Hybrid Control and Switched Systems

Lecture #1
Hybrid systems are everywhere: Examples

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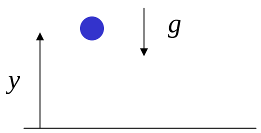
Summary

Examples of hybrid systems

1. Bouncing ball
2. Thermostat
3. Transmission
4. Inverted pendulum swing-up
5. Multiple-tank
6. Server
7. Supervisory control

Example #1: Bouncing ball



Free fall $\equiv \quad \ddot{y} = -g$

Collision $\equiv \quad y^+(t) = y^-(t) = 0$

$\dot{y}^+(t) = -c\dot{y}^-(t)$

$c \in [0,1] \equiv$ energy “reflected” at impact

Notation: given $x : [0,\infty) \rightarrow \mathbb{R}^n \equiv$ piecewise continuous signal

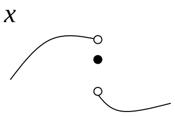
$$x^- : (0,\infty) \to \mathbb{R}^n \qquad x^-(t) := \lim_{\tau \uparrow t} x(t), \quad \forall t > 0$$

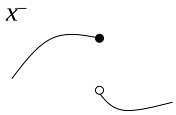
$$x^+ : [0,\infty) \to \mathbb{R}^n \qquad x^+(t) := \lim_{\tau \downarrow t} x(t), \quad \forall t \geq 0$$

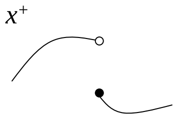
at points t where x is continuous $x(t) = x^-(t) = x^+(t)$

By convention we will generally assume right continuity, i.e.,

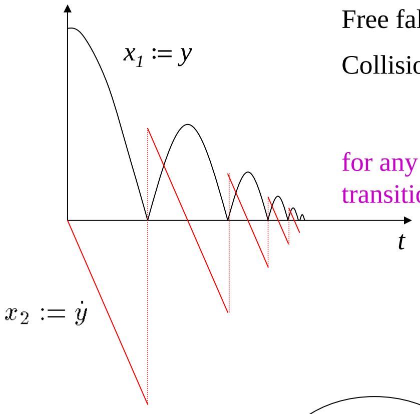
$$x(t) = x^+(t) \qquad \forall\, t \geq 0$$







Example #1: Bouncing ball

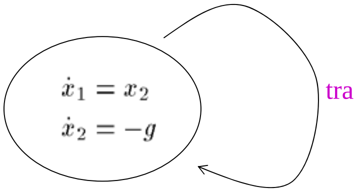


Free fall $\equiv \quad \ddot{y} = -g$

Collision $\equiv \quad y^+(t) = y^-(t) = 0$

$\dot{y}^+(t) = -c\dot{y}^-(t)$

for any c < 1, there are infinitely many transitions in finite time (Zeno phenomena)



guard or jump condition

$x_1 = 0$ & $x_2 < 0$?

$\dot{x}_1 = x_2$

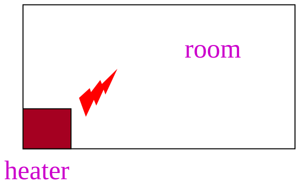
$\dot{x}_2 = -g$

transition

$x_2 := -\, c\, x_2^-$

state reset

Example #2: Thermostat



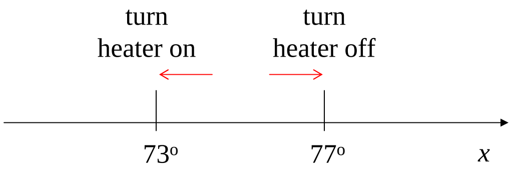
goal ≡ regulate temperature around 75^{o}

x ≡ mean temperature

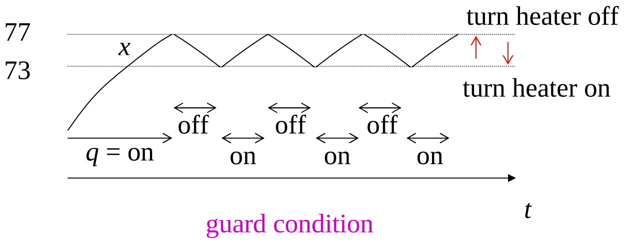
when heater is off: $\dot{x} \approx -x + 50$ $(x \to 50^{o})$

