Control Using Logic & Switching:
Part III
Supervisory Control

Tutorial for the 40th CDC

João P. Hespanha

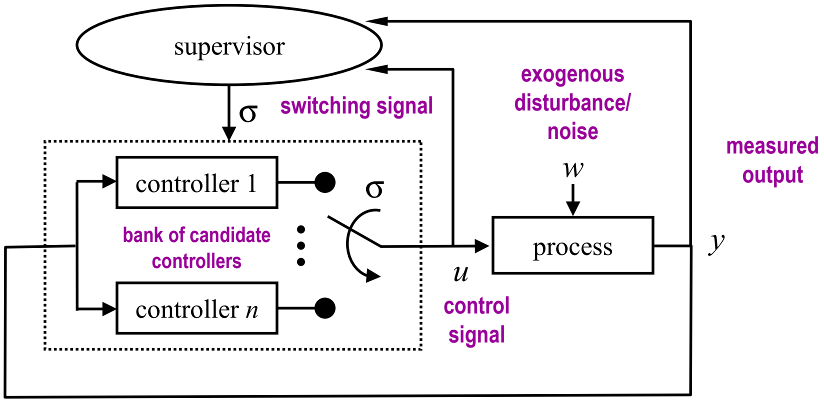
University of Southern California
University of California at Santa Barbara

Outline

- Supervisory control overview
- Estimator-based linear supervisory control
- Estimator-based nonlinear supervisory control
- Examples

Supervisory control

Motivation: *in the control of complex and highly uncertain systems, traditional methodologies based on a single controller do not provide satisfactory performance.*



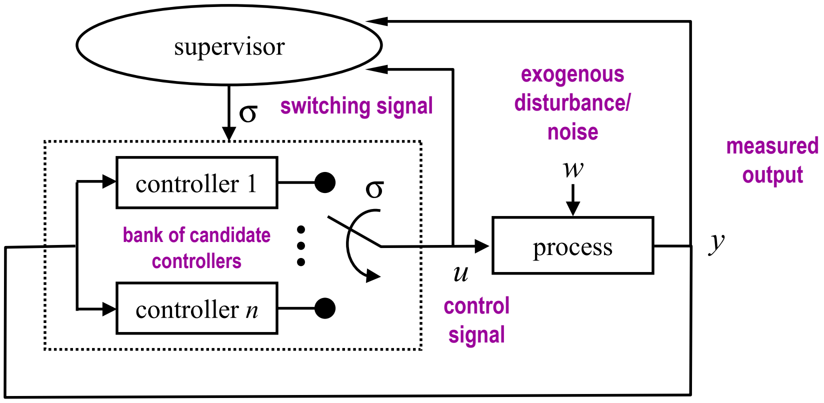
Key ideas:

1. Build a bank of alternative controllers
2. Switch among them online based on measurements

For simplicity we assume a stabilization problem, otherwise controllers should have a reference input r

Supervisory control

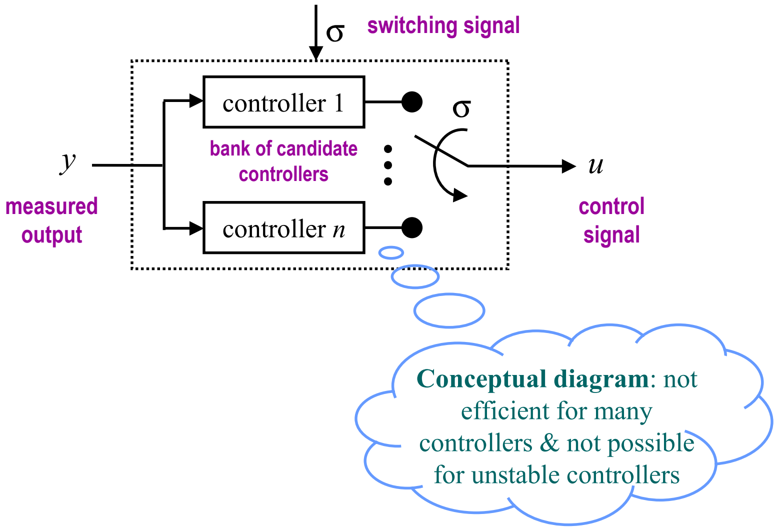
Motivation: *in the control of complex and highly uncertain systems, traditional methodologies based on a single controller do not provide satisfactory performance.*



Supervisor:

- places in the feedback loop the controller that seems more promising based on the available measurements
- typically *logic-based/hybrid system*

Multi-controller



Conceptual diagram: not efficient for many controllers & not possible for unstable controllers

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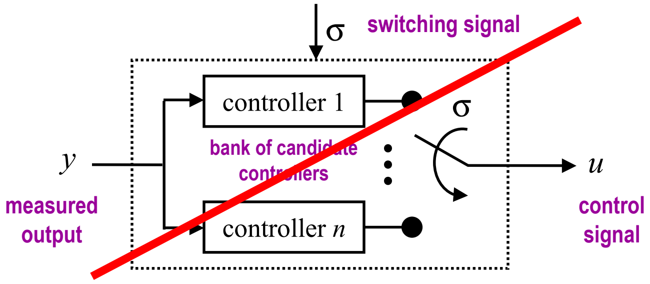
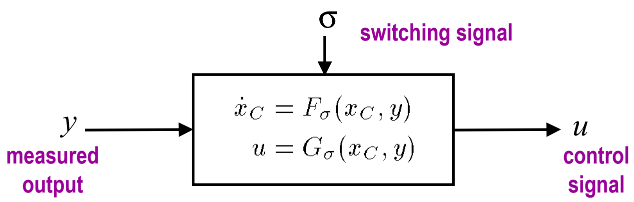
Multi-controller



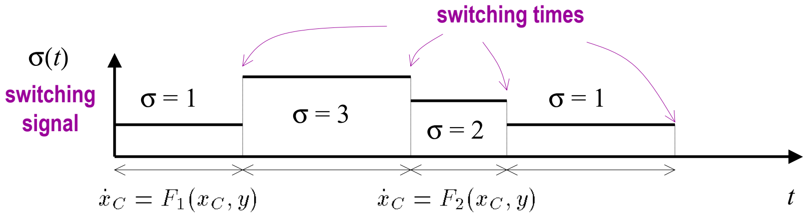
Given a family of (n-dimensional) candidate controllers

$$\mathcal{C} := \left\{ \dot{z}_q = F_q(z_q, y),\ u = G_q(z_q, y) : q \in \mathcal{Q} \right\}$$



$$\dot{x}_C = F_\sigma(x_C, y)$$
$$u = G_\sigma(x_C, y)$$

Multi-controller



$$\dot{x}_C = F_1(x_C, y) \qquad u = G_1(x_C, y)$$

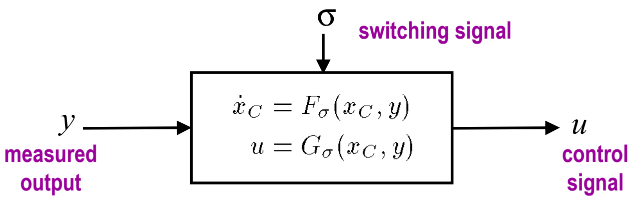
$$\dot{x}_C = F_3(x_C, y) \qquad u = G_3(x_C, y)$$

$$\dot{x}_C = F_2(x_C, y) \qquad u = G_2(x_C, y)$$

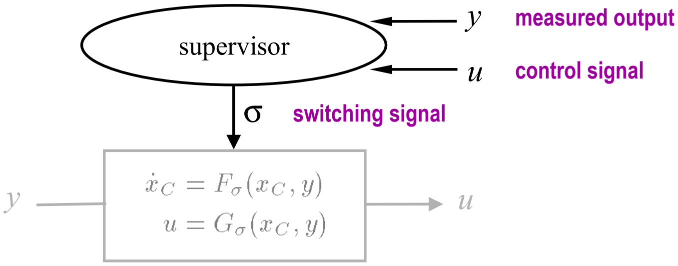
$$\dot{x}_C = F_1(x_C, y) \qquad u = G_1(x_C, y)$$

Given a family of (n-dimensional) candidate controllers

$$\mathcal{C} := \left\{ \dot{z}_q = F_q(z_q, y),\ u = G_q(z_q, y) : q \in \mathcal{Q} \right\}$$



Supervisor



Typically an hybrid system: φ ≡ continuous state
δ ≡ discrete state

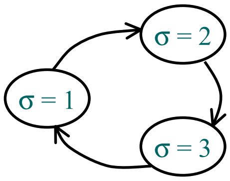
$$\delta^-(t) := \lim_{\tau \uparrow t} \delta(\tau)$$

$$\dot{\varphi} = f_\delta(\varphi, u, y) \qquad \delta = \phi(\varphi, \delta^-) \qquad \sigma = \psi_\delta(\varphi)$$

continuous vector field — discrete transition function — output function

Types of supervision

Pre-routed supervision



- try one controllers after another in a pre-defined sequence
- stop when the performance seems acceptable

↓

not effective when the number of controllers is large

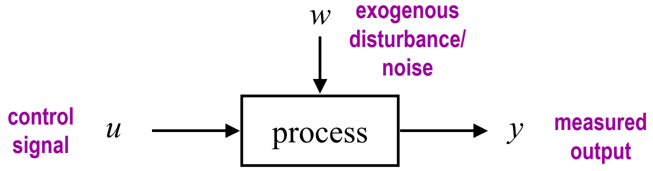
Performance-based supervision (direct)

- keep controller while observed performance is acceptable
- when performance of current controller becomes unacceptable, switch to controller that leads to best expected performance based on available data

Estimator-based supervision (indirect)

- estimate process model from observed data
- select controller based on current estimate – *Certainty Equivalence*

Estimator-based supervision's setup



w exogenous disturbance/ noise

control signal u → process → y measured output

Process is assumed to be in a family

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$$

parametric uncertainty

unmodeled dynamics

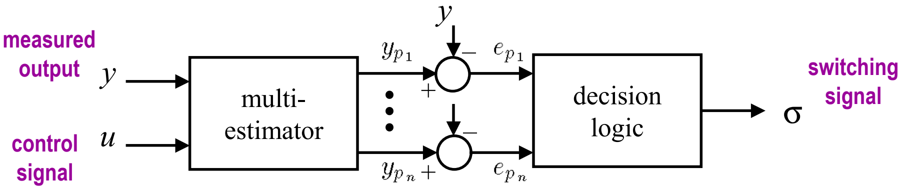
$\mathcal{M}_p \equiv$ small family of systems around a "nominal" process model N_p

for each process in a family $\mathcal{M}_p$, at least one *candidate controller* C_q, $q \in \mathcal{Q}$ provides adequate performance.

controller selection function

process in $\mathcal{M}_p, p\in \mathcal{P}$ → controller C_q with $q = \chi(p)$ provides adequate performance

Estimator-based supervisor



Process is assumed to be in family

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$$

process in $\mathcal{M}_p$, $p\in\mathcal{P}$ → controller C_q, $q = \chi(p)$ provides adequate performance

Multi-estimator

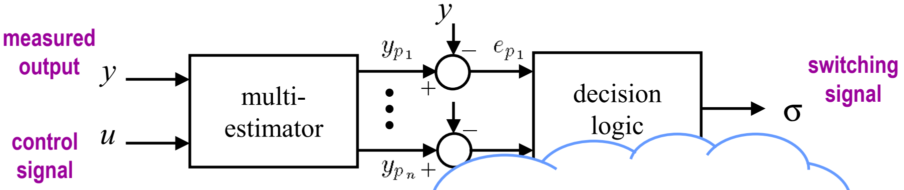
$y_p \equiv$ estimate of the process output y that would be correct if the process was N_p
$e_p \equiv$ output estimation error that would be small if the process was N_p

Decision logic:

e_p small → process likely in $\mathcal{M}_p$ → should use C_q, $q = \chi(p)$ → set $\sigma = \chi(p)$

Certainty equivalence inspired

Estimator-based supervisor



A stability argument cannot be based on this because typically
process in $\mathcal{M}_p \Rightarrow e_p$ small
but not the converse

Process is assumed to be in family

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$$

proce $\mathcal{M}_p$, p …p) …rformance

Multi-estimator

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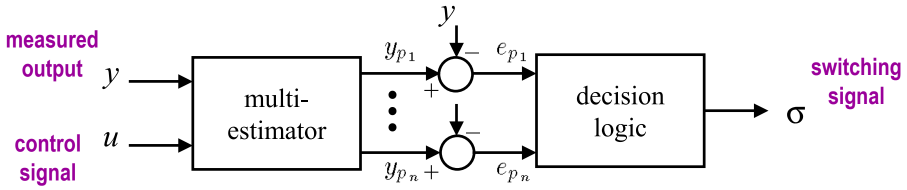
Estimator-based supervisor



Process is assumed to be in family

$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$ process in $\mathcal{M}_p, p\in\mathcal{P}$ ➡ controller C_q, $q=\chi(p)$ provides adequate performance

Multi-estimator

$y_p \equiv$ estimate of the process output y that would be correct if the process was N_p

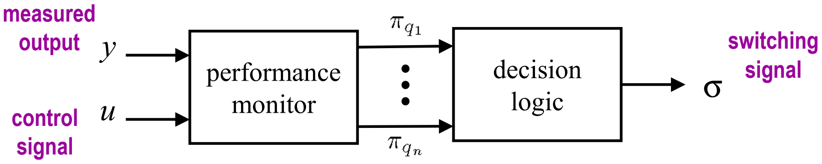
$e_p \equiv$ output estimation error that would be small if the process was N_p

Decision logic:

e_p small ➡ 🚫 ➡ set $\sigma = \chi(p)$ ➡ overall system is detectable through e_p ➡ overall state is small

Certainty equivalence inspired, but formally justified by detectability

Performance-based supervision



Candidate controllers: $\mathcal{C} := \{C_q : q \in \mathcal{Q}\}$

Performance monitor:

$\pi_q \equiv$ measure of the expected performance of controller C_q inferred from past data

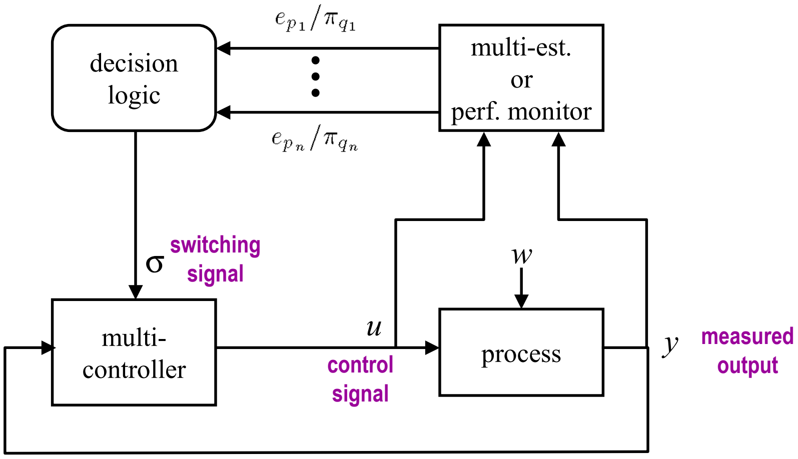
Decision logic:

π_σ is acceptable ➡ keep current controller

π_σ is unacceptable ➡ switch to controller C_q corresponding to best π_q

Abstract supervision

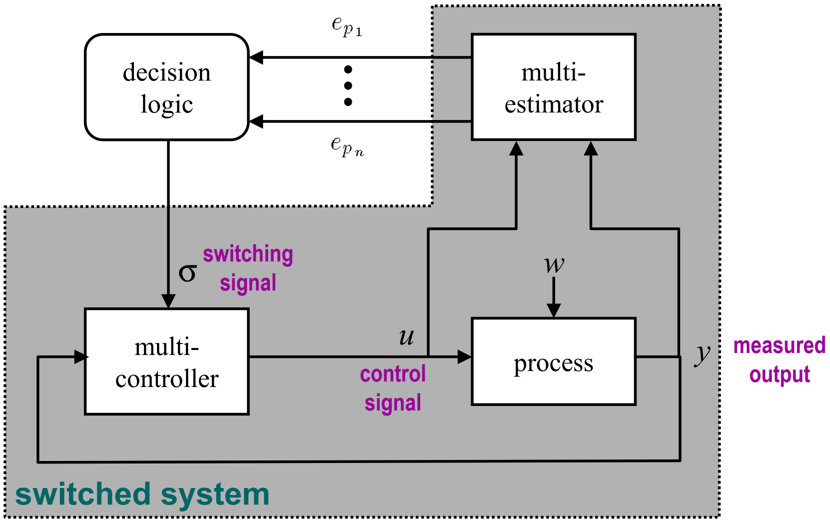
Estimator and performance-based architectures share the same common architecture



In this talk we will focus mostly on an estimator-based supervisor…

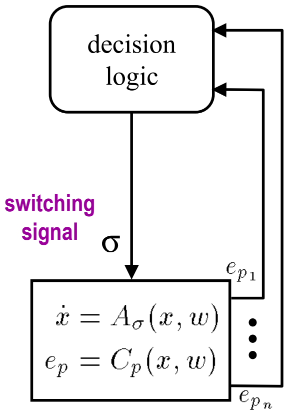
Abstract supervision

Estimator and performance-based architectures share the same common architecture



$$\dot{x} = A_\sigma(x, w) \qquad e_p = C_p(x, w) \quad p \in \mathcal{P}$$

The four basic properties (1-2)



Matching property:
At least one of the e_p is "small"

Why?

process in $\mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$ ➡ $\exists\, p^* \in \mathcal{P}$: process in $\mathcal{M}_{p^*}$ ➡ e_{p^*} is "small"

essentially a requirement on the multi-estimator

Detectability property:
For each $p \in \mathcal{P}$, the switched system is detectable through e_p when $\sigma = \chi(p)$

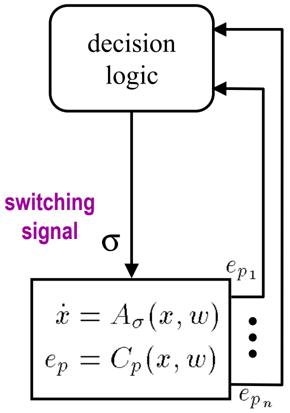
index of controller that stabilizes processes in $\mathcal{M}_p$

essentially a requirement on the candidate controllers

This property justifies using the candidate controller that corresponds to a small estimation error.

