\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m \times n}$ is the channel matrix (assumed to be known) with entries iid $\mathcal{N}(0, 1/n)$. and $\mathbf{z} \in \mathbb{R}^m$ the noise vector with entries iid $\mathcal{N}(0, 1)$. The normalization is such that the reciprocal of the noise variance $\sigma^2$ is equal to the SNR, i.e., $\text{SNR} = 1/\sigma^2$. The performance metric of interest is the bit-error rate (BER). For a detector which output $\hat{\mathbf{x}}$ as an estimate to $\mathbf{x}_0$, the BER is formally defined as

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \{\pm 1\}^n} \| y - A\mathbf{x} \|_2.$$  

We state our results in the large-system limit where $m, n \to \infty$, while the ratio of receive to transmit antennas is maintained fix to $\delta = m/n$. It is well known that in the worst case, solving (1) is an NP-hard combinatorial optimization problem in the number of users [15]. The BRO is a relaxation of (1) to an efficient convex quadratic program, namely $\hat{\mathbf{x}} = \text{sign} (\arg \min_{\|z\|_{\leq 1}} \| y - A\mathbf{x} \|_2)$. Its performance in the large-system limit has been recently analyzed in [10]. Regarding the performance of (1), Tse and Verdu [16] have shown that the BER approaches zero in the zero-noise limit. Beyond that, there is a now long literature that studied (1) using the replica method, developed in the field of spin-glasses. The use of the method in the context of multiuser detection was pioneered by Tanaka [11] and several extensions have followed up since then [13][17]. The replica method has the remarkable ability to yield highly nontrivial predictions, which in certain problem instances they can be formally shown to be correct (e.g., [18][19][20]). However, it is still lacking a complete rigorous mathematical justification. Specifically, to the best of our knowledge, justifying (or disproving) its prediction about the BER of (1) remains an open problem.

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† The proof of our main result Theorem [14] reveals that a non-asymptotic bound is also possible with only slight more effort.
3. RESULTS

3.1. Upper bound

This section contains our main result: a simple upper bound on the BER of (1). First, we introduce some useful notation. We say that an event $\mathcal{E}(n)$ holds with probability approaching 1 (wpa 1) if $\lim_{n \to \infty} \Pr(\mathcal{E}(n)) = 1$. Let $X_n$, a sequence of random variables indexed by $n$ and $X$ some constant. We write $X_n \overset{P}{=} X$ and $X_n \overset{p}{\leq} X$, if for all $\epsilon > 0$ the events $\{|X_n - X| \leq \epsilon\}$ and $\{|X_n \leq X + \epsilon\}$ hold wpa 1. Finally, let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the density of a standard normal variable, $Q(x) = \int_x^\infty \phi(\tau) d\tau$ its tail function and $Q^{-1}()$ the inverse of $Q()$.

**Theorem 3.1.** Fix constant noise variance $\sigma^2 > 0$ and $\delta > 0$. Let BER denote the bit-error-rate of the MAP detector in (1) for fixed but unknown $x_0 \in \{-1\}^n$. Define the function $\ell(\theta) : (0, 1) \to \mathbb{R}$:

$$\ell(\theta) := \sqrt{\delta} \sqrt{4\delta + \sigma^2} - \sqrt{\frac{2}{\pi}} e^{-\frac{(Q^{-1}(\theta))^2}{2}},$$

and let $\theta_0 \in (0, 1)$ be the largest solution to the equation $\ell(\theta) = \sigma \sqrt{\delta}$. Then, in the limit of $m, n \to \infty$, $m/n = \delta$, it holds $\operatorname{BER} \overset{p}{\leq} \theta_0$.

Proposition A.1 in the Appendix gathers several useful properties of the function $\ell()$. Notice that $\ell(0) > \ell(0^+) = \sqrt{\delta} \sigma$. Also, $\ell()$ is continuous and $\ell'(0^+) < 0$. Thus, $\theta_0$ in Theorem 3.1 is well-defined. Moreover, we show in Proposition A.1(iii) that if $\delta > 1.46$, then $\theta_0$ is the unique solution of the equation $\ell(\theta) = \sigma \sqrt{\delta}$ in $(0, 1)$, for all $\sigma > 0$. Numerical investigations shown in Figure 1 suggest that it is possible to relax the requirement $\delta > 1.46$.