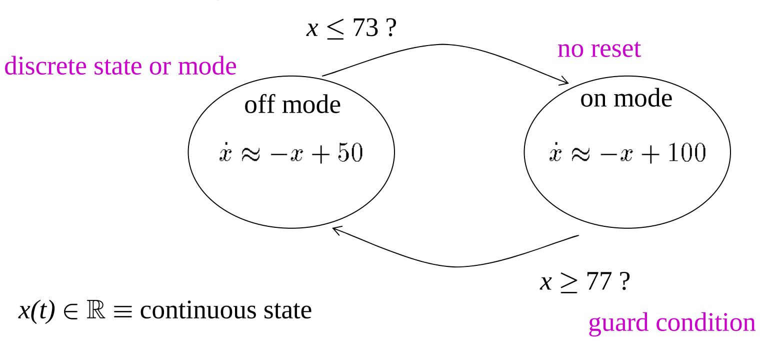
when heater is on: $\dot{x} \approx -x + 100$ $(x \to 100^{o})$

event-based control



Example #2: Thermostat

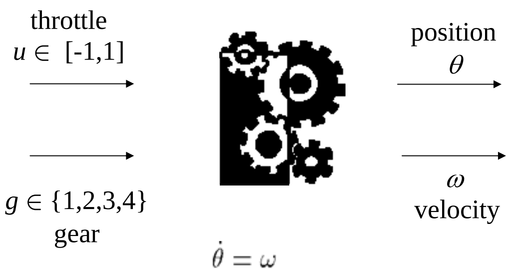




$x(t) \in \mathbb{R}$ ≡ continuous state

$q(t) \in$ { off, on } ≡ discrete state

Example #3: Transmission



throttle
$u \in [-1,1]$

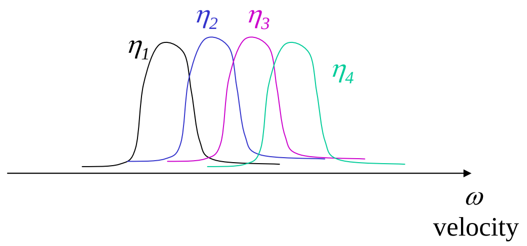
$g \in \{1,2,3,4\}$
gear

position
θ

ω
velocity

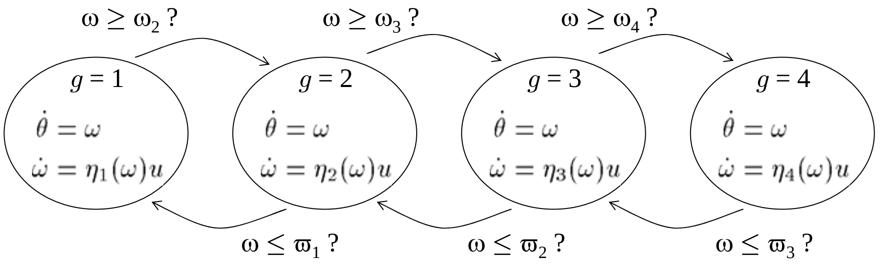
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_g(\omega)u$$

$\eta_k \equiv$ efficiency of the kth gear



[Hedlund, Rantzer 1999]

Example #3: Automatic transmission



$\omega \geq \omega_2$?

$\omega \geq \omega_3$?

$\omega \geq \omega_4$?

$g = 1$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_1(\omega)u$$

$g = 2$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_2(\omega)u$$

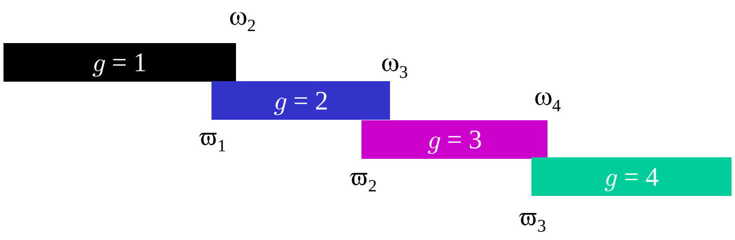
$g = 3$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_3(\omega)u$$

$g = 4$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_4(\omega)u$$

$\omega \leq \varpi_1$?