Why? Certainty equivalence stabilization theorem…

The four basic properties (3-4)



Small error property:
There is a process switching signal

$$\rho : [0,\infty) \to \mathcal{P}$$

$\rho(t)$ can be viewed as current parameter "estimate"

for which e_ρ is "small" compared to any fixed e_p and that is consistent with σ, i.e.,

$$\sigma = \chi(\rho)$$

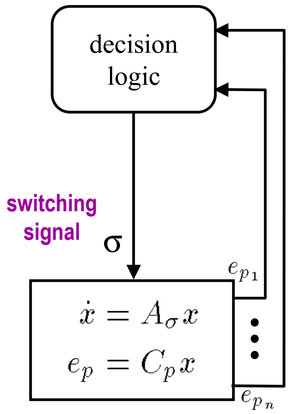
controller consistent with parameter "estimate"

Non-destabilization property:
Detectability is preserved for the time-varying switched system (not just for constant σ)

Typically requires some form of "slow switching"

Both are essentially (conflicting) properties of the decision logic

Analysis outline (linear case, $w = 0$)



1st by the Matching property:
$\exists\, p^* \in \mathcal{P}$ such that e_{p^*} is "small"

2nd by the Small error property:
$\exists\, \rho$ such that $\sigma = \chi(\rho)$ and e_ρ is "small"
(when compared with e_{p^*})

3rd by the Detectability property:
there exist matrices K_p such that the matrices
$A_q - K_p C_p,\ q = \chi(p)$
are asymptotically stable

4th the switched system can be written as

$$\dot{x} = (A_\sigma - K_\rho C_\rho)x + K_\rho e_\rho$$

injected system

asymptotically stable by non-destabilization property

"small" by 2nd step

$\therefore\, x$ is small (and converges to zero if, e.g., $e_\rho \in \mathcal{L}_2$)

Outline

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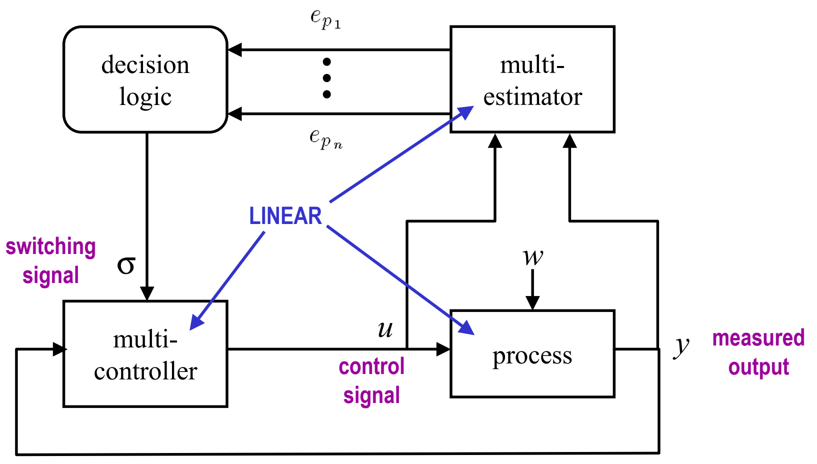
- ✗ Estimator-based linear supervisory control
- ✗ Estimator-based nonlinear supervisory control
- ✗ Examples

Estimator-based linear supervisory control


e_{p_1}

decision logic

multi-estimator

e_{p_n}

LINEAR

switching signal

σ

w

multi-controller

u

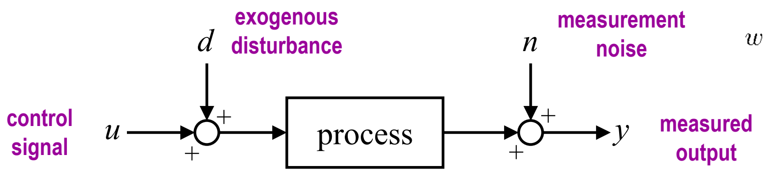
control signal

process

y

measured output

Class of admissible processes


exogenous disturbance

d

measurement noise

n

control signal

u

+

process

+

y

measured output

$w := \begin{bmatrix} d \\ n \end{bmatrix}$

The transfer function from *u* to *y* is an unknown element of

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$$

parametric uncertainty

$\mathcal{M}_p \equiv$ small family of systems around a "nominal" transfer function ν_p

Typically

multiplicative unmod. dynamics

additive unmod. dynamics

$$\mathcal{M}_p := \Big\{\nu_p(1+\delta_m)+\delta_a : \|\delta_m\|_{\infty,\lambda}\le\epsilon,\ \|\delta_a\|_{\infty,\lambda}\le\epsilon\Big\}\quad p\in\mathcal{P}$$

or

$$\mathcal{M}_p := \Big\{\frac{n_p+\delta_n}{d_p+\delta_d} : \|\delta_n\|_{\infty,\lambda}\le\epsilon,\ \|\delta_d\|_{\infty,\lambda}\le\epsilon\Big\}\quad p\in\mathcal{P}$$

co-prime factorization of ν_p (SISO)

A word on norms…

Given a signal y and $\lambda \geq 0$

$$\|y\|_{\lambda,[0,t)} = \left(\int_0^t e^{2\lambda\tau} y(\tau)^2 d\tau \right)^{\frac{1}{2}}$$

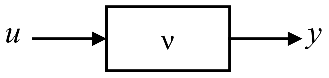
$e^{\lambda t}$ – weighted $\mathcal{L}_2$ norm of y truncated to $[0, t)$

Given a transfer function ν and $\lambda \geq 0$

$$\|\nu\|_{\infty,\lambda} := \sup_{\Re[s]\geq 0} |\nu(s-\lambda)|$$

$e^{\lambda t}$ – weighted $\mathcal{H}_\infty$ norm of ν (finite if all poles of ν have real part smaller than $-\lambda$)

The $e^{\lambda t}$ – weighted $\mathcal{L}_2$ induced norm of transfer function ν is numerically equal to the $e^{\lambda t}$ – weighted $\mathcal{H}_\infty$ norm of ν



$\| \nu \|_{\infty,\lambda} = \gamma$

$$\|y\|_{\lambda,[0,t)} \leq \gamma \|u\|_{\lambda,[0,t)} + c_0 \quad \forall t \geq 0$$

due to initial conditions

we will sometimes need a stability margin $\lambda > 0$ ...

Candidate controllers

Class of admissible processes

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p$$

$\mathcal{M}_p \equiv$ small family of systems around a "nominal" transfer function ν_p

Assume given a family of *candidate controller* transfer functions

$$\mathcal{C} := \left\{ \kappa_q : q \in \mathcal{Q} \right\}$$

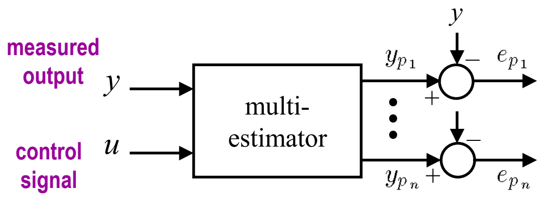
and a *controller selection function* $\chi : \mathcal{P} \to \mathcal{Q}$ such that

$\forall\, p \in \mathcal{P}$ controller κ_q, $q = \chi(p)$ stabilizes all processes in $\mathcal{M}_p$

χ maps parameter values with the corresponding stabilizing controller

No other constrain is posed on the candidate controllers:
κ_q can be designed using any (nonadaptive) technique
(e.g., pole placement, LQG/LQR, H-infinity, etc.)

Multi-estimator



$$\dot{x}_E = A_E x_E + D_E y + B_E u$$
$$y_p = C_p x_E \quad p \in \mathcal{P}$$

with A_E asymptotically stable

Matching property:

There exist positive constants c_0, c_w, c_ε, λ and some $p^* \in \mathcal{P}$ such that

$$\|e_{p^*}\|_{\lambda,[0,t)} \leq \underset{\text{initial cond.}}{c_0} + \overset{\text{noise/disturb.}}{c_w \|w\|_{\lambda,[0,t)}} + \underset{\text{unmodeled dynamics}}{\epsilon\, c_\epsilon \|u\|_{\lambda,[0,t)}} \qquad \forall t \geq 0$$

e_{p^*} is "$\mathcal{L}_2$" when noise/disturb. and unmodeled dynamics are "$\mathcal{L}_2$"

recall $\|y\|_{\lambda,[0,t)} = \left(\int_0^t e^{2\lambda\tau} y(\tau)^2 d\tau \right)^{\frac{1}{2}}$

A simple multi-estimator…

Class of admissible processes (2 elements)

$$\mathcal{M} := \left\{ c_p(sI - A_p)^{-1} b_p : p = 1, 2 \right\}$$

Assuming realizations are detectable, we could make **Luenberger observers**

$$\underset{\text{asymptotically stable}}{\dot{x}_p = (A_p - k_p c_p)} x_p + k_p y + b_p u \quad y_p = c_p x_p \quad p \in \mathcal{P} := \{1, 2\}$$

Multi-estimator:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overset{A_E}{\begin{bmatrix} A_1 - k_1 c_1 & 0 \\ 0 & A_2 - k_2 c_2 \end{bmatrix}} \overset{x_E}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} + \overset{D_E}{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}} y + \underset{B_E}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}} u$$

$$y_1 = \begin{bmatrix} c_1 & 0 \end{bmatrix} x_E \qquad y_2 = \begin{bmatrix} 0 & c_2 \end{bmatrix} x_E$$

process $\equiv c_{p^*} (s\,I - A_{p^*})^{-1}\, b_{p^*}$ ➡ $e_{p^*} = y_{p^*} - y \to 0$ (exp. fast)

with noise & unmodeled dynamics ➡ $\|e_{p^*}\|_{\lambda,[0,t)} \leq c_0 + c_w \|w\|_{\lambda,[0,t)} + \epsilon\, c_\epsilon \|u\|_{\lambda,[0,t)}$

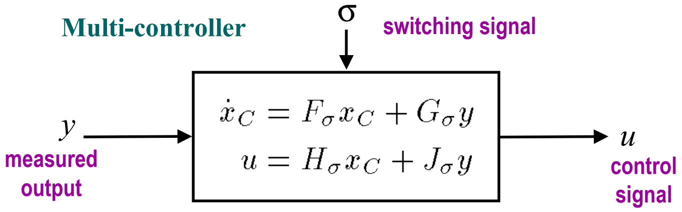
Multi-controller

Given a family of *candidate controller* transfer functions

$$\mathcal{C} := \left\{ \kappa_q : q \in \mathcal{Q} \right\}$$

Compute (n_C-dimensional) stabilizable and detectable realization

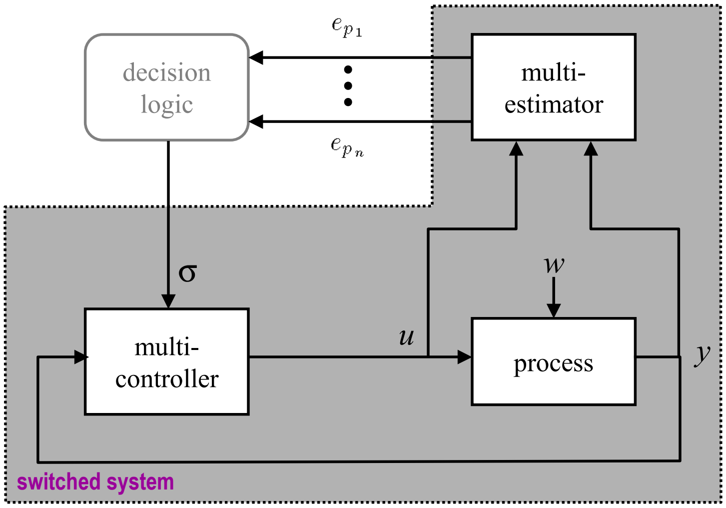
$$\left\{ (F_q, G_q, H_q, J_q) : q \in \mathcal{Q} \right\}$$


Multi-controller

$$\begin{aligned} \dot{x}_C &= F_\sigma x_C + G_\sigma y \\ u &= H_\sigma x_C + J_\sigma y \end{aligned}$$

detectability property?

Switched system



The switched system can be seen as the interconnection of the process *with the* "injected system"

essentially the multi-controller & multi-estimator but now quite…

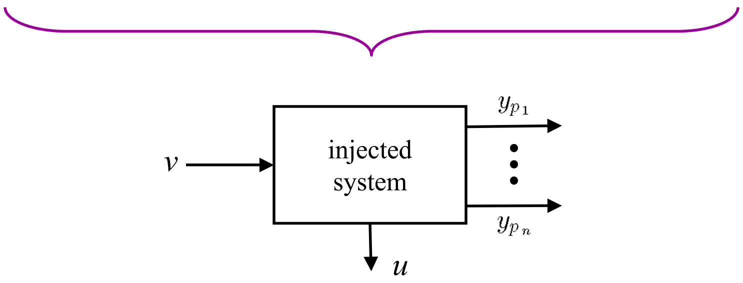
Constructing the injected system

1st Take an (arbitrary) process switching signal $\rho : [0,\infty) \to \mathcal{P}$.

2nd Define the signal $v := e_\rho = y_\rho - y$

3rd Replace y in the equations of the multi-estimator and multi-controller by $y_\rho - v$.

$$\begin{aligned} \dot{x}_E &= A_E x_E + D_E(y_\rho - v) + B_E u \\ y_p &= C_p x_E \quad p \in \mathcal{P} \end{aligned} \qquad \begin{aligned} \dot{x}_C &= F_\sigma x_C + G_\sigma(y_\rho - v) \\ u &= H_\sigma x_C + J_\sigma(y_\rho - v) \end{aligned}$$



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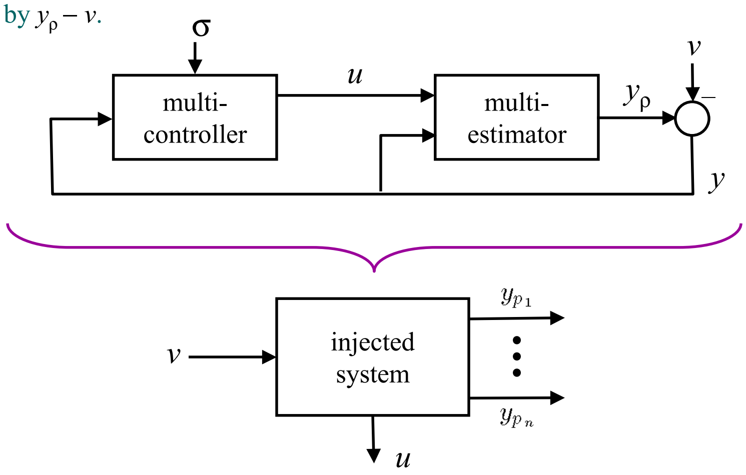
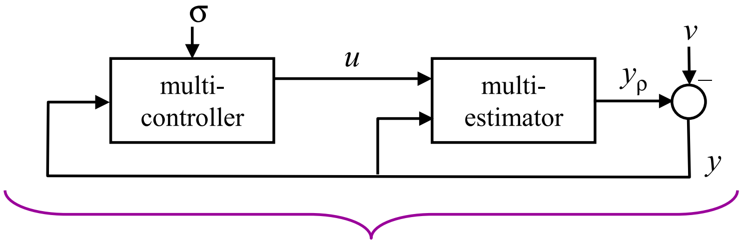


The injected system



$$\dot{x} = A_{\rho\sigma} x + B_\sigma v \qquad u = F_{\rho\sigma} x + G_\sigma v$$

$$y_p = C_p x \quad p \in \mathcal{P}$$

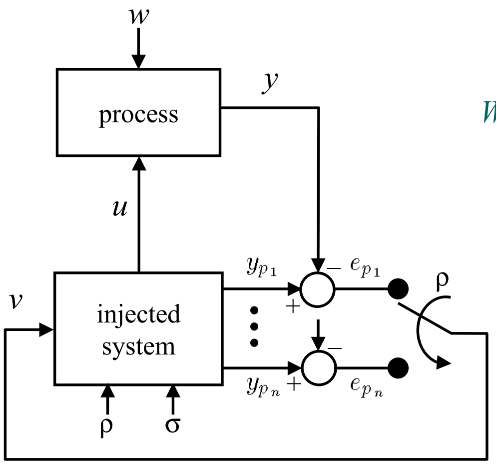
By inspection: for each $p \in \mathcal{P}$, $q \in \mathcal{Q}$

eigenvalues of $A_{p\,q} \equiv$ { subset of eigenvalues of A_E } ∪
{ poles of the feedback interconnection of ν_p with κ_q }

$\therefore$ $q = \chi(p)$ ➡ κ_q stabilizes ν_p ➡ $A_{p\,q}$ is asymptotically stable

If we choose ρ such that $\sigma = \chi(\rho)$ then $A_{\rho\,\sigma}$ is always stable

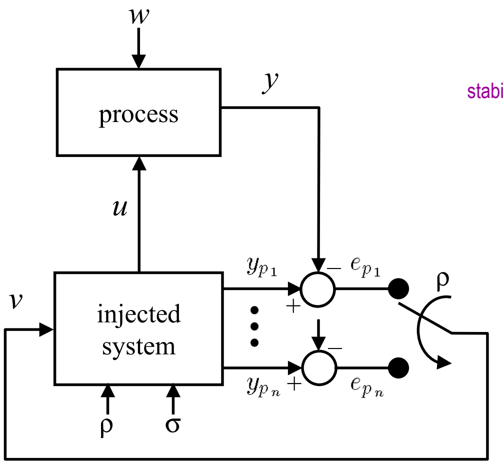
Switched system = process + injected system



Why? because $v := e_\rho = y_\rho - y$

Using this diagram one can
prove the detectability
property by inspection …

Detectability property



Suppose

- $\rho = p \in \mathcal{P}$ • $w = 0$
- $\sigma = \chi(p) \in \mathcal{Q}$ • $e_p = 0$ ($e_\rho = v = 0$)

stability of injected system ⇓

state & outputs u, y_p of injected system → 0

⇓

process input & output $u, y{=}y_p - e_p \to 0$

detectability of process ⇓

process state → 0

⇓

whole state of switched system → 0

The state of the switched system converges to zero along any solution compatible with zero input w & output e_p ➡ detectability

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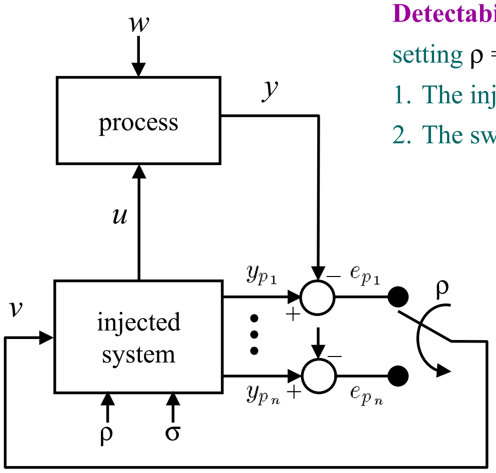
Detectability property



Detectability property: Given any $p \in \mathcal{P}$, setting $\rho = p \in \mathcal{P}$ and $\sigma = \chi(p) \in \mathcal{Q}$:

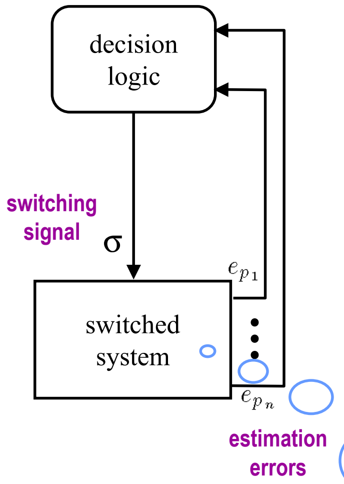
1. The injected system is asymptotically stable
2. The switched system is detectable through e_p

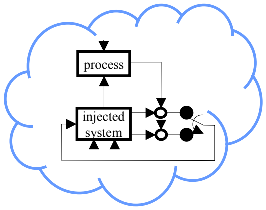
Also known as the Certainty Equivalence Stabilization Theorem

Stability of the injected system is not the only mechanism to achieve detectability:
e.g.: injected system i/o stable + process min. phase ⇒ detectability of switched system

(Certainty Equivalence Output Stabilization Theorem)

Decision logic





1. For boundedness one wants e_ρ small for some ρ consistent with σ (i.e., $\sigma = \chi(\rho)$)

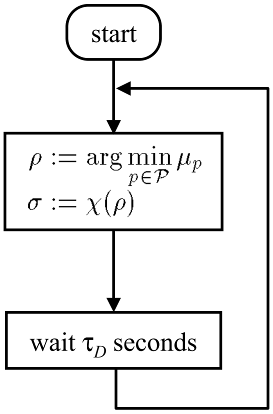
   "small error"

2. To recover the "static" detectability of the time-varying switched system one wants slow switching.

   "non-destabilization"

These are conflicting requirements:
1. ρ *should follow smallest* e_p
2. $\sigma = \chi(\rho)$ *should not vary*

Dwell-time switching



monitoring signals

$$\mu_p(t) := e^{-\lambda t}\|e_p\|_{\lambda,[0,t)} \qquad p \in \mathcal{P}$$

$$= \int_0^t e^{-2\lambda(t-\tau)}\|e_p(\tau)\|^2 d\tau$$

measure of the size of e_p *over a "window" of length* $1/\lambda$

forgetting factor

Non-destabilizing property:
The minimum interval between consecutive discontinuities of σ is $\tau_D > 0$.