**Remark 1 (On the function $\ell(\theta)$).** Let us elaborate on the operational role of the function $\ell()$. We partition the feasible vectors $x \in \{-1\}^n$ according to their Hamming distance from the true vector $x_0$. Specifically, for $\theta \in [0, 1]$ let $S_\theta := \{x \in \{-1\}^n : \|x - x_0\| = \theta n\}$ and consider the optimal cost of (1) for each partition, i.e.,

$$c_\theta := \min_{x \in S_\theta} \frac{1}{\sqrt{n}} \|y - Ax\|_2.$$  \hspace{1cm} (3)

Evaluating the BER of (1) is of course closely related to understanding the typical behavior of $c_\theta(\theta)$ in the large system limit. The proof of the theorem in Section 3.3 shows that $\ell(\theta)$ is a high-probability lower bound on $c_\theta(\theta)$. Hence, we get an estimate on the BER via studying $\ell(\theta)$ instead. In this direction, note that the value $\sigma \sqrt{\delta}$, to which $\ell(\theta)$ is compared to, is nothing but the typical value of $c_\theta(\theta) = \frac{1}{\sqrt{n}} \|y - Ax\|_2 = \frac{\|y\|_2}{\sqrt{n}}$. Finally, we make the following note for later reference: the value $\inf_{\theta \in (0, 1)} \ell(\theta)$ is a high-probability lower bound to the optimal cost of (1), i.e., $c_\theta = \inf_{\theta \in (0, 1)} c_\theta(\theta)$. An illustration of these is included in Figure 2.

**Remark 2 (A genie lower bound).** A lower bound on the BER of (1) can be obtained easily via comparison to the idealistic matched filter bound (MFB), where one assumes that all $n - 1$, but 1, bits of $x_0$ are known. In particular, the MFB corresponds to the probability of error in detecting (say) $x_{0,1} \in \{-1\}$ from $\tilde{y} = x_0 + A_1 z$, where $\tilde{y} = y - \sum_{i=2}^n x_{0, i}$ is assumed known, and $A_1$ is the $i^{th}$ column of $A$ (eqn., the MFB is the error probability of an isolated transmission of only the first bit over the channel). It can be shown (e.g., [10]) that the MFB is given by $Q(\sqrt{\delta} \operatorname{SNR})$. Combining this with (a straightforward re-parametrization of) Theorem 3.1 it follows that the BER of (1) satisfies

$$Q(\sqrt{\delta} \operatorname{SNR}) \leq \operatorname{BER} \leq Q(\tau_0),$$

where $\tau_0 \in \mathbb{R}$ is the smallest solution to the equation $\sqrt{\delta} \operatorname{SNR} + 2\phi(\tau) = \sqrt{\delta} \operatorname{SNR} + 1 + 4\operatorname{SNR} Q(\tau)$.

**Remark 3 (Behavior at high-SNR).** In Proposition A.1(ii) we prove that at high values of $\operatorname{SNR} > 1$, $\theta_0 \to 0$. Thus, from Theorem 3.1 we have that BER approaches zero. This thinking confirms already that our upper bound is non-trivial. In fact, Proposition A.1(ii) shows an even stronger statement, namely, at $\operatorname{SNR} > 1$, $\theta_0 \approx Q(\sqrt{\delta} \operatorname{SNR} - \eta)$ for an arbitrarily small $\eta > 0$ (see the appendix for exact statement). This, when combined with the MFB in 2 shows that our upper bound is approximately tight at high-SNR.