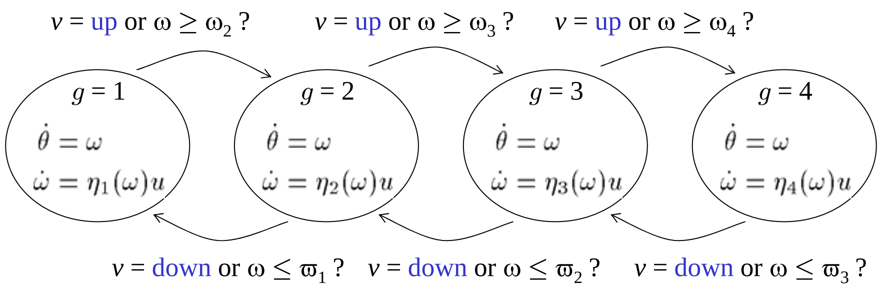
$\omega \leq \varpi_2$?

$\omega \leq \varpi_3$?



Example #3: Semi-automatic transmission

$v(t) \in \{$ up, down, keep $\} \equiv$ drivers input (discrete)



v = up or $\omega \geq \omega_2$?

v = up or $\omega \geq \omega_3$?

v = up or $\omega \geq \omega_4$?

$g = 1$: $\dot{\theta} = \omega$, $\dot{\omega} = \eta_1(\omega)u$

$g = 2$: $\dot{\theta} = \omega$, $\dot{\omega} = \eta_2(\omega)u$

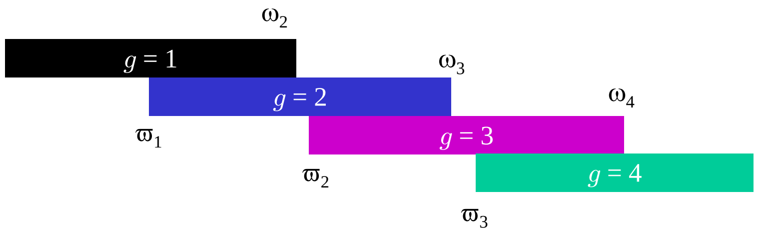
$g = 3$: $\dot{\theta} = \omega$, $\dot{\omega} = \eta_3(\omega)u$

$g = 4$: $\dot{\theta} = \omega$, $\dot{\omega} = \eta_4(\omega)u$

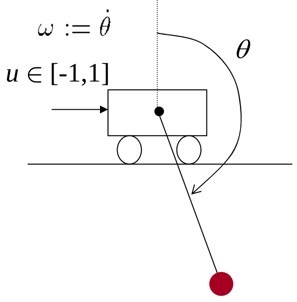
v = down or $\omega \leq \varpi_1$?

v = down or $\omega \leq \varpi_2$?

v = down or $\omega \leq \varpi_3$?



Example #4: Inverted pendulum swing-up



goal ≡ drive θ to 0 (upright position)

$$\ddot{\theta} = \sin\theta - u\cos\theta$$

$u \in$ [-1,1] ≡ force applied to the cart

$$\text{Total system energy} \equiv E := \underbrace{\frac{1}{2}\omega^2}_{\text{kinetic}} + \underbrace{(\cos\theta - 1)}_{\text{potential}}$$

normalized to be zero at stationary upright position

Feedback linearization controller: try to make

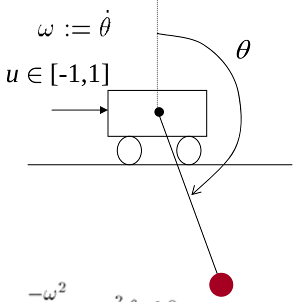
$$\ddot{\theta} + 2\dot{\theta} + \theta = 0 \quad \longleftarrow \quad \theta(t) = e^{-t}(\theta(0) + \theta(0)t + \omega(0)t)$$

$$\begin{cases} \ddot{\theta} = \sin\theta - u\cos\theta \\ \ddot{\theta} + 2\dot{\theta} + \theta = 0 \end{cases} \Rightarrow \quad u := \frac{2\dot{\theta} + \theta + \sin\theta}{\cos\theta}$$

only in [-1,1] close to upright position (θ = ω = 0)

[Astrom, Furuta 1999]

Example #4: Inverted pendulum swing-up



$$\ddot{\theta} = \sin\theta - u\cos\theta \qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$$

