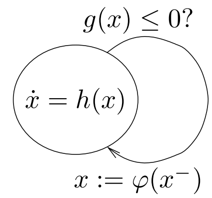
ECE229 Hybrid and switched systems: Homework #2

Instructor: João P. Hespanha

This homework requires the material covered in Lectures #2 and #3.

Exercise 1 (Existence of solution). Consider the following single-mode hybrid system:



where $h, \varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$.

1. How should one define f and Φ to write this hybrid system in the standard form

$$\dot{x} = f(x), \qquad\qquad x = \Phi(x^-). \tag{1}$$

Note: For this type of hybrid system is often convenient to define the following sets:

$$\mathcal{G} := \big\{x \in \mathbb{R}^n : x \neq \Phi(x)\big\}, \qquad \mathcal{R} := \Phi(\mathcal{G}) = \big\{\Phi(z) \in \mathbb{R}^n : z \in \mathcal{G}\big\},$$

called the guard *and the* reset *sets, respectively. It is instructive to express this type of hybrid systems as*

$$\begin{cases} \dot{x} = f(x) & x^- \in \mathbb{R}^n \setminus \mathcal{G}, \quad x \in \mathbb{R}^n \setminus \mathcal{G} \\ x = \Phi(x^-) & x^- \in \mathcal{G}, \qquad\quad x \in \mathcal{R} \end{cases}$$

to emphasize that a continuous flow can only occur when $x^- \in \mathbb{R}^n \setminus \mathcal{G}$, and that a discontinuous jump can only occur when $x^- \in \mathcal{G}$. Moreover, the state always enters $\mathcal{R}$ immediately after a jump occurs.

2. The following iterative algorithm can be used to construct a solution $x : [0, T) \to \mathbb{R}^n$, with $T \in (0, \infty]$ to the single-mode hybrid system (1).

Algorithm 1 (Construction of a solution to a single-mode hybrid system).

1 *Let $t_0 := 0$ and $k := 0$.*

2 *Let $\phi(t; t_k, x_k)$ denote the solution to the differential equation*

$$\dot{x} = f(x), \qquad\qquad x(t_k) = x_k, \tag{2}$$

and let $[t_k, T_k)$, $t_k < T_k \le \infty$ be the maximum interval on which the solution exists.

3 *Define*

$$t_{k+1} := \inf\big\{t \in (t_k, T_k] : \phi(t; t_k, x_k) \in \mathcal{G}\big\} \tag{3}$$

and assign the value of $x(t)$ on $[t_k, t_{k+1})$ using

$$x(t) := \phi(t; t_k, x_k), \quad \forall t \in (t_k, t_{k+1}).$$

Note that $x^-(t_{k+1}) = \phi(t_{k+1}; t_k, x_k)$, by continuity of $t \mapsto \phi(t; t_k, x_k)$.

4 *If $t_{k+1} < T_k$ and $x^-(t_{k+1}) = \phi\big(t_{k+1}; t_k, x_k\big)$ is bounded then assign*

$$x(t_{k+1}) = x_{k+1} := \Phi(x^-(t_{k+1}))$$

and repeat the cycle starting at 2. Otherwise, $T = T_k$ and the iteration stops.

In case the iteration does not stop for any finite k, $T := \sup_k T_k$.

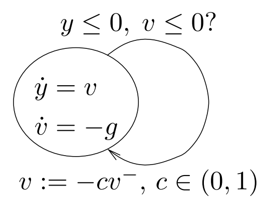
Provide conditions on f, $\mathcal{G}$, and $\mathcal{R}$ that guarantee that Algorithm 1 will succeed and lead to a unique solution.

Hint: Algorithm 1 implicitly makes several assumptions, which were not justified. You should identify these assumptions and choose conditions that guarantee that they hold.

3. Based on the answer to the previous question, state an existence of solution theorem for the single-mode hybrid system (1). Use the following template for your theorem:

   Theorem (Local existence for a single-mode hybrid system). *Assume* [$\cdots$ fill in the blank $\cdots$]. *For every x_0 such that* [$\cdots$ fill in the blank $\cdots$] *there exists some $T \in (0, \infty]$ such that the single-mode hybrid system* (1) *has a unique solution $x : [0, T) \to \mathbb{R}^n$ starting at $x(0) = x_0$.* □

4. Consider the following model for a bouncing ball:



   Verify whether or not it is possible to conclude local existence of solution using your theorem from question 3. If not, explain why Algorithm 1 still succeeds in producing a solution for every initial condition with $y(0) > 0$.

5. Adapt Algorithm 1 to construct a solution $q : [0, T) \to \mathcal{Q}$, $x : [0, T) \to \mathbb{R}^n$ to the following (multi-mode) hybrid system

$$\dot{x} = f(q, x), \qquad (q, x) = \Phi(q^-, x^-), \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n. \tag{4}$$

   How would you generalize your theorem from question 3 to this hybrid system.

   Hint: You may now find it useful to define the following sets:

$$\mathcal{G}_q := \{x \in \mathbb{R}^n : (q, x) \neq \Phi(q, x)\},$$
$$\mathcal{R}_q := \{x \in \mathbb{R}^n : \exists p \in \mathcal{Q}, z \in \mathcal{G}_p \ s.t.\ (q, x) = \Phi(p, z)\}.$$

   The following is true for these sets:

   (a) while the system is in model q, the continuous flow can one take place while $x \in \mathbb{R}^n \setminus \mathcal{Q}_q$;

   (b) if the system just transitioned into mode q, then $x \in \mathcal{R}_q$.

6. Provide a sufficient set of conditions on f, $\mathcal{G}$, and R for a unique solution $x : [0, \infty) \to \mathbb{R}^n$ to exist *globally* from any initial condition $x(0) = x_0 \in \mathbb{R}^n$ such that $g(x_0) < 0$.

   Do not be overly concerned if you conditions are very restrictive.