Small error property

Assume $\mathcal{P}$ finite and $\exists\ p^* \in \mathcal{P}$:

(e.g., $\mathcal{L}_2$ noise and no unmodeled dynamics)

$$\|e_{p^*}\|^2_{\lambda,[0,\infty)} = \int_0^\infty e^{2\lambda\tau}\|e_{p^*}(\tau)\|^2 d\tau \le C^* < \infty$$

$$\Downarrow$$

$$e^{2\lambda t}\mu_{p^*}(t) \le C^* \quad \forall t \ge 0$$

$$\Downarrow$$

when we select $\rho = p$ at time t we must have

$$e^{\lambda t}\mu_p(t) \le e^{\lambda t}\mu_{p^*}(t) \le C^* \qquad \Leftrightarrow \qquad \int_0^t e^{2\lambda\tau}\|e_p(\tau)\|^2 d\tau \le C^*$$

Two possible cases:

1. Switching will stop in finite time T at some $p \in \mathcal{P}$:

$$\int_0^\infty e^{2\lambda\tau}\|e_{\rho(\tau)}(\tau)\|^2 d\tau = \underbrace{\int_0^T e^{2\lambda\tau}\|e_{\rho(\tau)}(\tau)\|^2 d\tau}_{<\infty} + \underbrace{\int_T^\infty e^{2\lambda\tau}\|e_p(\tau)\|^2 d\tau}_{\le C^*} < \infty$$

Small error property

Assume $\mathcal{P}$ finite and $\exists\ p^* \in \mathcal{P}$:

(e.g., $\mathcal{L}_2$ noise and no unmodeled dynamics)

$$\|e_{p^*}\|^2_{\lambda,[0,\infty)} = \int_0^\infty e^{2\lambda\tau}\|e_{p^*}(\tau)\|^2 d\tau \le C^* < \infty$$

$$\Downarrow$$

$$e^{2\lambda t}\mu_{p^*}(t) \le C^* \quad \forall t \ge 0$$

$$\Downarrow$$

when we select $\rho = p$ at time t we must have

$$e^{\lambda t}\mu_p(t) \le e^{\lambda t}\mu_{p^*}(t) \le C^* \qquad \Leftrightarrow \qquad \int_0^t e^{2\lambda\tau}\|e_p(\tau)\|^2 d\tau \le C^*$$

Two possible cases:

2. After some finite time T switching will occur only among elements of a subset $\mathcal{P}^*$ of $\mathcal{P}$, each appearing in ρ infinitely many times:

$$\int_0^\infty e^{2\lambda\tau}\|e_{\rho(\tau)}(\tau)\|^2 d\tau = \underbrace{\int_0^T e^{2\lambda\tau}\|e_{\rho(\tau)}(\tau)\|^2 d\tau}_{<\infty} + \sum_{p\in\mathcal{P}^*} \underbrace{\int_T^\infty e^{2\lambda\tau}\|e_p(\tau)\|^2 d\tau}_{\le C^*} < \infty$$

Small error property

Assume $\mathcal{P}$ finite and $\exists\ p^* \in \mathcal{P}$:

(e.g., $\mathcal{L}_2$ noise and no unmodeled dynamics)

$$\|e_{p^*}\|^2_{\lambda,[0,\infty)} = \int_0^\infty e^{2\lambda\tau}\|e_{p^*}(\tau)\|^2 d\tau \le C^* < \infty$$

$$\Downarrow$$

$$e^{2\lambda t}\mu_{p^*}(t) \le C^* \quad \forall t \ge 0$$

$$\Downarrow$$

when we select $\rho = p$ at time t we must have

$$e^{\lambda t}\mu_p(t) \le e^{\lambda t}\mu_{p^*}(t) \le C^* \quad \Leftrightarrow \quad \int_0^t e^{2\lambda\tau}\|e_p(\tau)\|^2 d\tau \le C^*$$

Small error property: ($\mathcal{L}_2$ case)
Assume that $\mathcal{P}$ is a finite set. If $\exists\ p^* \in \mathcal{P}$ for which

$\|e_{p^*}\|_{\lambda,[0,\infty)} < \infty$ then $\|e_\rho\|_{\lambda,[0,\infty)} < \infty$

at least one error $\mathcal{L}_2$

“switched” error will be $\mathcal{L}_2$

Small error property

Small error property: ($\mathcal{L}_2$ case)
Assume that $\mathcal{P}$ is a finite set. If $\exists\ p^* \in \mathcal{P}$ for which

$\|e_{p^*}\|_{\lambda,[0,\infty)} < \infty$ then $\|e_\rho\|_{\lambda,[0,\infty)} < \infty$

at least one error $\mathcal{L}_2$

“switched” error will be $\mathcal{L}_2$
(for process switching signal ρ defined by the logic)

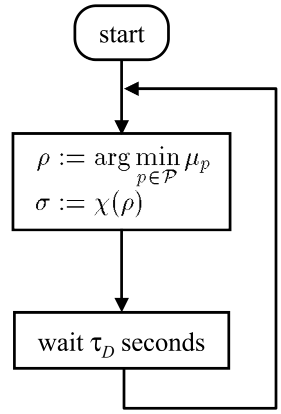
Small error property: (general case)
Assume that $\mathcal{Q}$ is a finite set with m element. For every $p \in \mathcal{P}$, $t \ge 0$, $\exists$ process switching signal $\rho_t : [0, t\,) \to \mathcal{P}$ such that:

1. $\sigma = \chi(\rho_t)$ except at most on m time intervals of length τ_D
2. $\|e_{\rho_t}\|_{\lambda,[0,t)} \le \sqrt{m}\,\|e_p\|_{\lambda,[0,t)}$

although the bound may not hold for e_ρ, it will hold for another process switching signal ρ_t that is “almost always” consistent with σ

The small error property can still be generalized for the case when $\mathcal{Q}$ is not finite (i.e., infinitely many controllers)

Implementation issues



monitoring signals

$$\mu_p(t) = \int_0^t e^{-2\lambda(t-\tau)}||e_p(\tau)||^2 d\tau \quad p \in \mathcal{P}$$

How to efficiently compute a large number of monitoring signals?

It is always possible to write:

appropriately defined function

$$||e_p||^2 = ||C_p x_E - y||^2 = k(p)' h(y, x_E) \;\; \forall p, y, x_E$$

From this and the definition of μ_p:

$$\dot{\mu}_p = -2\lambda \mu_p + k(p)' h(y, x_e) \qquad \mu_p(0) = 0$$

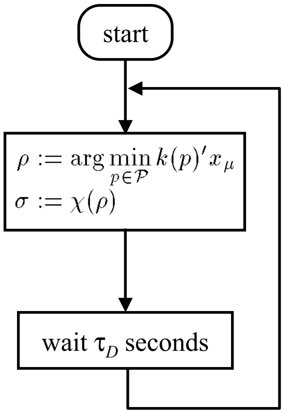
So, by linearity, all the μ_p can be generated by:

$$\dot{x}_\mu = -2\lambda x_\mu + h(y, x_E) \qquad x_\mu(0) = 0$$

$$\mu_p = k(p)' x_\mu \qquad p \in \mathcal{P}$$

dimension is independent of the number of element in $\mathcal{P}$

Implementation issues



monitoring signals

$$\mu_p(t) = \int_0^t e^{-2\lambda(t-\tau)}||e_p(\tau)||^2 d\tau \quad p \in \mathcal{P}$$

When $\mathcal{P}$ is a continuum (or very large), it may be issues with respect to the optimization for ρ.

Things are easy, e.g.,

1. $\mathcal{P}$ has a small number of elements
2. model is linearly parameterized on *p* (leads to *k*(*p*) quadratic)
3. there are closed form solutions (e.g., *k*(*p*) polynomial)
4. *k*(*p*) is convex on *p*

usual requirement in adaptive control

results still hold if there exists a computational delay τ_C *in performing the optimization, i.e.*

$$\rho(t) := \arg\min_{p\in\mathcal{P}} k(p)' x_\mu(t - \tau_C)$$

Analysis ($w = 0, \varepsilon = 0$)

Matching property: $\exists\, p^* \in \mathcal{P}$ such that

$$\|e_{p^*}\|_{\lambda,[0,t)} \le c_0 + c_w \cancel{\|w}\|_{\lambda,[0,t)} + \epsilon\, c_{\cancel{\epsilon}}\cancel{\|}u\|_{\lambda,[0,t)} \quad \forall t \ge 0$$

Detectability property: for frozen $\rho = p \in \mathcal{P}$ and $\sigma = \chi(p) \in \mathcal{Q}$ the injected system is asymptotically stable

Non-destabilizing property: The minimum interval between consecutive discontinuities of σ is $\tau_D > 0$.

Small error property ($\mathcal{L}_2$ case, $\mathcal{P}$ finite)**:** $\|e_{p^*}\|_{\lambda,[0,\infty)} < \infty \;\Rightarrow\; \|e_\rho\|_{\lambda,[0,\infty)} < \infty$

Analysis ($w = 0, \varepsilon = 0$)

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Small error property ($\mathcal{L}_2$ case, $\mathcal{P}$ finite)**:** $\|e_{p^*}\|_{\lambda,[0,\infty)} < \infty \;\Rightarrow\; \|e_\rho\|_{\lambda,[0,\infty)} < \infty$

Assumption (slow switching):

τ_D is large enough and λ is small enough so that the injected system is unif. exp. stable

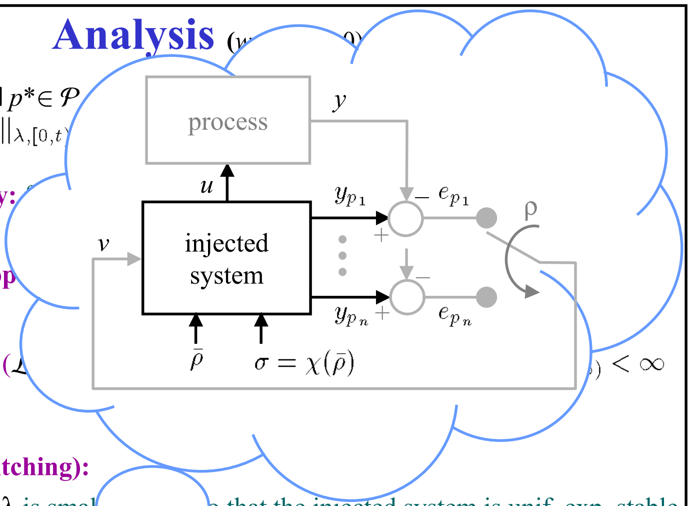
$$\exists c > 0,\ \bar{\lambda} > \lambda \qquad \|\Phi_{\bar{\rho}}(t,\tau)\| \le c e^{-\bar{\lambda}(t-\tau)} \qquad \forall t \ge \tau \ge 0$$

state transition matrix of injected system

$$\dot{z} = A_{\bar{\rho}\chi(\bar{\rho})} z$$

any process switching signal with interval between consecutive discontinuities no smaller than τ_D

Analysis (w 0)



Matching property: $\exists\, p^* \in \mathcal{P}$
$||e_{p^*}||_{\lambda,[0,t)}$

Detectability property:

Non-destabilizing prop

Small error property ($\mathcal{L}$ $) < \infty$

Assumption (slow switching):

τ_D is large enough and λ is smal o that the injected system is unif. exp. stable

$$\exists c > 0,\ \bar{\lambda} > \lambda \qquad ||\Phi_{\bar{\rho}}(t \quad \leq ce^{-\bar{\lambda}(t-\tau)} \qquad \forall t \geq \tau \geq 0$$

state transition matrix of injected system

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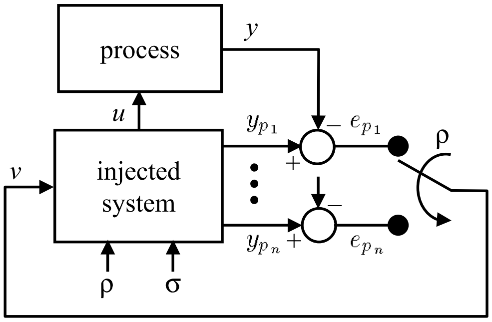
Analysis ($w = 0$, $\varepsilon = 0$)



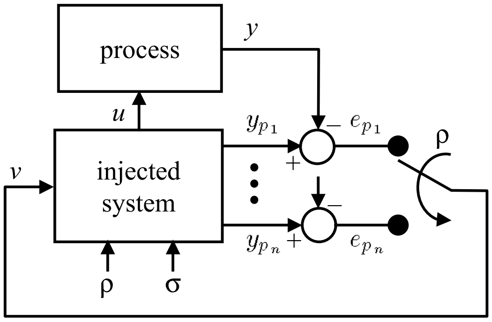
1st by the Matching property: $\exists\, p^* \in \mathcal{P}$ such that $||e_{p^*}||_{\lambda,[0,\infty)} < \infty$

2nd by the Small error property: $||v||_{\lambda,[0,\infty)} = ||e_\rho||_{\lambda,[0,\infty)} < \infty$

3rd by the Non-destabilization property & assumption the injected system is unif. exp. stable (state transition matrix decays faster then $e^{\lambda t}$)

$$\dot{x} = A_{\rho\sigma} x + B_\sigma v \qquad u = F_{\rho\sigma} x + G_\sigma v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

Analysis ($w = 0$, $\varepsilon = 0$)



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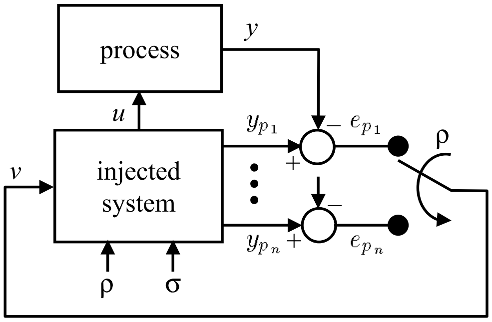
$$\dot{x} = A_{\rho\sigma}x + B_\sigma v \qquad u = F_{\rho\sigma}x + G_\sigma v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

4th by **2nd** and **3rd** $\|x\|_{\lambda,[0,\infty)} < \infty, \quad \lim_{t\to\infty} x(t) = 0$ (same for u and y_ρ)

Analysis ($w = 0$, $\varepsilon = 0$)



4th by **2nd** and **3rd** $\|x\|_{\lambda,[0,\infty)} < \infty, \quad \lim_{t\to\infty} x(t) = 0$ (same for u and y_ρ)

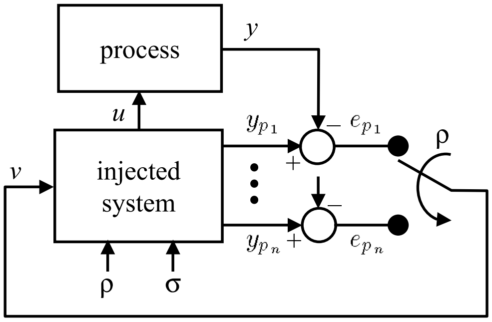
5th by the process' detectability:

$$\|u\|_{\lambda,[0,\infty)} < \infty \quad \|y\|_{\lambda,[0,\infty)} = \|y_\rho - e_\rho\|_{\lambda,[0,\infty)} < \infty$$

$$\Downarrow$$

state the process is also $\mathcal{L}_2$ and converges to zero

Analysis ($w = 0$, $\varepsilon = 0$)



Theorem:
Assuming that $\mathcal{P}$ is finite and in the absence of noise and unmodeled dynamics (i.e., $\varepsilon = 0$, $w(t) = 0$, $\forall\, t \ge 0$) the states of the process, the multi-estimator, and the multi-controller are all ($e^{\lambda t}$-weighted) $\mathcal{L}_2$ and converge to zero as $t \to \infty$.

Analysis (general case)

Matching property: $\exists\, p^* \in \mathcal{P}$ such that

$$\|e_{p^*}\|_{\lambda,[0,t)} \le c_0 + c_w \|w\|_{\lambda,[0,t)} + \epsilon\, c_\epsilon \|u\|_{\lambda,[0,t)} \qquad \forall t \ge 0$$

Detectability property: for $\rho = p \in \mathcal{P}$ and $\sigma = \chi(p) \in \mathcal{Q}$ the injected system is asymptotically stable

Non-destabilizing property: The minimum interval between consecutive discontinuities of σ is $\tau_D > 0$.