**Remark 4 (Gordon’s comparison inequality).** The proof of Theorem 3.1 uses Gordon’s comparison inequality for Gaussian processes (also known as the Gaussian min-max Theorem (GMT)). In essence, the GMT provides a simple lower bound on the typical value of $c_\theta(\theta)$ in (3) in the large-system limit. Gordon’s inequality is classically used to establish (non)-asymptotic probabilistic lower bounds on the minimum singular value of Gaussian matrices [21], and has a number of other applications in high-dimensional convex geometry [22]. In general, the inequality is not tight. Recently, Stojnic [23] proved that the inequality is tight when applied to convex problems. The result was refined in [24] and has been successfully exploited to precisely analyze the BER of the BRO [10]. Unfortunately, the minimization in (3) is not convex, thus there are no immediate tightness guarantees regarding the lower bound $\ell(\theta)$. Interestingly, in Section 3.4 we show that if GMT was (asymptotically) tight then it would result in a prediction that matches the replica prediction in 25.

**Remark 5 (Replica prediction).** The replica prediction on the BER of (1) is given by [11] (based on the ansatz of replica-symmetry (RS))
as the solution to a system of nonlinear equations. In its general form the BER formula can exhibit complicated behavior, such as anomalous, non-monotonic dependence on the SNR, and there are regimes where the RS solution is unstable \([11, 12]\). However, for large enough values of \(\delta\), it is reported in [25] Eqn. (15) that such complications don’t arise and the prediction of the BER reduces to the solution to the following fixed-point equation:\(^2\)

\[
\theta = Q\left(\sqrt{\frac{\delta}{\sigma^2 + 4\theta}}\right) .
\]  

(5)

In Proposition A.1(ii), we prove that for \(\delta > 1.46\) Equation (5) has a unique solution for all values of the noise variance \(\sigma^2\). Moreover, Proposition A.1(v) shows that when \(\delta > 1.46\) and SNR \(\gg 1\), the unique solution of (5) satisfies \(\theta_0 = Q(\sqrt{\delta} \text{SNR})\). This suggests that at high-SNR, the BER of the (1) decreases at an optimal rate. We note that our numerical investigations suggest that the requirement \(\delta > 1.46\) in the statements above can possibly be relaxed (see also Section 3.2). We will investigate this further in the future.

3.2. Numerical Evaluations

Figure 3 includes numerical illustrations that help visualize the prediction of Theorem 3.1 and several of the remarks that followed. For two values of \(\delta\), we plot BER as a function of SNR = 1/\(\text{SNR}\). Each plot includes four curves: (i) the MFB; (ii) the solution to (5) corresponding to the replica prediction; (iii) the upper bound \(\theta_0\) of Theorem 3.1; (iv) the BER of the BRO according to [10, Thm. II]. We make several observations. First, it is interesting to note that our numerical investigations suggest that the requirement \(\delta > 1.46\) is not entirely to be blamed for this behavior, since the replica prediction experiences the same very one. There is no contradiction here: the MAP detector is not optimal for minimizing the BER (e.g., \([11\text{ Sec. 2]}\), thus it is likely that its convex approximation (aka, the BRO) shows better performance at low-SNR. On the other hand, for high-SNR the upper bound takes values significantly smaller than the BER of the BRO. This proves that at high-SNR the latter is still quite far from that of the combinatorial optimization it tries to approximate.

3.3. Proof Theorem 3.1

Let \(\hat{x}\) be the solution to (1). First, observe that \(\|\hat{x} - x_0\|^2 = 2n - 2(n - \sum_{i=1}^n 1_{\{x_i \neq x_0\}}) = 4n\) BER. Hence, we will prove that

\[
\|\hat{x} - x_0\|^2 \leq \alpha_0 =: 2\sqrt{\theta_0} \in (0, 1) .
\]  

(6)

Second, due to rotational invariance of the Gaussian measure we can assume without loss of generality that \(x_0 = 0\). For convenience, define the (normalized) error vector \(w := n^{-1/2}(x - \hat{x})\) and consider the set of feasible such vectors that do not satisfy (6), i.e.,

\[
S(\alpha_0) := \left\{ w \in \{-2/\sqrt{n}, 0\}^n : \|w\|_2 \geq \alpha_0 + \epsilon \right\} ,
\]