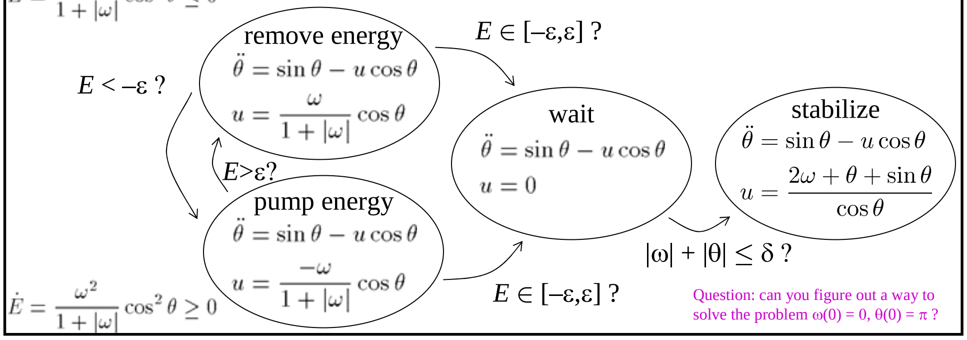
Hybrid controller:

1st pump/remove energy into/from the system by applying maximum force, until E ≈ 0 (energy control) $\dot{E} = -u\omega\cos\theta$

2nd wait until pendulum is close to the upright position

3th next to upright position use feedback linearization controller

$$\dot{E} = \frac{-\omega^2}{1+|\omega|}\cos^2\theta \le 0$$



remove energy $\ddot{\theta} = \sin\theta - u\cos\theta$ $u = \frac{\omega}{1+|\omega|}\cos\theta$

$E \in [-\varepsilon,\varepsilon]$?

$E < -\varepsilon$?

$E > \varepsilon$?

wait $\ddot{\theta} = \sin\theta - u\cos\theta$ $u = 0$

stabilize $\ddot{\theta} = \sin\theta - u\cos\theta$ $u = \frac{2\omega + \theta + \sin\theta}{\cos\theta}$

pump energy $\ddot{\theta} = \sin\theta - u\cos\theta$ $u = \frac{-\omega}{1+|\omega|}\cos\theta$

$|\omega| + |\theta| \le \delta$?

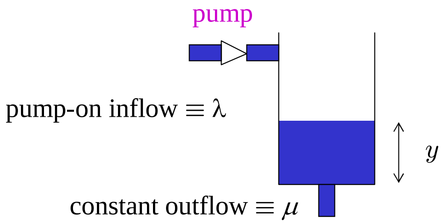
$E \in [-\varepsilon,\varepsilon]$?

$$\dot{E} = \frac{\omega^2}{1+|\omega|}\cos^2\theta \ge 0$$

Question: can you figure out a way to solve the problem $\omega(0) = 0$, $\theta(0) = \pi$?

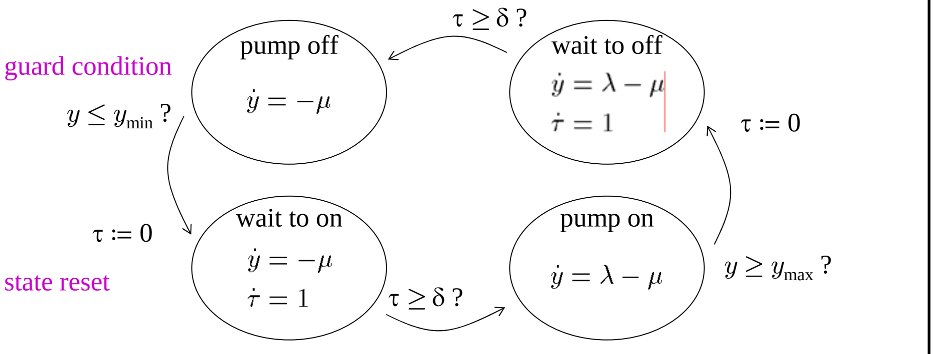
Example #5: Tank system

*



goal ≡ prevent the tank from emptying or filling up

δ ≡ delay between command is sent to pump and the time it is executed



guard condition

pump off $\dot{y} = -\mu$

$\tau \ge \delta$?

wait to off $\dot{y} = \lambda - \mu$ $\dot{\tau} = 1$

$y \le y_{\min}$?