Small error property ($\mathcal{Q}$ finite)**:** $\forall\, p \in \mathcal{P}$, $t \ge 0$, $\exists\, \rho_t : [0, t) \to \mathcal{P}$ such that:

1. $\sigma = \chi(\rho_t)$ except at most on m time intervals of length τ_D
2. $\|e_{\rho_t}\|_{\lambda,[0,t)} \le \sqrt{m}\, \|e_p\|_{\lambda,[0,t)}$

we will start by cheating and assuming that $\sigma = \chi(\rho_t)$ ***all the time*** *…*

Assumption (slow switching): τ_D is large enough and λ is small enough so that

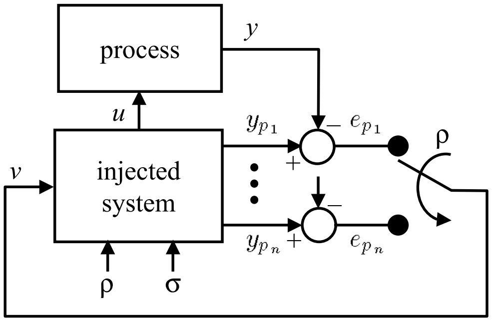
$$\exists c > 0,\ \bar{\lambda} > \lambda \qquad \|\Phi_{\bar{\rho}}(t,\tau)\| \le c e^{-\bar{\lambda}(t-\tau)} \qquad \forall t \ge \tau \ge 0$$

state transition matrix of injected system

$$\dot{z} = A_{\bar{\rho}\chi(\bar{\rho})} z$$

any process switching signal with interval between consecutive discontinuities no smaller than τ_D

Analysis (general case)



Consider a fixed interval $[0, t)$

1st by the Matching property: $\exists\, p^* \in \mathcal{P}$ such that

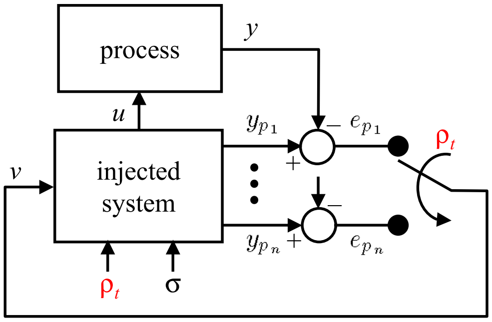
$$||e_{p^*}||_{\lambda,[0,t)} \leq c_0 + c_w ||w||_{\lambda,[0,t)} + \epsilon\, c_\epsilon ||u||_{\lambda,[0,t)}$$

2nd by the Small error property: $\exists\, \rho_t$ such that "$\sigma = \chi(\rho_t)$" &

$$||e_{\rho_t}||_{\lambda,[0,t)} \leq \sqrt{m}||e_{p^*}||_{\lambda,[0,t)} \leq c_0\sqrt{m} + c_w\sqrt{m}||w||_{\lambda,[0,t)} + \epsilon\, c_\epsilon\sqrt{m}||u||_{\lambda,[0,t)}$$

3rd use ρ_t from Small error property to construct the injected system

Analysis (general case)



2nd by the Small error property: $\exists\, \rho_t$ such that "$\sigma = \chi(\rho_t)$" &

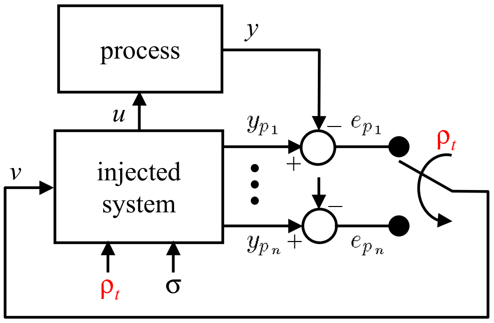
$$||v||_{\lambda,[0,t)} = ||e_{\rho_t}||_{\lambda,[0,t)} \leq c_0\sqrt{m} + c_w\sqrt{m}||w||_{\lambda,[0,t)} + \epsilon\, c_\epsilon\sqrt{m}||u||_{\lambda,[0,t)}$$

3rd use ρ_t from Small error property to construct the injected system
since "$\sigma = \chi(\rho_t)$" the injected system switches among stability matrices

4th by the Non-destabilization property & assumption the injected system is unif. exp. stable (state transition matrix decays faster then $e^{\lambda t}$)

$$\dot{x} = A_{\rho\sigma}x + B_\sigma v \qquad u = F_{\rho\sigma}x + G_\sigma v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

Analysis (general case)



2nd by the Small error property: ∃ ρ_t such that "$\sigma = \chi(\rho_t)$" &

$$\|v\|_{\lambda,[0,t)} = \|e_{\rho_t}\|_{\lambda,[0,t)} \le c_0\sqrt{m} + c_w\sqrt{m}\|w\|_{\lambda,[0,t)} + \epsilon\, c_\epsilon\sqrt{m}\|u\|_{\lambda,[0,t)}$$

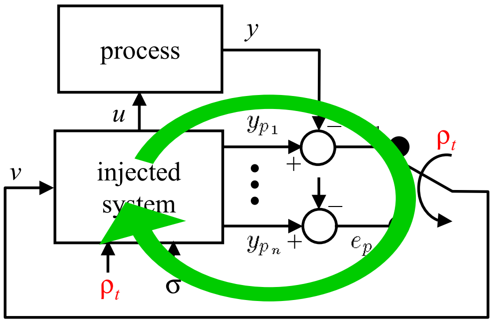
4th by the Non-destabilization property & assumption the injected system is unif. exp. stable (state transition matrix decays faster then $e^{\lambda t}$)

$$\dot{x} = A_{\rho\sigma}x + B_\sigma v \qquad u = F_{\rho\sigma}x + G_\sigma v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

$$\Downarrow$$

finite $\|\cdot\|_{\lambda,[0,t)}$ induced norm from v to u : $\|u\|_{\lambda,[0,t)} \le \gamma\|v\|_{\lambda,[0,t)} + \bar{c}_0$

Analysis (general case)



2nd by the Small error property: ∃ ρ_t such that "$\sigma = \chi(\rho_t)$" &

$$\|v\|_{\lambda,[0,t)} = \|e_{\rho_t}\|_{\lambda,[0,t)} \le c_0\sqrt{m} + c_w\sqrt{m}\|w\|_{\lambda,[0,t)} + \epsilon\, c_\epsilon\sqrt{m}\|u\|_{\lambda,[0,t)}$$

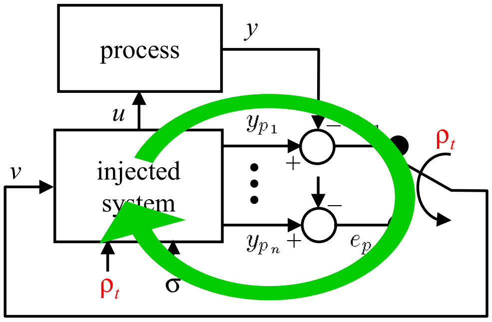
4th by the Non-destabilization property & assumption: $\|u\|_{\lambda,[0,t)} \le \gamma\|v\|_{\lambda,[0,t)} + \bar{c}_0$

5th by small-gain argument (using **2nd** and **4th**)

$$\epsilon < \frac{1}{\gamma\, c_\epsilon\sqrt{m}} \quad \Rightarrow \quad \|v\|_{\lambda,[0,t)} \le \frac{(c_0 + \epsilon\, c_\epsilon \bar{c}_0)\sqrt{m}}{1 - \epsilon\,\gamma\, c_\epsilon\sqrt{m}} + \frac{c_w\sqrt{m}}{1 - \epsilon\,\gamma\, c_\epsilon\sqrt{m}}\|w\|_{\lambda,[0,t)}$$

just as before: v bounded & injected system stable ⇒ … ⇒ all signals bounded

Analysis (general case)



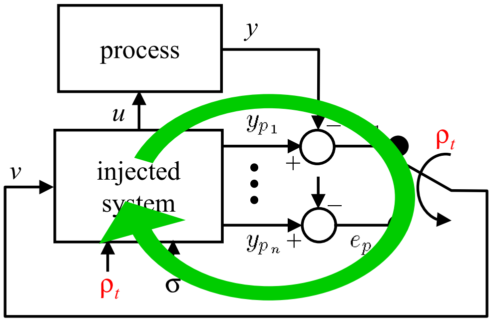
Where did we use "$\sigma = \chi(\rho_t)$" ?

3rd use ρ_t from Small error property to construct the injected system
since "$\sigma = \chi(\rho_t)$" the injected system switches among stability matrices

4th by the Non-destabilization property & assumption the injected system is unif. exp. stable (state transition matrix decays faster then $e^{\lambda t}$)

$$\dot{x} = A_{\rho\sigma}x + B_{\sigma}v \qquad u = F_{\rho\sigma}x + G_{\sigma}v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

Analysis (general case)



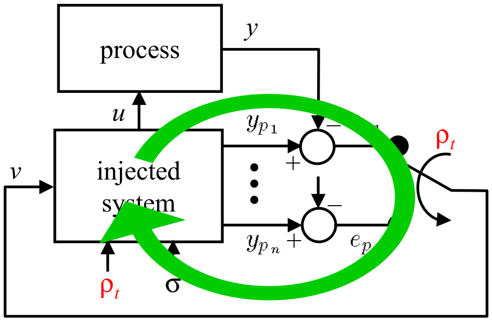
Where did we use "$\sigma = \chi(\rho_t)$" ?

3rd use ρ_t from Small error property to construct the injected system
since $\sigma = \chi(\rho_t)$ except at most on m time intervals of length τ_D,
the injected system switches among stability matrices except at most on m time intervals of length τ_D

4th by the Non-destabilization property & assumption the injected system is still unif. exp. stable (state transition matrix decays faster then $e^{\lambda t}$)

$$\dot{x} = A_{\rho\sigma}x + B_{\sigma}v \qquad u = F_{\rho\sigma}x + G_{\sigma}v \qquad y_p = C_p x \quad p \in \mathcal{P}$$

Analysis (general case)



Theorem:
Assuming that Q is a finite set with m element and

$$\epsilon < \frac{1}{\gamma \, c_\epsilon \sqrt{m}}$$

the $\| \cdot \|_{\lambda,[0,t)}$ norm of all signals can be bounded by expressions of the form

$$\bar{c}_0 + \bar{c}_w ||w||_{\lambda,[0,t)}$$

(finite induced norms)

Moreover:

- $w(t)$ is uniformly bounded for $t \in [0,\infty)$ ⇒ all signals uniformly bounded
- $w(t) \to 0$ as $t \to \infty$ ⇒ all signals converge to zero as $t \to \infty$

Fast switching

So far…

Assumption (slow switching): τ_D is large enough and λ is small enough so that

$$||\Phi_{\bar{\rho}}(t,\tau)|| \leq c e^{-\bar{\lambda}(t-\tau)} \qquad \forall t \geq \tau \geq 0$$

state transition matrix of injected system

$$\dot{z} = A_{\bar{\rho}\chi(\bar{\rho})} z$$

any process switching signal with interval between consecutive discontinuities no smaller than τ_D

Can be relaxed to …

Assumption (fast switching):
λ is small enough so that all matrices $A_{p\,\chi(p)} + \lambda I, p \in \mathcal{P}$ are asymptotically stable

(any dwell-time τ_D will do)

Fast switching

Assumption (fast switching):

λ is small enough so that all matrices $A_{p\,\chi(p)} + \lambda I, p \in \mathcal{P}$ are asymptotically stable

(any dwell-time τ_D will do)

Theorem:

Assuming that the process is SISO and that the multi-controller is realized as

$$\dot{x}_C = (A_C + d_C f_\sigma) x_C + b_C y, \qquad u = f_\sigma x_C + j_\sigma y,$$

(no loss of generality)

there exists a constant ϵ^* such that when

$$\epsilon < \epsilon^*$$

the $\|\cdot\|_{\lambda,[0,t)}$ norm of all signals can be bounded by expressions of the form

$$\bar{c}_0 + \bar{c}_w \|w\|_{\lambda,[0,t)}$$

(finite induced norms)

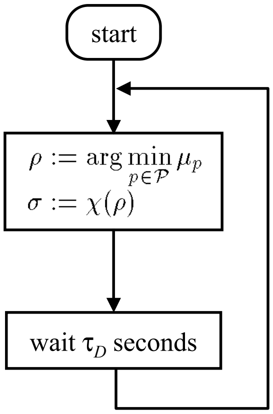
Moreover:

- $w(t)$ is uniformly bounded for $t \in [0,\infty)$ $\Rightarrow$ all signals uniformly bounded
- $w(t) \to 0$ as $t \to \infty$ $\Rightarrow$ all signals converge to zero as $t \to \infty$

proof: utilize internal structure of injected system & inject more errors to "boost rate of decay"…

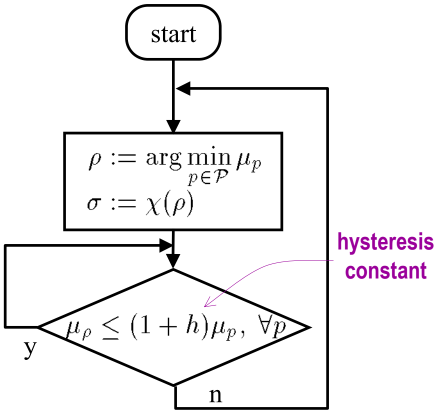
Other logics

Dwell-time switching



wait fixed amount of time

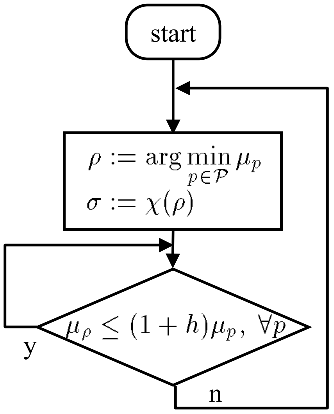
Scale-independent hysteresis switching



wait until current monitoring signal becomes significantly larger than some other one

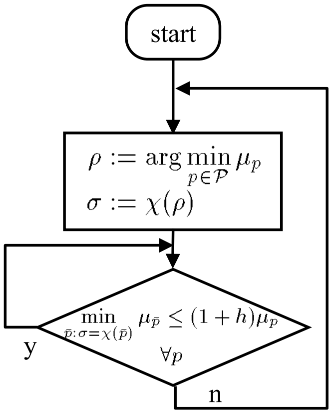
Other logics

Scale-independent hysteresis switching



wait until current monitoring signal becomes significantly larger than some other one

Hierarchical hysteresis switching



*wait until current monitoring signal, **or another one consistent with** σ, becomes significantly larger than some other one*

Other logics

For both logics we have:

Small error property:

For every $p \in \mathcal{P}$, $t \geq 0$, ∃ process switching signal $\rho_t : [0, t) \rightarrow \mathcal{P}$ such that:

$$\sigma = \chi(\rho_t) \qquad \|e_{\rho_t}\|^2_{\lambda,[0,t)} \le (1+h)m\,\mu_p(t)$$

all the time !

Non-destabilizing property: For every $p \in \mathcal{P}$

$$N_\sigma(\tau,t) \le 1 + m + \frac{m\log\left(\frac{\mu_p(t)}{\epsilon+e^{-\lambda t}\epsilon_0}\right)}{\log(1+h)} + \frac{m\lambda(t-\tau)}{\log(1+h)} \qquad \forall t > \tau \geq 0$$

number of switchings in the interval $[\tau, t)$

number of elements in $\mathcal{P}$ (scale-indep.) or in $\mathcal{Q}$ (hierarchical)

average dwell-time type growth

$\Downarrow$

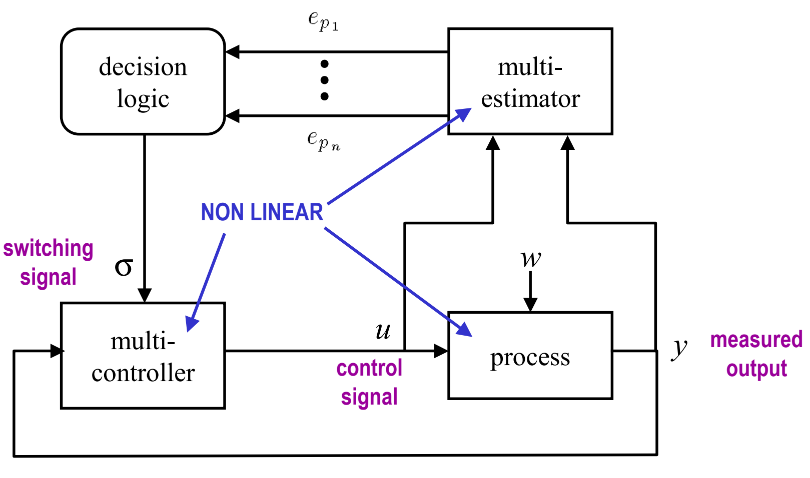
either
- large h (hysteresis constant) or
- small λ (forgetting factor)

leads to stability of injected system

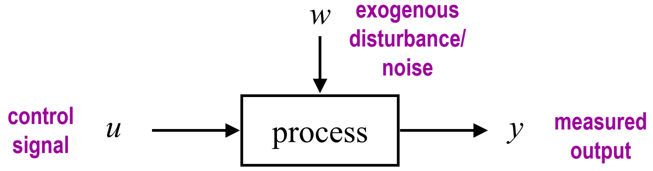
Outline

- ✓ Supervisory control overview
- ✓ Estimator-based linear supervisory control
- ✗ Estimator-based nonlinear supervisory control
- ✗ Examples

Estimator-based nonlinear supervisory control



Class of admissible processes



$$\dot{x}_P = A(x_P, w, u) \qquad y = C(x_P, w)$$

Process is assumed to be in a family

unmodeled dynamics

$$\mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$$

parametric uncertainty

$\mathcal{M}_p \equiv$ small family of systems around a nominal process model N_p

Typically $$\mathcal{M}_p := \left\{ \mathrm{M}_p : \mathrm{d}(\mathrm{M}_p, \mathrm{N}_p) \leq \epsilon_p \right\}$$

metric on set of state-space model (?)