\(^2\)For the reader’s convenience we note the following mapping between notation here and [25]: \(\alpha \leftrightarrow \alpha, \sigma^2 \leftrightarrow \beta \epsilon^{-1}\) and BER \(\leftrightarrow (1 - m)/2\).

for some fixed (but arbitrary) \(\epsilon > 0\). Also, denote the (normalized) objective function of (1) as \(F(w) = F(w; x, G) := n^{-1/2}\|x - Gw\|_2\), where \(G = \sqrt{n}A\) has entries iid standard normal. With this notation, our goal towards establishing (6) is proving that there exists constant \(\eta := \eta(\epsilon) > 0\) such that the following holds wpa 1,

\[
\min_{w \in S(\alpha_0)} F(w) \geq \min_{w \in \{-2/\sqrt{n}, 0\}^n} F(w) + \eta .
\]  

(7)

Our strategy in showing the above is as follows.

First, we use Gordon’s inequality to obtain a high-probability lower bound on the left-hand side (LHS) of (7). In particular, it can be shown (see for example [10]) that the primary optimization (PO) in the (LHS) of (7) can be lower bounded with high-probability by an auxiliary optimization (AO) problem, which is defined as follows:

\[
\min_{w \in S(\alpha_0)} G(w; g, h) := \sqrt{\|w\|_2^2 + \sigma^2 \|g\|_2^2} - h^T w ,
\]  

(8)

where \(g \in \mathbb{R}^n\) and \(h \in \mathbb{R}^n\) have entries iid Gaussian \(N(0, 1/n)\). The AO can be easily simplified as follows

\[
\min_{1 \leq i \leq n} \sqrt{\alpha^2 + \sigma^2} \|g_i\|_2 - \frac{2}{\sqrt{n}} \sum_{i=1}^{(\sigma^2/4)n} h_i^+ ,
\]  

(9)
where, $h^1_i ≥ h^2_i ≥ \ldots ≥ h^4_i$ denotes the ordered statistics of the entries of $h$ and we have used the fact that for $w ∈ \{-2/\sqrt{\pi}, 0\}^n$ it holds $|w||2| = \alpha ⇔ |w||0| = \alpha^2/4$. Furthermore, note that $||g||2^2 \to 3/2$ and $\ell$ for any fixed $θ ∈ (0, 1) : \frac{1}{\sqrt{\pi}} \sum_{i=1}^{g} h^i_i \to \frac{1}{2}e^{-\frac{(\alpha^2/4)^2}{2}}$. Thus, the objective function in (9) converges in probability, point-wise on $α$, to $\ell(α^2/4)$ (cf. (2)). In fact, since the minimization in (9) is over a compact set, uniform convergence holds and the minimum value converges to $\min_{1 ≥ α ≥ α_0} + \ell(α^2/4)$. Combining the above, shows that for all $θ > 0$ the following event holds wpa 1:

$$\min_{w ∈ S(α_0)} G(w; g, h) ≥ \min_{1 ≥ α ≥ α_0} \ell(α^2/4) - η.$$  \hspace{1cm} (10)

Next, we obtain a simple upper bound on the RHS in (7):

$$\min_{w ∈ \{-2/\sqrt{\pi}, 0\}^n} F(w) ≤ F(0) = \frac{∥z∥2}{\sqrt{π}},$$  \hspace{1cm} (11)

which we combine with the fact that wpa 1 it holds $||z||2/\sqrt{π} ≤ \sqrt{3} σ + η$.

Combining the two displays in (10) and (11), we have shown that (7) holds as long as there exists $θ ≥ 0$ such that

$$\min_{1 ≥ α ≥ α_0} \ell(α^2/4) ≥ \sqrt{3} σ + 3 η.$$  \hspace{1cm} (12)

At this point, recall that $α^2/4 = θ^0$ and the definition of $θ^0$ as the largest solution to the equation $\ell(θ) = \sqrt{3} σ$. By this definition and the fact that $\ell(θ)$ is continuous and satisfies $\ell(1^-) > \sqrt{3} σ$ we have that $\ell(θ) > \sqrt{3} σ$ for all $θ > θ^0$. Thus there always exist η(ε) satisfying (12) and the proof is complete.