Most results presented here:

- independent of metric d (e.g., detectability)
- or restricted to case $\varepsilon_p = 0$ (e.g., matching)

Candidate controllers

Class of admissible processes

$$\mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$$

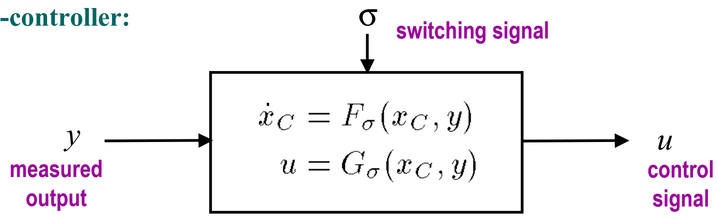
$\mathcal{M}_p \equiv$ small family of systems around a nominal process model N_p

Assume given a family of *candidate controllers*

$$\mathcal{C} := \left\{ \dot{z}_q = F_q(z_q, y),\ u = G_q(z_q, y) : q \in \mathcal{Q} \right\}$$

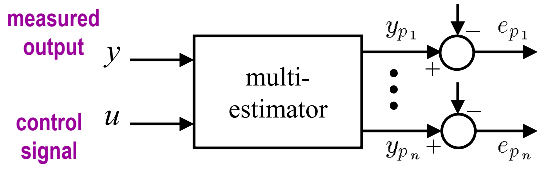
(without loss of generality all with same dimension)

Multi-controller:



$$\dot{x}_C = F_\sigma(x_C, y)$$
$$u = G_\sigma(x_C, y)$$

Multi-estimator



$$\dot{x}_E = A_E(x_E, y, u)$$
$$y_p = C_p(x_E) \quad p \in \mathcal{P}$$

How to design a multi-estimator?

we want: **Matching property:** there exist some $p^* \in \mathcal{P}$ such that e_{p^*} is "small"

Typically obtained by:

process in $\mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$ ➡ $\exists\, p^* \in \mathcal{P}$: process in $\mathcal{M}_{p^*}$ ➡ e_{p^*} is "small"

when process "matches" $\mathcal{M}_{p^*}$ the corresponding error must be "small"

Designing multi-estimators - I

(state accessible)

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot{z} = A_p(z, u) \qquad y = z \qquad p \in \mathcal{P}$$

no exogenous input w **state accessible**

Multi-estimator:

$$\dot{z}_p = A(z_p - y) + A_p(y, u) \qquad y_p = z_p \qquad p \in \mathcal{P}$$

asymptotically stable A

When process matches the nominal model N_{p^*}

exponentially

$$e_{p^*} := y_{p^*} - y = z_{p^*} - z \quad \Rightarrow \quad \dot{e}_{p^*} = A e_{p^*} \quad \Rightarrow \quad e_{p^*} \to 0 \text{ as } t \to \infty$$

Matching property: Assume $\mathcal{M} = \{\, \mathrm{N}_p : p \in \mathcal{P} \,\}$

$$\exists\, p^* \in \mathcal{P},\, c_0,\, \lambda^* > 0 : \qquad \| e_{p^*}(t) \| \leq c_0\, e^{-\lambda^* t} \qquad t \geq 0$$

State-sharing

Multi-estimator:

$$\dot z_p = A(z_p - y) + A_p(y,u) \qquad y_p = z_p \qquad p \in \mathcal{P}$$

state of multi-estimator is $x_E := \{ z_p : p \in \mathcal{P} \}$

can be large when $\mathcal{P}$ has many elements

Suppose $A_p(y, u)$ is *separable*, i.e.,

$$A_p(y,u) = \overset{\text{matrix}}{M(y,u)}\ \overset{\text{vector}}{k(p)} \qquad \forall p,\ u,\ y$$

true, e.g., is process is linearly parameterized

By linearity, the y_p can be generated by:

$$\dot X_E = A(X_E - Y) + M(y,u) \qquad y_p = X_E k(p) \quad p \in \mathcal{P}$$

matrix with the size of M with every column equal to y

The dimension of the multi-estimator is independent of the number of elements in $\mathcal{P}$ (could even by infinity)

Designing multi-estimators - II

(output-injection away from stable linear system)

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot z = A_p z + B_p w + H_p(y,u) \qquad y = C_p z + D_p w \qquad p \in \mathcal{P}$$

asymptotically stable A_p

nonlinear output injection

(generalization of case I)

Multi-estimator:

$$\dot z_p = A_p z_p + H_p(y,u) \qquad y_p = C_p z_p \qquad p \in \mathcal{P}$$

When process matches the nominal model N_{p^*}

$$\tilde z_{p^*} := z_{p^*} - z \quad \Rightarrow \quad \begin{aligned} \dot{\tilde z}_{p^*} &= A_p \tilde z_{p^*} - B_p w \\ e_{p^*} &= C_p \tilde z_{p^*} - D_p w \end{aligned}$$

Matching property: Assume $\mathcal{M} = \{ N_p : p \in \mathcal{P} \}$

$$\exists\, p^* \in \mathcal{P}, c_0, c_w, \lambda^* > 0 : \quad \| e_{p^*}(t) \| \le c_0\, e^{-\lambda^* t} + c_w \qquad t \ge 0$$

with $c_w = 0$ in case $w(t) = 0$, $\forall\, t \ge 0$

State-sharing is possible when all A_p are equal an $H_p(y, u)$ is *separable*:

$$H_p(y,u) = M(y,u)\,k(p) \quad \forall p, u, y$$

Designing multi-estimators - III

(output-inj. and coord. transf. away from stable linear system)

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot{\bar{z}} = \zeta_p(\bar{z})\Big(A_p \xi_p^{-1}(\bar{z}) + B_p w + H_p(y,u)\Big) \qquad y = C_p \xi_p^{-1}(\bar{z}) + D_p w \qquad p \in \mathcal{P}$$

asymptotically stable A_p

(generalization of case I & II)

$\bar{z} = \xi_p(z) \equiv$ cont. diff. coordinate transformation with continuous inverse ξ_p^{-1} (may depend on unknown parameter p)

$\zeta_p := \xi_p' \circ \xi_p^{-1}$

The Matching property is an input/output property so the same multi-estimator can be used:

$$\dot{z}_p = A_p z_p + H_p(y,u) \qquad y_p = C_p z_p \qquad p \in \mathcal{P}$$

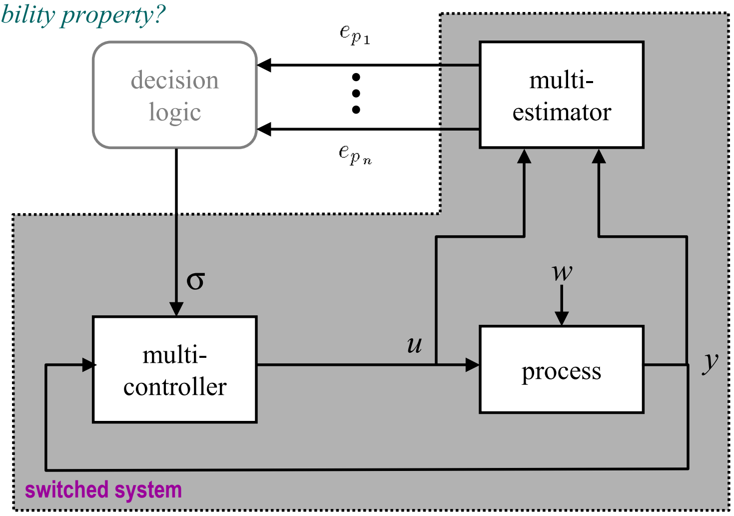
Matching property: Assume $\mathcal{M} = \{ N_p : p \in \mathcal{P} \}$

$$\exists\, p^* \in \mathcal{P},\ c_0,\ c_w,\ \lambda^* > 0 : \qquad \| e_{p^*}(t) \| \leq c_0\, e^{-\lambda^* t} + c_w \qquad t \geq 0$$

with $c_w = 0$ in case $w(t) = 0$, $\forall\, t \geq 0$

Switched system

detectability property?



Also now the switched system can be seen as the interconnection of the process *with the* injected system

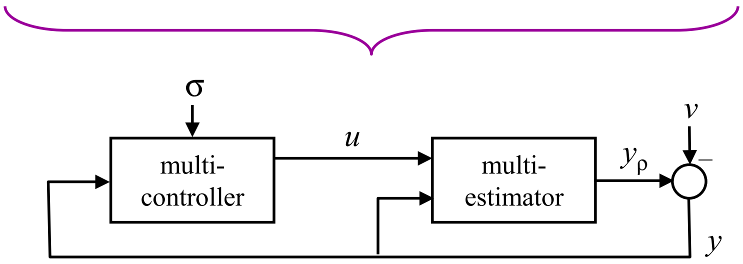
Constructing the injected system

1st Take a process switching signal $\rho : [0,\infty) \rightarrow \mathcal{P}$.

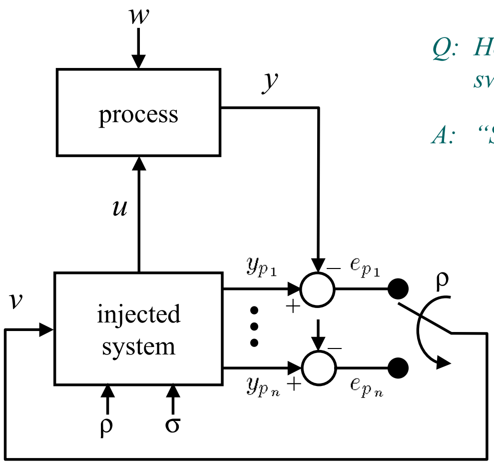
2nd Define the signal $v := e_\rho = y_\rho - y$

3rd Replace y in the equations of the multi-estimator and multi-controller by $y_\rho - v$.

$$\begin{aligned} \dot{x}_E &= A_E(x_E, y_\rho - v, u) & \qquad \dot{x}_C &= F_\sigma(x_C, y_\rho - v) \\ y_p &= C_p(x_E) \quad p \in \mathcal{P} & u &= G_\sigma(x_C, y_\rho - v) \end{aligned}$$



Switched system = process + injected system



Q: How to get "detectability" on the switched system ?

A: "Stability" of the injected system

Stability & detectability of nonlinear systems

Stability: input u "small" ⇒ state x "small"

$$\dot{x} = A(x,u) \qquad A(0,0) = 0$$

Input-to-state stable (ISS) if $\exists\ \beta\in\mathcal{KL},\ \gamma\in\mathcal{K}$

$$||x(t)|| \le \beta(||x(0)||,t) + \sup_{\tau\in[0,t)} \gamma(||u(\tau)||) \qquad \forall t \ge 0$$

Integral input-to-state stable (iISS) if $\exists\ \alpha\in\mathcal{K}_\infty,\ \beta\in\mathcal{KL},\ \gamma\in\mathcal{K}$

$$\alpha(||x(t)||) \le \beta(||x(0)||,t) + \int_{\tau\in[0,t)} \gamma(||u(\tau)||) \qquad \forall t \ge 0$$

strictly weaker

Notation:

$\alpha:[0,\infty) \to [0,\infty)$ is class $\mathcal{K}$ ≡ continuous, strictly increasing, $\alpha(0) = 0$

is class $\mathcal{K}_\infty$ ≡ class $\mathcal{K}$ and unbounded

$\beta:[0,\infty)\times[0,\infty) \to [0,\infty)$ is class $\mathcal{KL}$ ≡ $\beta(\cdot,t) \in \mathcal{K}$ for fixed t &
$\lim_{t\to\infty} \beta(s,t) = 0$ (monotonically) for fixed s

---

Stability & detectability of nonlinear systems

Stability: input u "small" ⇒ state x "small"

$$\dot{x} = A(x,u) \qquad A(0,0) = 0$$

Input-to-state stable (ISS) if $\exists\ \beta\in\mathcal{KL},\ \gamma\in\mathcal{K}$

$$||x(t)|| \le \beta(||x(0)||,t) + \sup_{\tau\in[0,t)} \gamma(||u(\tau)||) \qquad \forall t \ge 0$$

Integral input-to-state stable (iISS) if $\exists\ \alpha\in\mathcal{K}_\infty,\ \beta\in\mathcal{KL},\ \gamma\in\mathcal{K}$

$$\alpha(||x(t)||) \le \beta(||x(0)||,t) + \int_{\tau\in[0,t)} \gamma(||u(\tau)||) \qquad \forall t \ge 0$$

strictly weaker

One can show:

1. for ISS systems: $u \to 0$ ⇒ solution exist globally & $x \to 0$
2. for iISS systems: $\int_0^\infty \gamma(||u||) < \infty$ ⇒ solution exist globally & $x \to 0$

Stability & detectability of nonlinear systems

Detectability: input u & output y "small" ⇒ state x "small"

$$\dot{x} = A(x,u) \qquad y = C(x,u)$$

Detectability (or input/output-to-state stability IOSS) if $\exists\ \beta\in\mathcal{KL},\ \gamma_u, \gamma_y\in\mathcal{K}$

$$\|x(t)\| \le \beta(\|x(0)\|,t) + \sup_{\tau\in[0,t)} \gamma_u(\|u(\tau)\|) + \sup_{\tau\in[0,t)} \gamma_y(\|y(\tau)\|) \qquad \forall t \ge 0$$

Integral detectable (iIOSS) if $\exists\ \alpha\in\mathcal{K}_\infty,\ \beta\in\mathcal{KL},\ \gamma_u, \gamma_y\in\mathcal{K}$

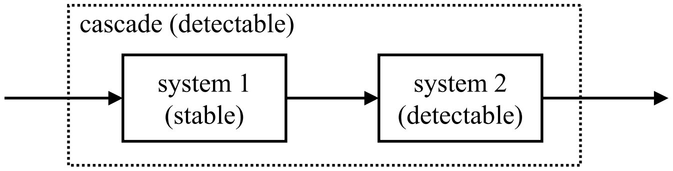
strictly weaker

$$\alpha(\|x(t)\|) \le \beta(\|x(0)\|,t) + \int_{\tau\in[0,t)} \gamma_u(\|u(\tau)\|) + \int_{\tau\in[0,t)} \gamma_y(\|y(\tau)\|) \qquad \forall t \ge 0$$

One can show:

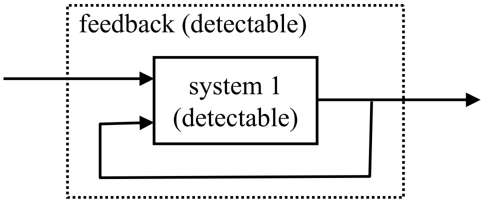
1. for IOSS systems: $u, y \to 0 \Rightarrow x \to 0$
2. for iIOSS systems: $\int_0^\infty \gamma_u(\|u\|),\ \int_0^\infty \gamma_y(\|y\|) < \infty \Rightarrow x \to 0$

Interconnecting stable & detectable systems



Lemma 1 (cascade):

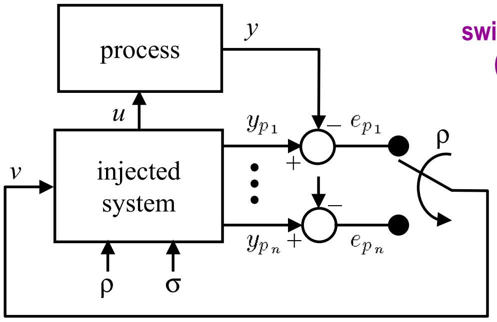
i. system 1 ISS & system 2 detectable ⇒ cascade detectable
ii. system 1 integral ISS & system 2 detectable ⇒ cascade integral detectable



Lemma 2 (feedback):

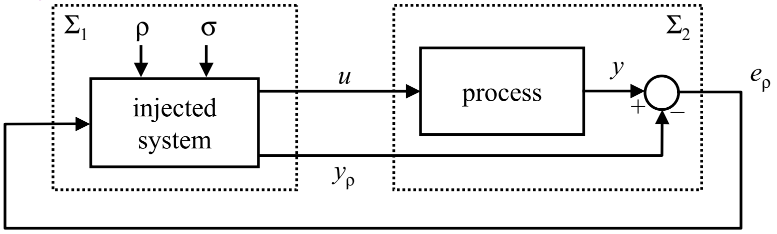
i. system 1 detectable ⇒ feedback detectable
ii. system 1 integral detectable ⇒ feedback integral detectable

Certainty Equivalence Stabilization Theorem



switched system
(version 1)

switched system
(version 2)



Certainty Equivalence Stabilization Theorem

1st *process detectable* ⇒ system Σ_2 detectable

2nd *injected system ISS* & **1st** ⇒ cascade $\Sigma_1 - \Sigma_2$ detectable

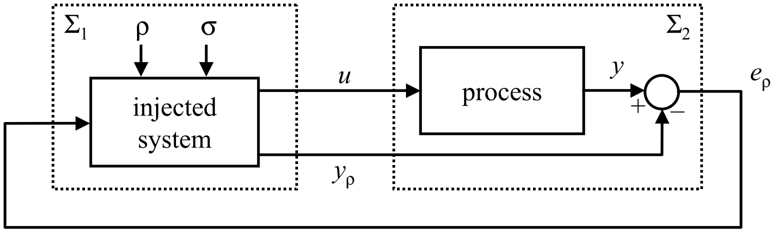
3rd cascade $\Sigma_1 - \Sigma_2$ detectable ⇒ switched system detectable

or

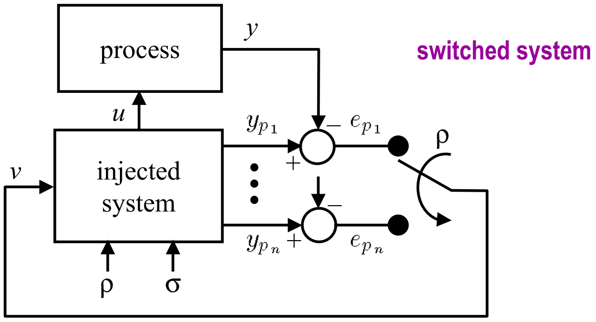
2nd' *injected system integral ISS* & **1st** ⇒ cascade $\Sigma_1 - \Sigma_2$ integral detectable

3rd' cascade $\Sigma_1 - \Sigma_2$ integral detectable ⇒ switched system integral detectable

switched system
(version 2)



Certainty Equivalence Stabilization Theorem



Theorem: (Certainty Equivalence Stabilization Theorem)

Suppose the process is detectable and take fixed $\rho = p \in \mathcal{P}$ and $\sigma = q \in \mathcal{Q}$

1. injected system ISS ⇒ switched system detectable.
2. injected system integral ISS ⇒ switched system integral detectable

Stability of the injected system is not the only mechanism to achieve detectability:
e.g., injected system i/o stable + process "min. phase" ⇒ detectability of switched system

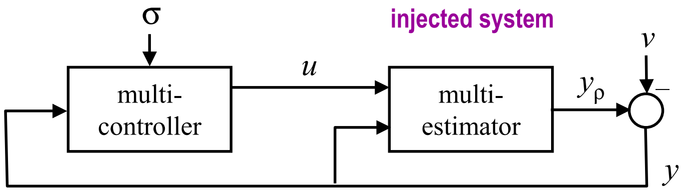
(Nonlinear Certainty Equivalence Output Stabilization Theorem)

Achieving ISS for the injected system

Theorem: (Certainty Equivalence Stabilization Theorem)

Suppose the process is detectable and take fixed $\rho = p \in \mathcal{P}$ and $\sigma = q \in \mathcal{Q}$

1. injected system ISS ⇒ switched system detectable.
2. injected system integral ISS ⇒ switched system integral detectable



$\rho = p \in \mathcal{P}$
$\sigma = q \in \mathcal{Q}$

We want to design candidate controllers *that make the injected system (at least) integral ISS with respect to the "disturbance" input v*

Nonlinear robust control design problem, but…

- "disturbance" input v can be measured ($v = e_\rho = y_\rho - y$)
- the whole state of the injected system is measurable (x_C, x_E)

Designing candidate controllers - I

(feedback linearizable)

Suppose the multi-estimator is of the form

$$\dot{x}_E = Ax_E + B(\Psi(x_E, y) + u) + Dy \qquad y_p = C_p x_E \qquad p \in \mathcal{P}$$

with ($A+D\,C_p$, B) stabilizable

To obtain the injected system, we make $y = y_p - v$:

$$\dot{x}_E = (A + DC_p)x_E + B\big(\Psi(x_E, C_p x_E - v) + u\big) - Dv$$

must be input-to-state stabilized by the $q = \chi(p)$
candidate controller (with respect to “disturbance” v)

Candidate controller $q = \chi(p)$:

with $A+D\,C_p+B\,F_p$ asymptotically stable

$$u = -\Psi(x_E, C_p x_E - v) + F_p x_E$$

$$\Downarrow$$

injected system ISS: $\dot{x}_E = (A + DC_p + F_p)x_E - Dv$

Also applicable if the multi-estimator is a coordinate transformation away from this form…

Designing candidate controllers - II

(input/output feedback linearizable)

Suppose for each p $\in \mathcal{P}$ we can

1. partition the state of the multi-estimator as $x_E = (x_p, \bar{x}_p)$
2. write its dynamics as

with ($A+D_p\,C_p$, B_p) stabilizable

$$\dot{x}_p = A_p x_p + B_p(\Psi_p(x_E, y) + u) + D_p y \qquad y_p = C_p x_p$$
$$\dot{\bar{x}}_p = \bar{A}_p(\bar{x}_p, x_p, u, y)$$

ISS with respect to “input” (x_p, u, y)

Also applicable if the multi-estimator is a coordinate transformation (possibly p-dependent) away from this form…

This form is quite common because
typically one can linearize the system
with respect to each individual y_p but not
with respect to all the y_p simultaneously

Designing candidate controllers - II

(input/output feedback linearizable)

Suppose for each p ∈ $\mathcal{P}$ we can

1. partition the state of the multi-estimator as $x_E = (x_p, \bar{x}_p)$
2. write its dynamics as

with ($A+D_p C_p$, B_p) stabilizable

$$\dot{x}_p = A_p x_p + B_p(\Psi_p(x_E, y) + u) + D_p y \qquad y_p = C_p x_p$$
$$\dot{\bar{x}}_p = \bar{A}_p(\bar{x}_p, x_p, u, y)$$

ISS with respect to "input" (x_p, u, y)

Also applicable if the multi-estimator is a coordinate transformation (possibly p-dependent) away from this form…

Candidate controller $q = \chi(p)$:

with $A+D_p C_p + B_p F_p$ asymptotically stable

$$u = -\Psi(x_E, C_p x_E - v) + F_p x_E$$

$$\Downarrow$$

injected system ISS:

$$\dot{x}_p = (A_p + D_p C_p + B_p F_p) x_p - D_p v,$$
$$\dot{\bar{x}}_p = \bar{A}_p(\bar{x}_p, x_p, -\Psi(x_E, C_p x_p - v) + F_p x_p, C_p x_p - v),$$

(cascade of ISS systems)

Detectability property

Detectability property:

For any of the previous multi-estimators and candidate controller

$$u = -\Psi(x_E, C_p x_E - v) + F_p x_E$$

1. The injected system is ISS (and also integral ISS)
2. The switched system is detectable through e_p (and also integral detectable)

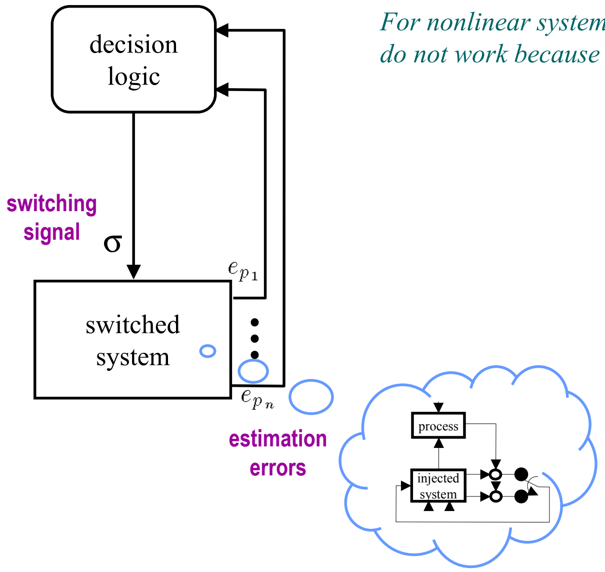
$$\alpha_p(||\bar{x}(t)||) \le \beta_p(||\bar{x}(0)||, t) + \int_{\tau\in[0,t)} \gamma_p(||e_p(\tau)||) + \int_{\tau\in[0,t)} \varphi_p(||w(\tau)||) \qquad t \ge 0$$

Remarks:

a. Other controllers also lead to detectability, e.g., one could
  - *1st use the feedback linearizing controller to find an ISS control Lyapunov function*
  - *2nd use the ISS control Lyapunov to construct a more robust controller (e.g., using an inverse optimal design)*

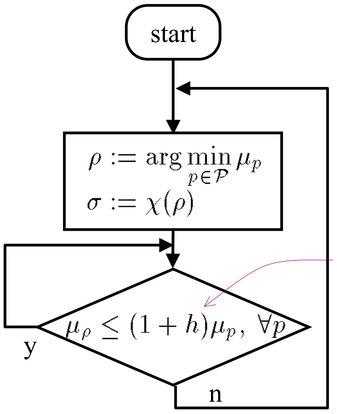
b. It is possible to achieve iISS for much larger classes of systems (e.g., systems that cannot even be controlled by smooth time-invariant feedback)

Decision logic



For nonlinear systems dwell-time logics do not work because of finite escape

Scale-independent hysteresis switching



hysteresis constant

wait until current monitoring signal becomes significantly larger than some other one

class $\mathcal{K}$ function from detectability property

monitoring signals

$$\mu_p(t) := e^{-\lambda t}\epsilon_0 + \int_0^t e^{-\lambda(t-\tau)}\gamma_p(\|e_p(\tau)\|)d\tau$$

$$p \in \mathcal{P}$$

measure of the size of e_p over a "window" of length $1/\lambda$

forgetting factor

All the μ_p can be generated by a system with small dimension if $\gamma_p(\|e_p\|)$ is separable. i.e.,

$$\gamma_p(\|e_p\|) = k(p)'h(y,u,x_E) \quad \forall p,u,y,x_E,$$

Scale-independent hysteresis switching

Theorem: Let $\mathcal{P}$ be finite with m elements. For every $p \in \mathcal{P}$

$$N_\sigma(\tau, t) \le 1 + m + \frac{m \log\left(\epsilon_0^{-1} e^{\lambda t} \mu_p(t)\right)}{\log(1+h)} + \frac{m\lambda(t-\tau)}{\log(1+h)} \qquad \forall t > \tau \ge 0$$

number of switchings in $[\tau, t)$

and

$$\int_0^t e^{-\lambda(t-\tau)} \gamma_\rho(\|e_\rho(\tau)\|) d\tau \le (1+h) m \mu_p(t) \qquad \forall t > 0$$

Assume $\mathcal{P}$ is finite, the γ_p are locally Lipschitz and

$$\exists p^* \in \mathcal{P}, c_0 > 0, \lambda^* > \lambda : \qquad \|e_{p^*}(t)\| \le c_0 e^{-\lambda^* t} \qquad \forall t \in [0, T_{\max})$$

maximum interval of existence of solution

$$\Downarrow$$

$$e^{\lambda t} \mu_p(t) = \epsilon_0 + \int_0^t e^{\lambda \tau} \gamma_p(\|e_p(\tau)\|) d\tau \quad \Rightarrow \quad N_\sigma(t, \tau) \ \& \int_0^t e^{\lambda \tau} \gamma_\rho(\|e_\rho(\tau)\|) d\tau$$

uniformly bounded on $[0, T_{\max})$ ⇒ uniformly bounded on $[0, T_{\max})$

Scale-independent hysteresis switching

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number of switchings in $[\tau, t)$

and

$$\int_0^t e^{-\lambda(t-\tau)} \gamma_\rho(\|e_\rho(\tau)\|) d\tau \le (1+h) m \mu_p(t) \qquad \forall t > 0$$

Assume $\mathcal{P}$ is finite, the γ_p are locally Lipschitz and

$$\exists p^* \in \mathcal{P}, c_0 > 0, \lambda^* > \lambda : \qquad \|e_{p^*}(t)\| \le c_0 e^{-\lambda^* t} \qquad \forall t \in [0, T_{\max})$$

maximum interval of existence of solution

Non-destabilizing property: Switching will stop at some finite time $T^* \in [0, T_{\max})$

Small error property: $\displaystyle\int_0^t e^{\lambda \tau} \gamma_\rho(\|e_\rho(\tau)\|) d\tau \le C^* < \infty \qquad \forall t \in [0, T_{\max})$

Analysis

($w = 0$, no unmodeled dynamics)

1st by the Matching property: $\exists\, p^* \in \mathcal{P}$ such that $\| e_{p^*}(t) \| \leq c_0\, e^{-\lambda^* t} \quad t \geq 0$

2nd by the Non-destabilization property: switching stops at a finite time

$$T^* \in [0, T_{\max}) \;\Rightarrow\; \rho(t) = p \;\&\; \sigma(t) = \chi(p) \;\; \forall\, t \in [T^*, T_{\max})$$

3rd by the Small error property:

$$\int_{T^*}^{T_{\max}} e^{\lambda \tau} \gamma_\rho(\|e_\rho(\tau)\|) d\tau < \infty$$

4th by the Detectability property:

the state x of the switched system is bounded on $[T^*, T_{\max})$

$$\Downarrow$$

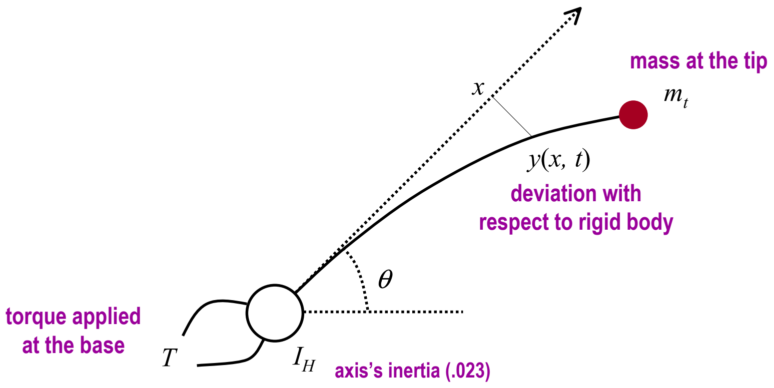
solution exists globally $T_{\max} = \infty$ & $x \to 0$ as $t \to \infty$

Theorem: Assume that $\mathcal{P}$ is finite and all the γ_p are locally Lipschitz.
The state of the process, multi-estimator, multi-controller, and all other signals converge to zero as $t \to \infty$.

Outline

- ✓ Supervisory control overview
- ✓ Estimator-based linear supervisory control
- ✓ Estimator-based nonlinear supervisory control
- ✗ Examples

Example: One-link flexible manipulator



PDE (small bending):

beam's elasticity

transversal slice's inertia

$$\ddot{y}(x,t) + \frac{EI}{\rho} y''''(x,t) = -x\ddot{\theta}(t)$$

beam's mass density (.68Kg total mass)

Boundary conditions:

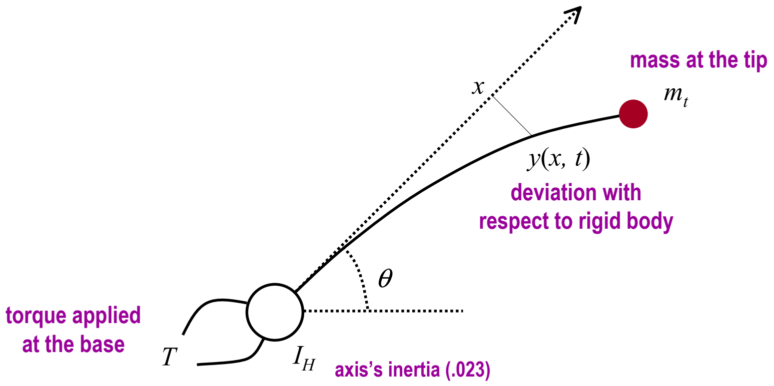
beam's length (113 cm)

$$y(0,t) = y'(0,t) = 0$$

$$y''(L,t) = y'''(L,t) + \frac{m_t}{\rho} y''''(L,t) = 0$$

$$T(t) = I_H \ddot{\theta}(t) - EIy''(0,t)$$

Example: One-link flexible manipulator



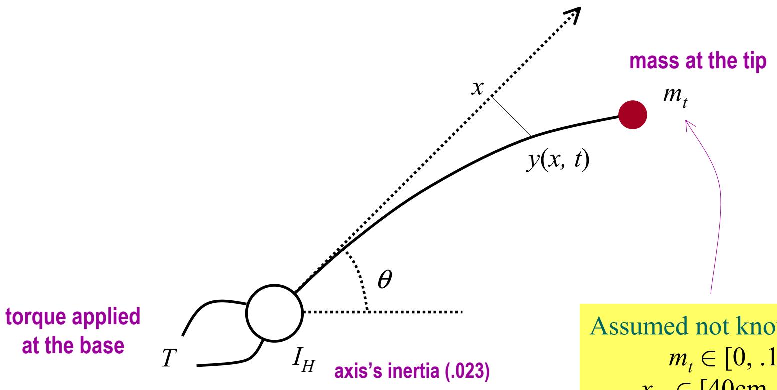
Series expansion and truncation:

$$y(x,t) \approx \sum_{k=1}^{n} \phi_k(x) q_k(t)$$

eigenfunctions of the beam

$$\dot{q} = Aq + bu$$

Example: One-link flexible manipulator



Assumed not known a priori:
$m_t \in$ [0, .1Kg]
$x_{sg} \in$ [40cm, 60cm]

Control measurements:

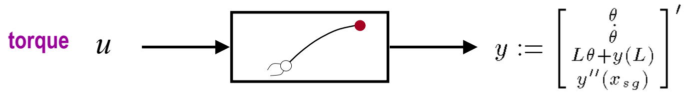
$\theta(t)$ ≡ base angle

$\dot{\theta}(t)$ ≡ base angular velocity

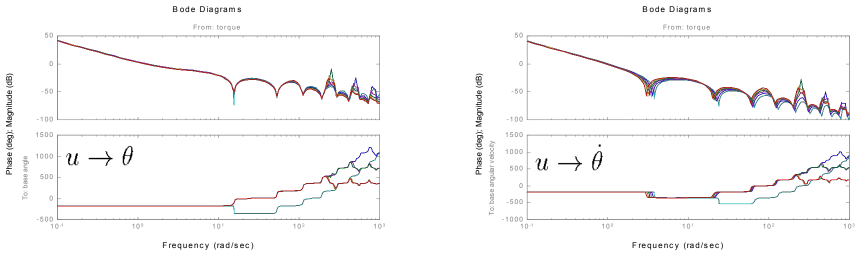
$L\theta(t) + y(L,t)$ ≡ tip position

$y''(x_{sg}, t)$ ≡ bending at position x_{sg} (measured by a strain gauge attached to the beam at position x_{sg})

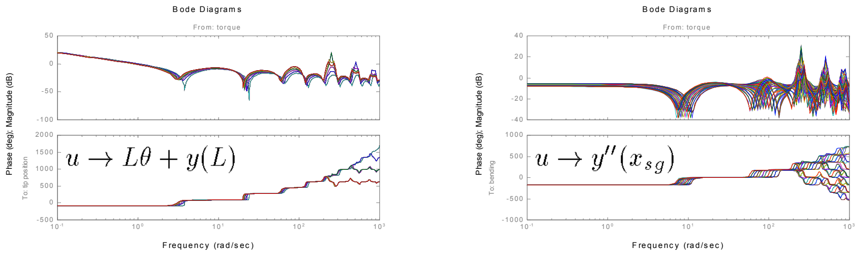
Example: One-link flexible manipulator



$$y := \begin{bmatrix} \theta \\ \dot{\theta} \\ L\theta + y(L) \\ y''(x_{sg}) \end{bmatrix}'$$

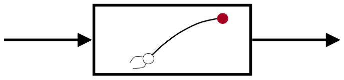


transfer functions as m_t ranges over [0, .1Kg] and x_{sg} ranges over [40cm, 60cm]



Example: One-link flexible manipulator

torque u



$$y := \begin{bmatrix} \theta \\ \dot{\theta} \\ L\theta + y(L) \\ y''(x_{sg}) \end{bmatrix}'$$

Class of admissible processes:

$$\mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$$

$\mathcal{M}_p \equiv$ family around a nominal transfer function corresponding to parameters $p := (m_t, x_{sg})$

unknown parameter
$p := (m_t, x_{sg})$

parameter set:
grid of 18 points in
$[0, .1] \times [40, 60]$

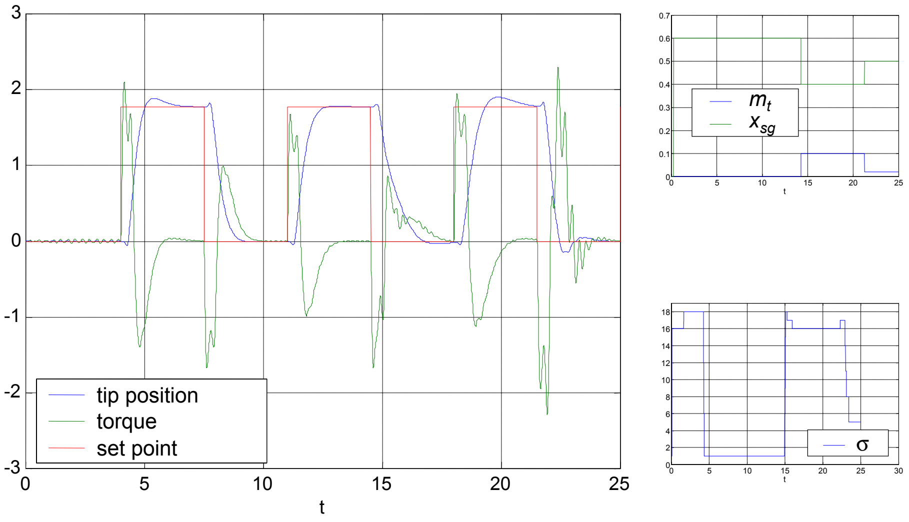
For this problem it is not possible to write the coefficients of the nominal transfer functions as a function of the parameters because these coefficients are the solutions to transcendental equations that must be computed numerically.