3.4. Gordon’s prediction meets Tanaka

Inspecting the proof of Theorem 3.1 reveals two possible explanations for why the resulting upper bound might be loose. First, recall that we obtain a lower bound in the LHS of (7) via Gordon’s inequality. As mentioned, in Remark 3 the inequality is not guaranteed to be tight in this instance. Second, recall that in upper bounding the RHS of (7) we use the crude bound in (11). Specifically, we upper bound the optimal cost $c_*$ of the MAP optimization in (1) simply by the value of the objective function at a known feasible solution, namely $x = x_0$.

In this section, we make the following leap of faith. We assume that $\inf_{θ ∈ (0, 1)} \ell(θ)$ is asymptotically tight high-probability lower bound of $c_*$, i.e., for all $θ > 0$ wpa 1:

$$\min_{x ∈ \{±1\}^n} \frac{1}{\sqrt{n}} |y - Ax|2 ≤ \inf_{θ ∈ (0, 1)} \ell(θ) + η.$$  \hspace{1cm} (13)

Assuming (13) is true and repeating the arguments of Section 3.3 leads to the following conclusion: the BER of the MAP detector is upper bounded by $θ^* = \arg \min_{θ ∈ (0, 1)} \ell(θ)$. This can be also be expressed as the solution to the fixed-point equation $\ell'(θ^*) = 0$, which is shown in Proposition A.1 (iv) to be equivalent to (5). Specifically, we show that for $θ > 1.46$ and all values of SNR the minimizer $θ^*$ of $\ell(θ)$ is unique and coincides with the unique solution of the replica prediction in (5).

4. CONCLUSION

In this paper, we prove a simple yet highly non-trivial upper bound on the BER of the MAP detector in the case of BPSK signals and of equal-power condition. Theorem 3.1 naturally extends to allow for general constellation types and power control and it also enjoys a non-asymptotic version. Furthermore, we show that the replica prediction matches the MFB at high-SNR for $θ > 1.46$. Relaxing the requirement on $θ > 1.46$ (as suggested by our numerical simulations to be possible) is an interesting future direction. Finally, it is an exciting (albeit potentially challenging) question proving under what conditions (if any) our assumption in (13) is indeed true, which would confirm the validity of the replica prediction in (5).

A. PROPERTIES OF $\ell(θ)$

Proposition A.1. Let $\ell : (0, 1) → R$ and $θ_0$ be defined as in Theorem 3.1 Also, let $θ^* = \arg \min_{θ ∈ (0, 1)} \ell(θ)$. Then,

(i) $θ ∈ (0, 1)$ is a critical point of $\ell$ if and only if it solves (5).

(ii) $\ell(θ)$ has a unique critical point

(iii) $θ_0$ is the unique solution of the equation $\ell(θ) = σ/\sqrt{π}$ in (0, 1).

(iv) The unique solution of (5) is the unique $θ^* = \arg \min_{θ} \ell(θ)$. Finally, if $θ > 1.46$ it holds that:

(v) The unique solution $θ^* = θ^*(σ)$ of (5) satisfies $\frac{θ}{Q(\sqrt{π}/σ)} → 1$, in the limit of $σ^2 → 0$:

(vi) For $θ > 0, \theta_0 = θ_0(σ)$ satisfies $\limsup_{σ → 0} \frac{θ_0}{Q(\sqrt{π}/σ)} ≤ 1$.

Proof. (i) We observe that for any $θ ∈ (0, 1)$,

$$\ell'(θ) = \frac{2 \sqrt{3}}{\sqrt{4θ + σ^2}} - \sqrt{\frac{2}{π}} \left( e^{-\frac{(Q(1^-)(θ))^2}{2}} \right)' = \frac{2 \sqrt{3}}{\sqrt{4θ + σ^2}} - \sqrt{\frac{2}{π}} \left( Q^{-1}(θ) \right)' Q^{-1}(θ) \left( e^{-\frac{(Q(1^-)(θ))^2}{2}} \right)' = \frac{2 \sqrt{3}}{\sqrt{4θ + σ^2}} - 2 Q^{-1}(θ)$$

where for the last equation we have used that for all $θ$:

$$\left( Q^{-1}(θ) \right)' Q^{-1}(θ) = 1,$$

and the definition of $Q'$. Therefore indeed $\ell'(θ) = 0$ is equivalent to (5). Now for the second part observe that for the continuous function $F(θ) := θ - Q \left( \sqrt{\frac{θ}{σ+θ}} \right)$, $F(0) = 0 - Q \left( \sqrt{\frac{θ}{σ}} \right) < 0$ and $0 < 1 - Q \left( \sqrt{\frac{θ}{σ+θ}} \right) = F(1)$, implying that by the mean value theorem there is always a root for $F(θ) = 0$ for some $θ ∈ (0, 1)$, yielding a critical point for $ℓ$.

(ii) We know from (i) that $ℓ(θ)$ has at least one critical point. We argue by contradiction for the uniqueness. Assuming the existence of two critical points, we have two roots in (0, 1) for $θ - Q \left( \sqrt{\frac{θ}{σ+θ}} \right)$ (cf. (5)). This implies by Rolle’s theorem, that for some $θ ∈ (0, 1), 1 = \left[ Q \left( \sqrt{\frac{θ}{σ+θ}} \right) \right]'$. We conclude the proof
with the following claim.  

Claim: For any \( \delta > 1.46 \) and \( \sigma \geq 0 \) there is no \( \theta \in (0, 1) \) such that
\[
1 = \left[ Q\left( \frac{\sqrt{\delta}}{\sigma + 4\theta} \right) \right]^\prime. \]

Proof of the claim: Assume there is such a \( \theta \in (0, 1) \). The relation can be equivalently written as
\[
Q\left( \sqrt{\frac{\delta}{\sigma^2 + 4\theta}} \right) = \frac{1}{2} \left( \frac{\alpha^2 + 4\theta^2}{2\sqrt{\delta}} \right),
\]
or by definition of \( Q' \),
\[
\frac{\delta}{\sigma^2 + 4\theta} = \frac{\sqrt{\delta}}{2} \frac{\sqrt{\delta}}{\sqrt{\sigma^2 + 4\theta}}. \]

Changing variables to \( y = \frac{\delta}{\sigma^2 + 4\theta} \) we conclude that for some \( y > 0 \) it must be true that
\[
e^y y^\frac{\sigma}{2} = 2 \sqrt{\frac{\pi}{\delta}}. \tag{14}\]

By elementary calculus we get that the left hand side of (14) obtains global minimum over all \( y > 0 \) equal to \( \left( \frac{2\sqrt{\pi}}{\delta} \right)^\frac{\sigma}{2} \) and therefore (14) implies that \( \delta \) should satisfy \( 2\sqrt{\frac{\pi}{\delta}} \geq \left( \frac{2\sqrt{\pi}}{\delta} \right)^\frac{\sigma}{2} \) or \( \delta \leq 2\sqrt{\frac{\pi}{\delta}} \). The first digits of the last quantity can be easily checked to be 1.45311 yielding a contradiction with \( \delta > 1.46 \).

(iii) We can continuously extend \( \ell(\cdot) \) to include the endpoints of the interval \([0, 1]\). Note that \( \ell(0) = \ell(0^+) = \sigma \sqrt{\delta} \). For the sake of contradiction, suppose that there exists \( 0 < \theta_1 < \theta_0 \) such that \( \ell(\theta_1) = \sigma \sqrt{\delta} \). Then, by Rolle’s theorem \( \ell(\cdot) \) would have two distinct critical points, which contradicts (ii).