Family of candidate controllers:

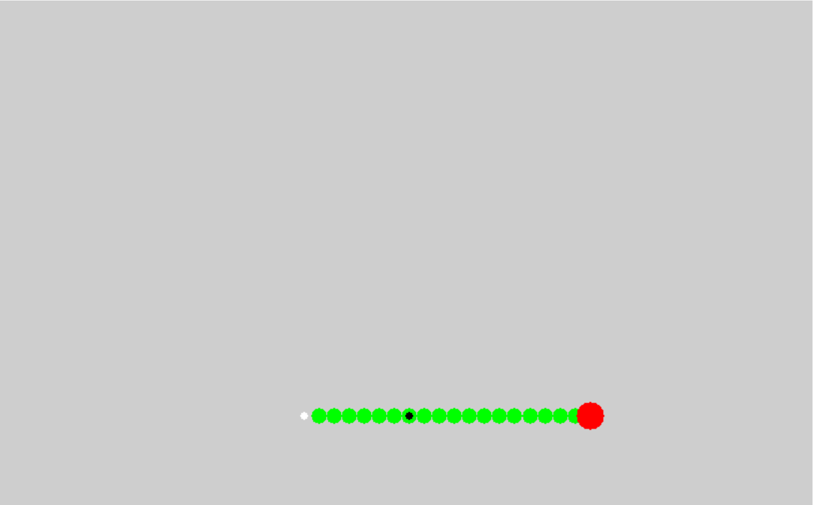
$$\mathcal{C} := \left\{ \kappa_q : q \in \mathcal{Q} \right\} \qquad \mathcal{Q} := \{1, 2, \ldots, 18\}$$

18 controllers designed using LQR/LQE, one for each nominal model

Example: One-link flexible manipulator

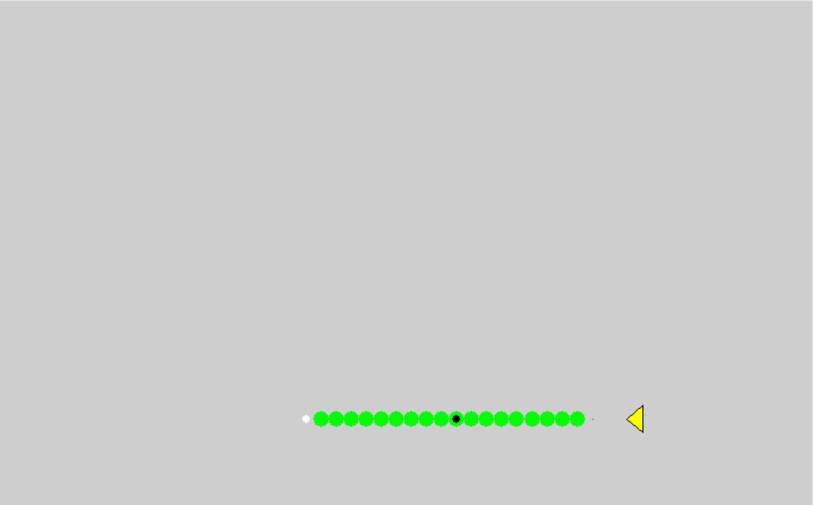


Example: One-link flexible manipulator



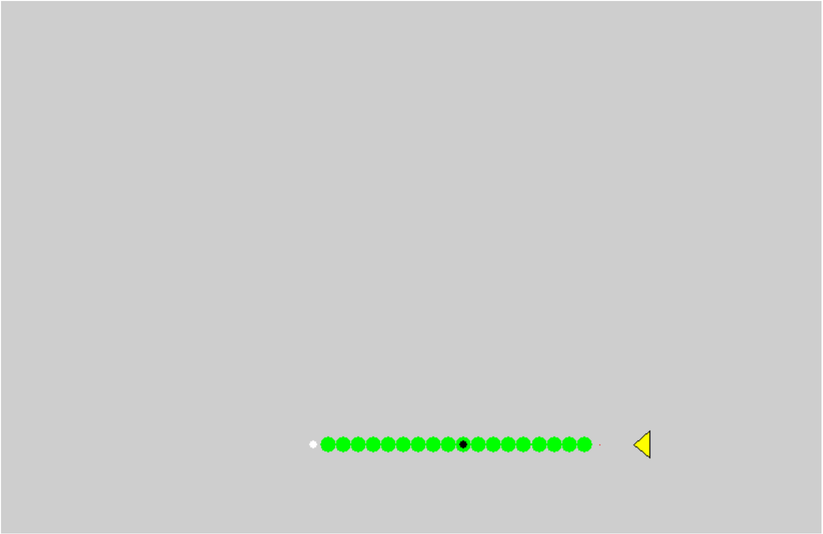
(open-loop)

Example: One-link flexible manipulator



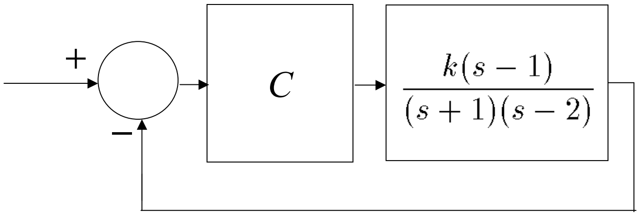
(closed-loop with fixed controller)

Example: One-link flexible manipulator



(closed-loop with supervisory control)

Example: Uncertain gain

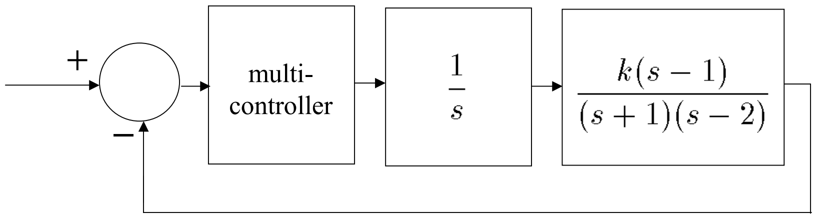


$$\boxed{1 \le k < 4}$$

The maximum gain margin achievable by a single linear time-invariant controller is 4

Doyle, Francis, Tannenbaum, Feedback Control Theory, 1992

Example: Uncertain gain

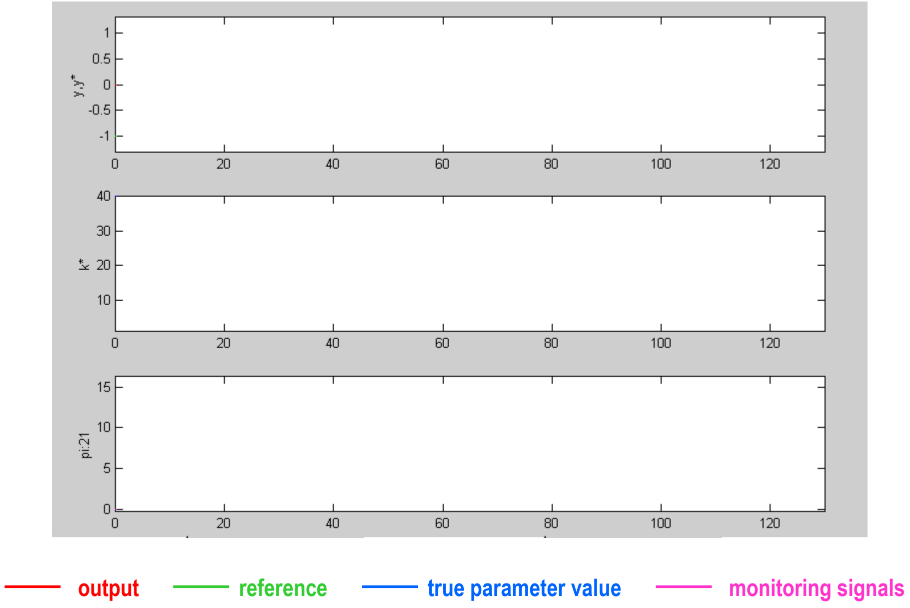


$\boxed{1 \leq k < 40}$

$$\kappa_p = \frac{1}{p}\,\frac{448s^2 + 450s - 18}{31s(s-9)} \qquad p \in \{(1.2)^{i-1} : i = 1, 2, \ldots, 21\}$$

Anderson et. al., “Multiple Model Adaptive Control, Part 1: Finite Controller Coverings,” *Special George Zames Issue of IJRNC,* 2000.

Example: Uncertain gain

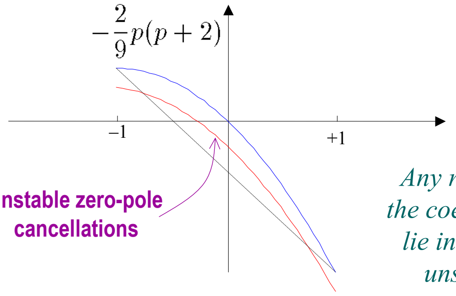


Example: 2-dim SISO linear process

Class of admissible processes:

$$\mathcal{M} := \bigcup_{p\in\mathcal{P}} \{\nu_p\} \qquad \nu_p(s) := \frac{s-\frac{1}{6}(p+2)}{s^3+ps^2-\frac{2}{9}\underbrace{p(p+2)}s} \qquad p\in\mathcal{P} := [-1,1]$$

nominal transfer function nonlinear parameterized on p

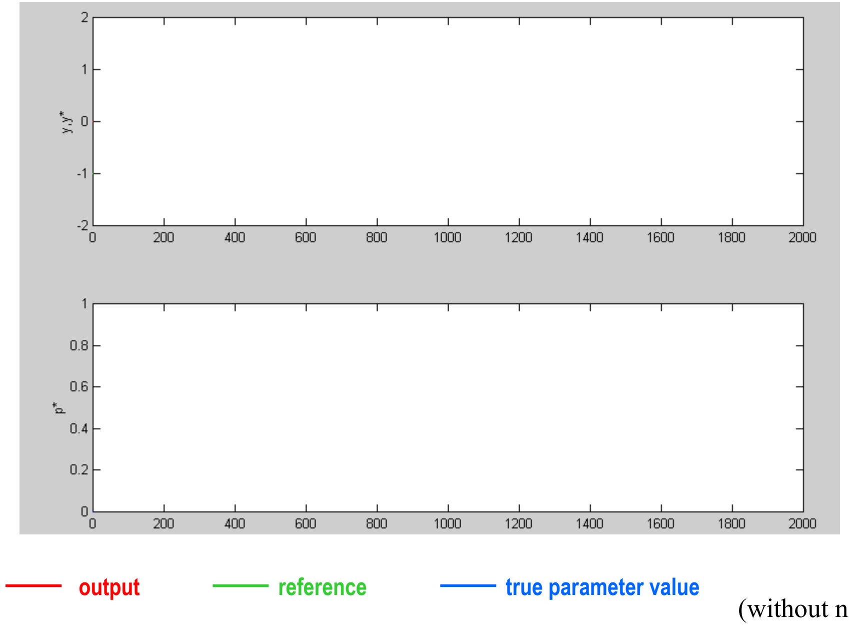


unstable zero-pole cancellations

Any re-parameterization that makes the coefficients of the transfer function lie in a convex set will introduce an unstable zero-pole cancellation

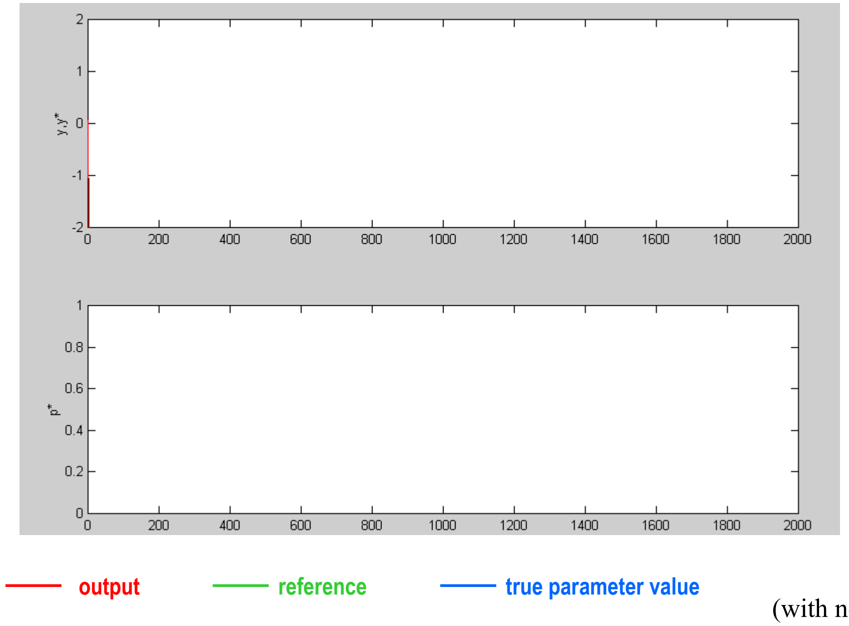
But the multi-estimator is still separable and state-sharing can be used …

Example: 2-dim SISO linear process



(without noise)

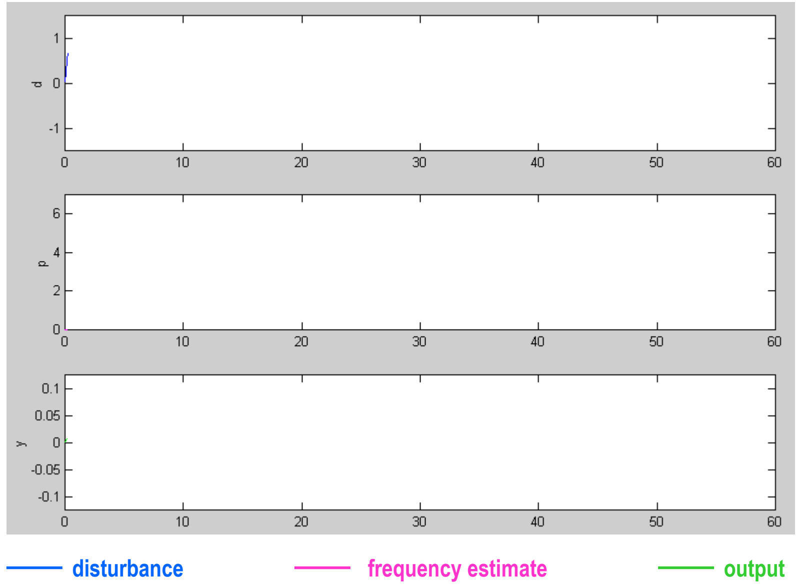
Example: 2-dim SISO linear process



(with noise)

Example: Disturbance Rejection

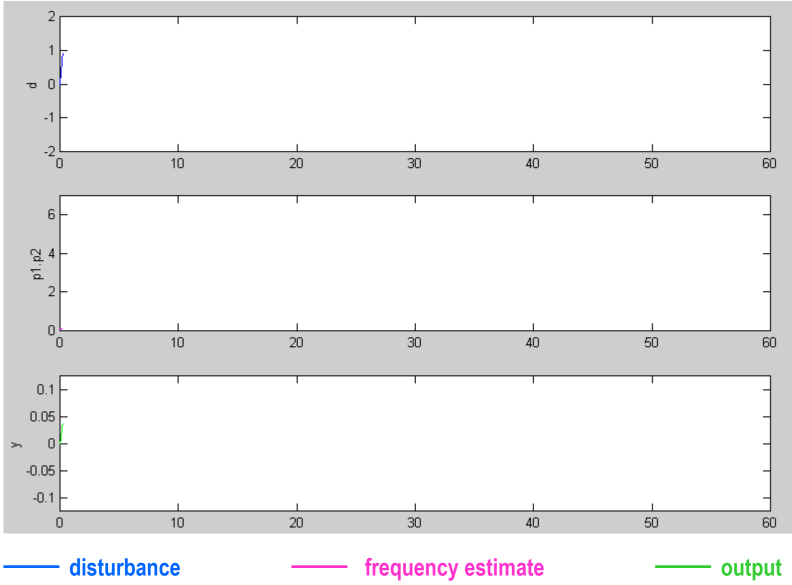
(unknown frequency)



(rejection of one sinusoid)

Example: Disturbance Rejection

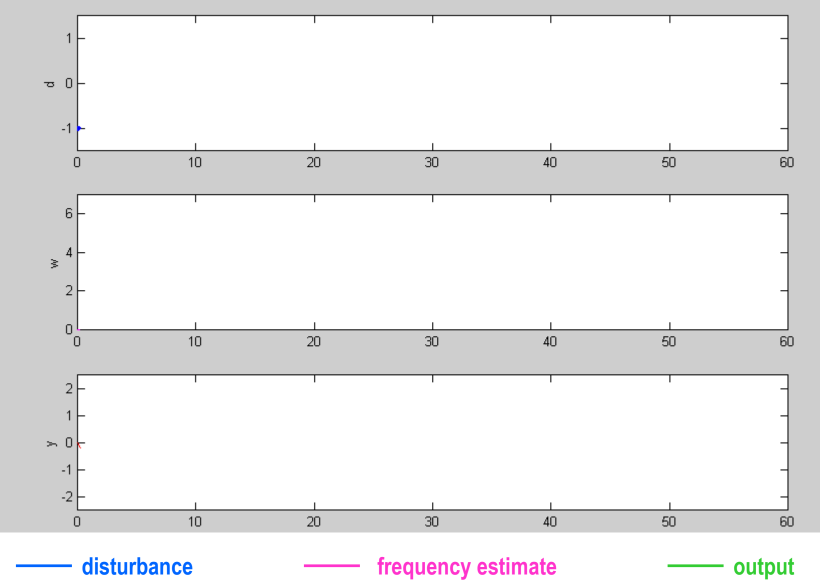
(unknown frequency)



(rejection of two sinusoids)

Example: Disturbance Rejection

(unknown frequency)



(rejection of a square wave)

Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot\alpha = p_1\alpha^3 + p_2\beta \qquad\qquad p := \{p_1, p_2\} \in \mathcal{P} \subset [-1,1]\times[-1,1]\setminus\{0\}$$

$$\dot\beta = u$$

state accessible

Multi-estimator (option I):

$$\begin{aligned}\dot\alpha_p &= p_1\alpha^3 + p_2\beta - (\alpha_p - \alpha)\\ \dot\beta_p &= u - (\beta_p - \beta)\end{aligned} \qquad p \in \mathcal{P}$$

it is separable so we can do state-sharing

When process matches the nominal model N_{p^*}

exponentially

$$e_{p^*} := \begin{bmatrix}\tilde\alpha_p\\ \tilde\beta_p\end{bmatrix} = \begin{bmatrix}\alpha_p - \alpha\\ \beta_p - \beta\end{bmatrix} \quad\Rightarrow\quad \begin{aligned}\dot{\tilde\alpha}_{p^*} &= -\tilde\alpha_{p^*}\\ \dot{\tilde\beta}_{p^*} &= -\tilde\beta_{p^*}\end{aligned} \quad\Rightarrow\quad e_{p^*} \to 0 \text{ as } t \to \infty$$

Matching property: Assume $\mathcal{M} = \{\, \mathrm{N}_p : p \in \mathcal{P} \,\}$

$$\exists\, p^* \in \mathcal{P},\ c_0,\ \lambda^* > 0 : \qquad \| e_{p^*}(t) \| \le c_0\, e^{-\lambda^* t} \qquad t \ge 0$$

Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot\alpha = p_1\alpha^3 + p_2\beta \qquad\qquad p := \{p_1, p_2\} \in \mathcal{P} \subset [-1,1]\times[-1,1]\setminus\{0\}$$

$$\dot\beta = u$$

state accessible

To facilitate the controller design, one can first "back-step" the system to simplify its stabilization:

$$\dot\alpha = -\alpha + \gamma \qquad \gamma := \alpha + p_1\alpha^3 + p_2\beta$$

$$\dot\gamma = -\gamma + p_2\big(u - \Psi_p(\alpha,\gamma)\big) \qquad \Psi_p(\alpha,\gamma) := \frac{(\alpha-\gamma)(1+3p_1\alpha^2)}{p_2}$$

$$y := \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ \frac{1}{p_2}(\gamma - \alpha - p_1\alpha^3)\end{bmatrix}$$

after the coordinate transformation the new state is no longer accessible

now the control law $u = \Psi_p(\alpha,\gamma)$ stabilizes the system

Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot\alpha = -\alpha + \gamma \qquad p \in \mathcal{P} \subset [-1,1]\times[-1,1]\setminus\{0\}$$

$$\dot\gamma = -\gamma + p_2\big(u - \Psi_p(\alpha,\gamma)\big)$$

$$y := \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha \\ \frac{1}{p_2}(\gamma-\alpha-p_1\alpha^3)\end{bmatrix} \qquad \gamma = \alpha + p_1\alpha^3 + p_2\beta$$

Multi-estimator (option II):

$$\dot\alpha_p = -\alpha_p + \gamma_p$$

$$\dot\gamma_p = -\gamma_p + p_2\big(u - \Psi_p(\alpha,\gamma)\big) \qquad p \in \mathcal{P}$$

it is separable so we can do state-sharing

When process matches the nominal model N_{p*}

$$e_p := \begin{bmatrix}\tilde\alpha_p\\ \tilde\gamma_p\end{bmatrix} = \begin{bmatrix}\alpha_p - \alpha \\ \gamma_p - \alpha - p_1\alpha^3 - p_2\beta\end{bmatrix} \Rightarrow \begin{matrix}\dot{\tilde\alpha}_{p*} = -\tilde\alpha_{p*} + \tilde\gamma_{p*} \\ \dot{\tilde\gamma}_{p*} = -\tilde\gamma_{p*}\end{matrix} \xRightarrow{\text{exponentially}} e_{p*} \to 0$$

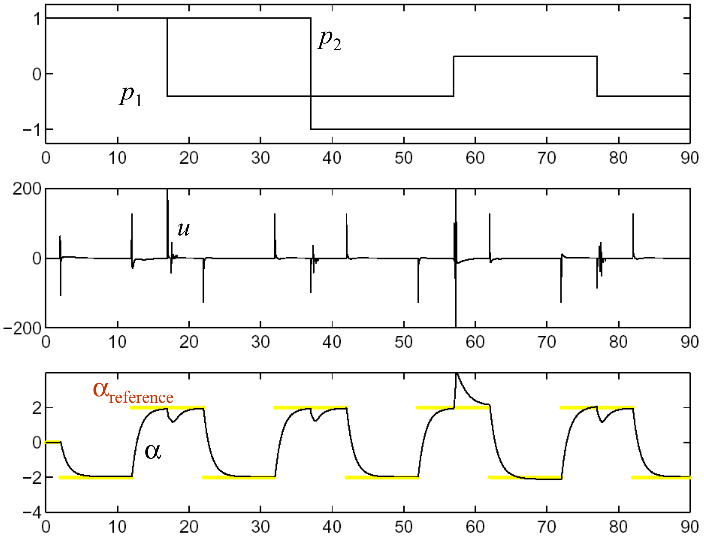
Matching property

Candidate controller $q = \chi(p)$:

$u = \Psi_p(\alpha,\gamma)$ **makes injected system ISS** $\Rightarrow$ **Detectability property**

Example: System in strict-feedback form

$$\dot\alpha = p_1\alpha^3 + p_2\beta \qquad \dot\beta = u$$



Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot{\alpha} = p_1\alpha^3 + p_2\beta \qquad\qquad p := \{p_1, p_2\} \in \mathcal{P} \subset [-1,1] \times [-1,1] \setminus \{0\}$$

$$\dot{\beta} = u$$

state accessible

In the previous back-stepping procedure:

$$\dot{\alpha} = -\alpha + \gamma \qquad\qquad \gamma = \alpha + p_1\alpha^3 + p_2\beta$$

$$\dot{\gamma} = -\gamma + p_2\big(u - \Psi_p(\alpha,\gamma)\big)$$

the controller

$u = \Psi_p(\alpha,\gamma)$ $\Rightarrow$ $p_2\beta \to -\alpha - p_1\alpha^3$

drives $\gamma \to 0$

nonlinearity is cancelled (even when $p_1 < 0$ **and it introduces damping)**

One could instead make

$$p_2\beta \to \varphi_p(\alpha) := \begin{cases} 0 & p_1\alpha^2 \leq -1 \\ -\alpha - p_1\alpha^3 & p_1\alpha^2 > -1 \end{cases}$$

pointwise min-norm design

still leads to exponential decrease of α (without canceling nonlinearity when $p_1 < 0$)

Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\dot{\alpha} = p_1\alpha^3 + p_2\beta \qquad\qquad p := \{p_1, p_2\} \in \mathcal{P} \subset [-1,1] \times [-1,1] \setminus \{0\}$$

$$\dot{\beta} = u$$

state accessible

A different recursive procedure:

$$\dot{\alpha} = p_1\alpha^3 + \varphi_p(\alpha) + \gamma \qquad\qquad \gamma := p_2\beta - \varphi_p(\alpha)$$

$$\dot{\gamma} = \Psi_p(\alpha,\gamma) + p_2 u$$

In this case

$$u = \frac{1}{p_2}\begin{cases} 0 & \gamma\Psi_p(\alpha,\gamma) \leq -\gamma^2 \\ -\gamma + \Psi_p(\alpha,\gamma) & \gamma\Psi_p(\alpha,\gamma) > -\gamma^2 \end{cases} \quad \Rightarrow \quad \gamma \to 0 \ \text{ exponentially}$$

$$\Downarrow$$

$$p_2\beta \to \varphi_p(\alpha)$$

$$\Downarrow$$

$$\alpha \to 0 \ \text{ exponentially}$$

pointwise min-norm recursive design

Example: System in strict-feedback form

Suppose nominal models N_p, $p \in \mathcal{P}$ are of the form

$$\begin{aligned} \dot{\alpha} &= p_1\alpha^3 + \varphi_p(\alpha) + \gamma \\ \dot{\gamma} &= \Psi_p(\alpha,\gamma) + p_2 u \\ y &:= \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}\alpha \\ \frac{1}{p_2}(\gamma + \varphi_p(\alpha))\end{bmatrix} \end{aligned} \qquad p \in \mathcal{P} \subset [-1,1]\times[-1,1]\setminus\{0\}$$

$$\gamma = p_2\beta - \varphi_p(\alpha)$$

Multi-estimator (option III):

$$\begin{aligned} \dot{\alpha}_p &= p_1\alpha^3 + \varphi_p(\alpha) + \gamma - (\alpha_p - \alpha) \\ \dot{\gamma}_p &= \Psi_p(\alpha,\gamma) + p_2 u - (\gamma_p - \gamma) \end{aligned} \qquad p \in \mathcal{P}$$

When process matches the nominal model N_{p^*}

$$e_p := \begin{bmatrix}\tilde{\alpha}_p \\ \tilde{\gamma}_p\end{bmatrix} = \begin{bmatrix}\alpha_p - \alpha \\ \gamma_p - p_2\beta + \varphi_p(\alpha)\end{bmatrix} \Rightarrow \begin{aligned} \dot{\tilde{\alpha}}_{p^*} &= -\tilde{\alpha}_{p^*} \\ \dot{\tilde{\gamma}}_{p^*} &= -\tilde{\gamma}_{p^*} \end{aligned} \Rightarrow \overset{\text{exponentially}}{e_{p^*} \to 0}$$

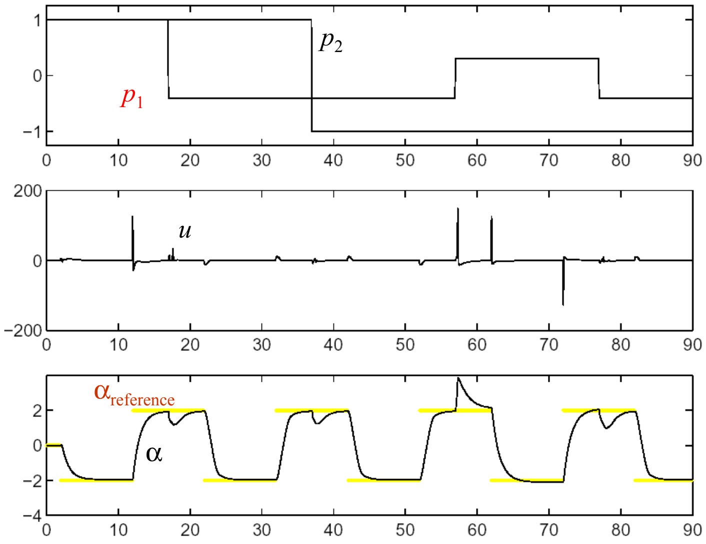
Matching property

Candidate controller $q = \chi(p)$:

$$u = \frac{1}{p_2}\begin{cases} 0 & \gamma\Psi_p(\alpha,\gamma) \le -\gamma^2 \\ -\gamma + \Psi_p(\alpha,\gamma) & \gamma\Psi_p(\alpha,\gamma) > -\gamma^2 \end{cases} \Rightarrow \textbf{Detectability property}$$

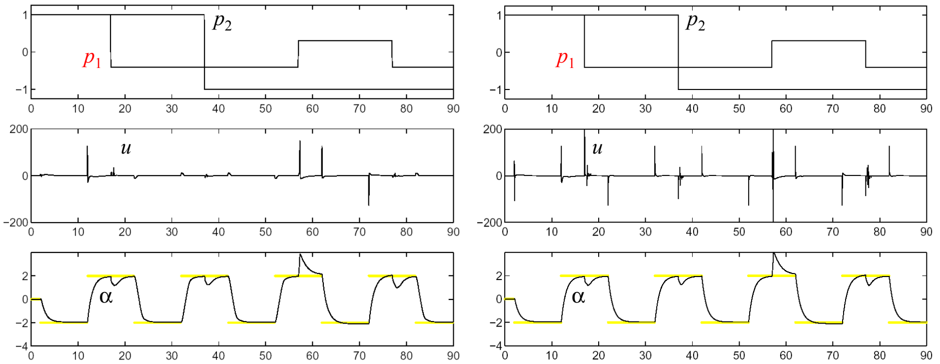
Example: System in strict-feedback form

$$\dot{\alpha} = p_1\alpha^3 + p_2\beta \qquad \dot{\beta} = u$$



Example: System in strict-feedback form

$$\dot{\alpha} = p_1 \alpha^3 + p_2 \beta \qquad \dot{\beta} = u$$

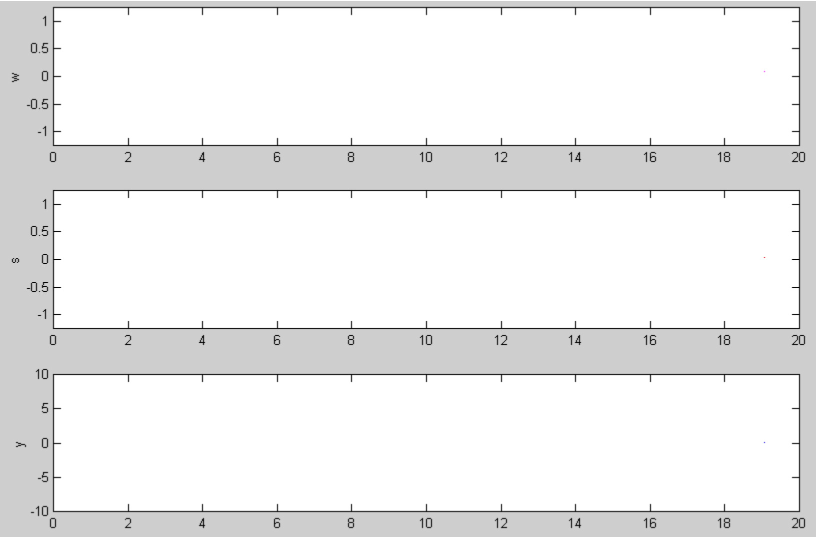


pointwise min-norm design

feedback linearization design

Example: Unstable-zero dynamics

$$\dot{x} = -py^2 + u \qquad \dot{y} = x + py^2 - u$$

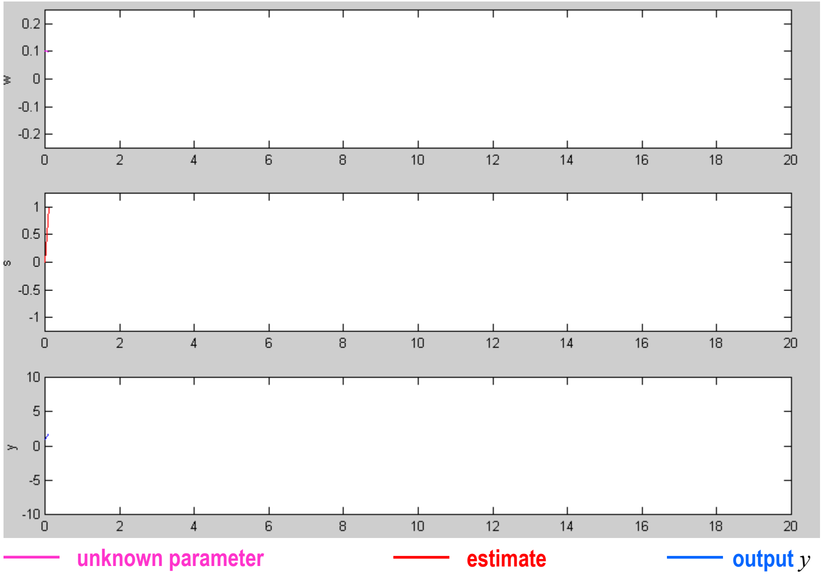


— **unknown parameter** — **estimate** — **output y**

(stabilization)

Example: Unstable-zero dynamics

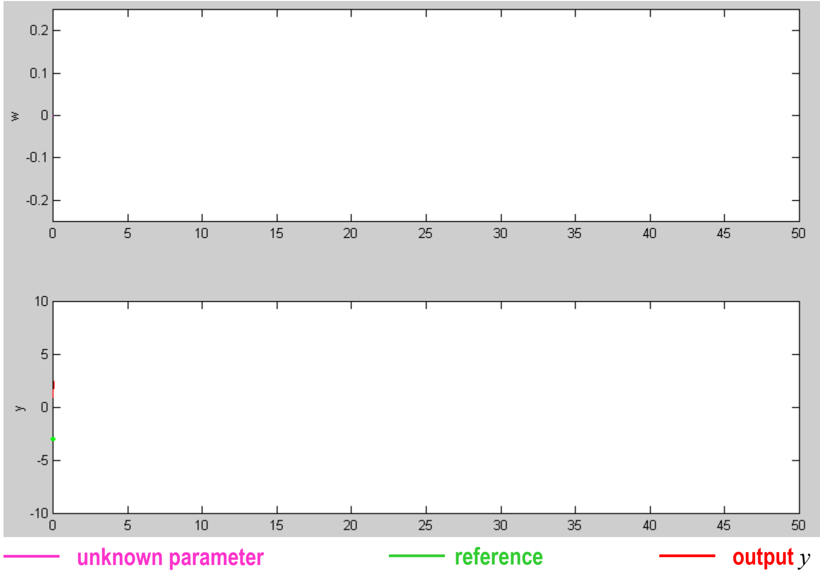
$\dot{x} = -py^2 + u$ $\qquad$ $\dot{y} = x + py^2 - u$



(stabilization with noise)

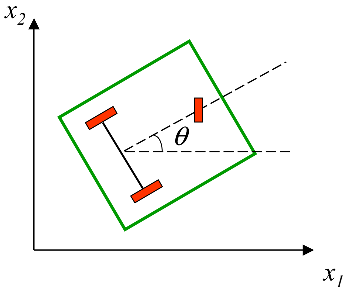
Example: Unstable-zero dynamics

$\dot{x} = -py^2 + u$ $\qquad$ $\dot{y} = x + py^2 - u$



(set-point with noise)

Example: Kinematic unicycle robot



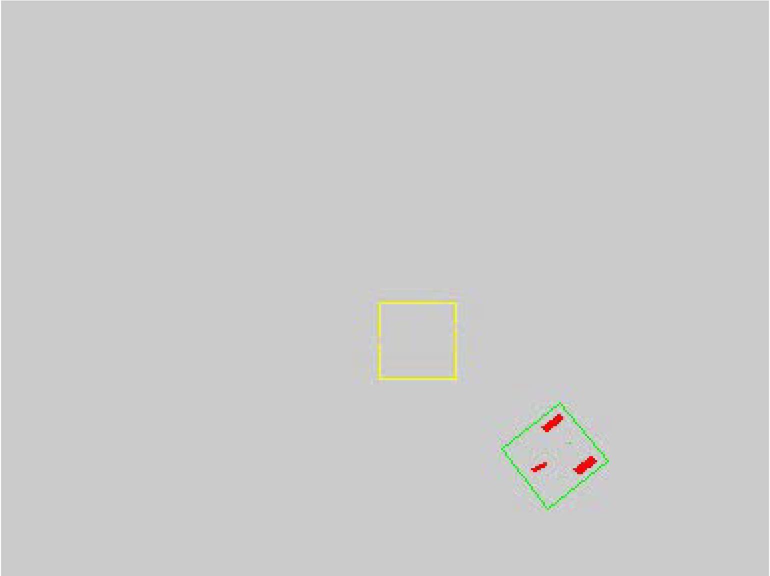
$$\dot{x}_1 = p_1 u_1 \cos\theta$$
$$\dot{x}_2 = p_1 u_1 \sin\theta$$
$$\dot{\theta} = p_2 u_2$$

$u_1 \equiv$ forward velocity
$u_2 \equiv$ angular velocity

$p_1, p_2 \equiv$ unknown parameters determined by the radius of the driving wheel and the distance between them

This system cannot be stabilized by a continuous time-invariant controller. The candidate controllers were themselves hybrid

Example: Kinematic unicycle robot



Example: Induction motor in current-fed mode

$$\dot{\lambda} = -R\lambda + Ru$$
$$\dot{\omega} = \tau - \tau_L$$
$$\tau = u' J \lambda$$

$\lambda \in \mathbb{R}^2$ ≡ rotor flux
$u \in \mathbb{R}^2$ ≡ stator currents
ω ≡ rotor angular velocity
τ ≡ torque generated

ω is the only measurable output

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Unknown parameters:
$\tau_L \in [\tau_{\min}, \tau_{\max}]$ ≡ load torque
$R \in [R_{\min}, R_{\max}]$ ≡ rotor resistance

"Off-the-shelf" *field-oriented* candidate controllers:

$$\dot{\rho} = \frac{R}{\beta_d^2}\tau_d$$
$$\tau_d = -\tau_d\Big(K_p + K_I \textstyle\int \cdot\Big)(\omega - \omega_d)$$
$$u = e^{\rho J}\begin{bmatrix} \beta_d \\ \frac{\tau_d}{\beta_d} \end{bmatrix}$$