(iv) Let \( \theta_0 \) the critical point of \( \ell \) in \((0, 1)\). We show that \( \theta_0 = \theta_\ast \). From the derivation for part (i), we know for all \( \theta \in (0, 1) \),
\[
\ell'(\theta) = \frac{2 \sqrt{\delta}}{\sqrt{\sigma^2 + 4\theta}} - 2Q^{-1}(\theta). \tag{16}\]

As for \( \theta \) approaching 0, \( Q^{-1}(\theta) \) approaches \( +\infty \), we can conclude that for \( \theta \) sufficiently small, \( \ell'(\theta) < 0 \). Similarly, as for \( \theta \) approaching 1, \( Q^{-1}(\theta) \) approaches \( -\infty \), we conclude that for \( \theta \) sufficiently close to 1, \( \ell'(\theta) > 0 \). Given that \( \ell'(\theta) = 0 \) has a unique solution \( \theta_0 \), we conclude that \( \theta < \theta_0 \) implies \( \ell'(\theta) < 0 \) and \( \theta > \theta_0 \), \( \ell'(\theta) > 0 \). In particular, \( \theta_0 \) is the global minimum of \( \ell(\cdot) \), i.e. \( \theta_0 = \theta_\ast \).

(v) We first establish that \( \theta_\ast \to 0 \) as \( \sigma \to 0 \). To see this, consider by contradiction a limiting point, \( \theta_L > 0 \) of the function \( \theta_L(\sigma) \). By (15) it must be true by taking limits, \( \theta_L = \ell(\frac{\sqrt{\delta}}{\sigma_L}) \).

Therefore, the function defined by \( F(t) = t - Q\left( \sqrt{\frac{\delta}{t}} \right) \) for \( t > 0 \) and \( F(0) = 0 \) has \( \theta_L \) a root in \((0, 1)\). In particular that implies a \( \theta \in (0, 1) \) with \( F(\theta) = 0 \) or 1 = \( \left[ Q\left( \sqrt{\frac{\delta}{\sigma_L}} \right) \right]^\prime \), which contradicts the claim proven in part (ii) for \( \sigma^2 = 0 \).

Now by mean value theorem, for some \( \sigma_T \in (0, \theta_\ast) \),
\[
\theta_\ast - Q\left( \frac{\sqrt{\delta}}{\sigma} \right) = \left[ Q\left( \frac{\sqrt{\delta}}{\sigma + 4\theta} \right) \right]^\prime \theta_\ast = \frac{2 \sqrt{\delta}}{(4\theta + \sigma^2)^\frac{\sigma}{2}} \left[ Q\left( \frac{\sqrt{\delta}}{\sigma + 4\theta} \right) \right] \theta_\ast = \frac{\sqrt{2} \sqrt{\delta}}{\sqrt{\pi}(4\theta + \sigma^2)^\frac{\sigma}{2}} e^{-\frac{\sigma}{2(\sigma^2 + 4\theta)^\frac{\sigma}{2}}}, \tag{17}\]

Therefore,
\[
\limsup_{\sigma \to 0} \left| 1 - \frac{Q\left( \frac{\sqrt{\delta}}{\theta_\ast} \right)}{\theta_\ast} \right| \leq \limsup_{\sigma \to 0} \frac{\sqrt{2} \sqrt{\delta}}{\sqrt{\pi}(4\theta_T + \sigma^2)^\frac{\sigma}{2}} e^{-\frac{\sigma}{2(\sigma^2 + 4\theta_T)^\frac{\sigma}{2}}}. \tag{18}\]

Since \( 0 < \theta_T < \theta_\ast \), we know that \( \theta_T \) also goes to zero as \( \sigma \) goes to zero. In particular \( \sigma^2 + 4\theta_T \) goes to zero and as \( \delta \) is fixed, (18) implies the desired result.

(vi) By (iii) \( \theta_0 \) is the unique solution in \((0, 1)\) of \( \ell(\theta) = \sigma \sqrt{\delta} = \ell(0^+) \). Now we have \( \theta_0 > 0 \) satisfies \( \ell(\theta_0) = \sigma \sqrt{\delta} \) or
\[
\sqrt{\delta} \sqrt{4\theta_0 + \sigma^2} - \frac{2 \sqrt{\delta}}{\pi} e^{-(Q^{-1}(\theta_0))^\prime} = \sqrt{\delta} \sigma. \tag{19}\]

We first prove that \( \theta_0 \to 0 \), as \( \sigma \to 0 \). Indeed, if not, suppose \( \theta_N > 0 \) is a positive limiting point of \( \theta_0 \) as \( \sigma \) goes to zero. Then (19) implies \( \sqrt{\delta} \sqrt{4\theta_0 + \sigma^2} - \frac{2 \sqrt{\delta}}{\pi} e^{-(Q^{-1}(\theta_0))^\prime} = \sqrt{\delta} \sigma \). Since \( L(0) = 0 \) by Rolle’s theorem we have for some \( \theta_L \in (0, 1) \), \( L'(\theta_L) = 0 \) which gives \( \theta_L = Q\left( \frac{\sqrt{\delta}}{2\theta_L} \right) \). Therefore, the function defined by \( F(t) = t - Q\left( \frac{\sqrt{\delta}}{2\theta_L} \right) \) for \( t > 0 \) and \( F(0) = 0 \) has \( \theta_L \) a root in \((0, 1)\). In particular that implies a \( \theta \in (0, 1) \) with \( F(\theta) = 0 \) or 1 = \( \left[ Q\left( \sqrt{\frac{\delta}{\sigma_L}} \right) \right]^\prime \), which contradicts the claim proven in part (ii) for \( \sigma^2 = 0 \).

Now by (16) by rearranging and the Gaussian density, we have
\[
\sqrt{\delta} \left( \sqrt{4\theta_0 + \sigma^2} - \sigma \right) = \phi(Q^{-1}(\theta_0)). \tag{20}\]

Taylor expansion around 0 and the fact that \( \theta_0 = o(1) \) give \( \phi(Q^{-1}(\theta_0)) = 2Q^{-1}(\theta_0)\theta_0 + o(\theta_0) \).

Hence we have
\[
\sqrt{\delta} \frac{1}{2\theta_0} \sqrt{4\theta_0 + \sigma^2} = \sqrt{\delta} \sigma - \eta < 0 \tag{21}\]

or
\[
\theta_0 < Q\left( \frac{2 \sqrt{\delta}}{\sqrt{4\theta_0 + \sigma^2} - \sigma} - \eta \right). \tag{22}\]

Finally, by (22) and the mean value theorem we have that for some \( \sigma_T \in (0, \theta_0) \),
\[
\theta_0 - Q\left( \frac{\sqrt{\delta}}{\sigma} - \eta \right) < Q\left( \frac{2 \sqrt{\delta}}{\sqrt{4\theta_0 + \sigma^2} - \sigma} - \eta \right) = \left[ Q\left( \frac{2 \sqrt{\delta}}{\sqrt{4\theta_T + \sigma^2} - \sigma} - \eta \right) \right] \theta_0 = \frac{2 \sqrt{\delta}}{\sqrt{4\theta_T + \sigma^2} - \sigma} - \eta \right) \theta_0 = \frac{2 \sqrt{\delta}}{\sqrt{4\theta_T + \sigma^2} - \sigma} - \eta \left( \frac{\sigma}{\sqrt{\sigma^2 + 4\theta_T}} \right)^2 / \theta_0. \tag{23}\]
Therefore,
\[
1 - \lim_{\sigma \to 0} \inf_{\theta_0} \frac{Q(\frac{\sqrt{2\sigma + 4\theta_T}}{\theta_0} - \eta)}{\theta_0} \leq \lim_{\sigma \to 0} \sup_{\theta_0} \sqrt[4]{\frac{\pi}{\sqrt{2(4\theta_T + \sigma^2 - \eta)^2}}} - \frac{\sqrt{2\sigma + 4\theta_T}}{\sqrt[4]{\pi(4\theta_T + \sigma^2 - \eta)^2}}. \tag{18}
\]

Since \( 0 < \theta_T < \theta_0 \) we know that \( \theta_T \) also goes to zero as \( \sigma \) goes to zero. In particular \( \sigma^2 + 4\theta_T \) goes to zero and as \( \delta \) is fixed, \( \theta_T \) implies the desired result.

\[\square\]

B. REFERENCES