$\tau := 0$

$\tau := 0$

state reset

wait to on $\dot{y} = -\mu$ $\dot{\tau} = 1$

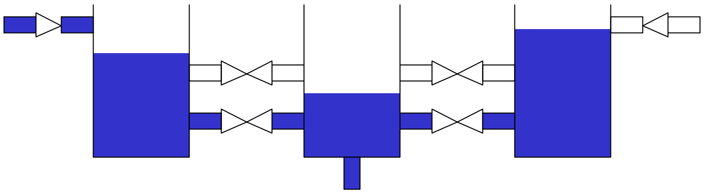
$\tau \ge \delta$?

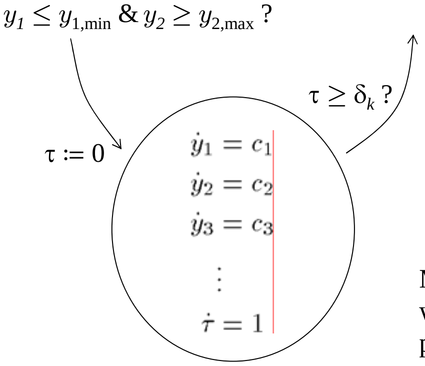
pump on $\dot{y} = \lambda - \mu$

$y \ge y_{\max}$?

How to choose $y_{\min}$ and $y_{\max}$ for given μ, λ, δ ?

Example #5: Multiple-tank system





$y_1 \leq y_{1,\min}$ & $y_2 \geq y_{2,\max}$?

$\tau \geq \delta_k$?

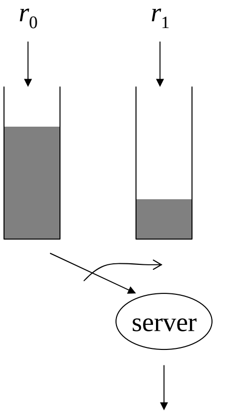
$\tau := 0$

$$\dot{y}_1 = c_1$$
$$\dot{y}_2 = c_2$$
$$\dot{y}_3 = c_3$$
$$\vdots$$
$$\dot{\tau} = 1$$

Initialized rectangular hybrid automata
- all differential equations have constant r.h.s.
- all jump cond. are of the form:
  state var. 1 ∈ fixed interval 1 &
  state var. 2 ∈ fixed interval 2 & etc.
- all resets have constant r.h.s.

Most general class of hybrid systems for which there exist completely automated procedures to compute the set of reachable states

Example #6: Switched server



rates of incoming parts

buffers

$\delta_{ij} \equiv$ setup time needed to move from buffer *i* to *j*

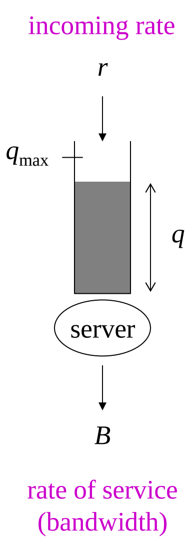
$r > r_0 + r_1 + r_2$

rate of service

Scheduling algorithm (cyclic scheduling):
1. start in buffer 0
2. work on a buffer until empty
3. when buffer *j* is empty move to buffer *j* + 1 (mod 3)

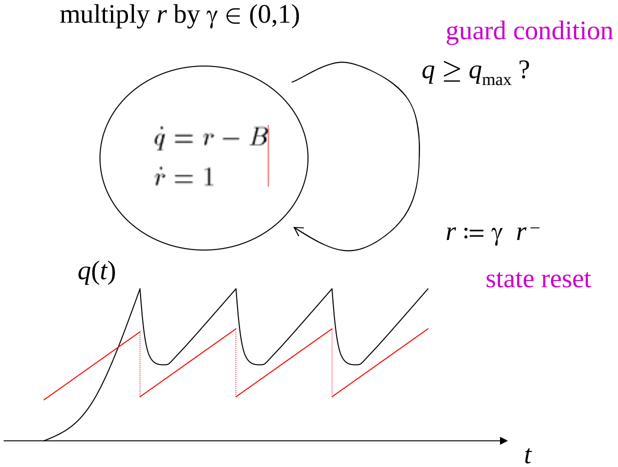
Often also an initialized rectangular hybrid automata …

Example #7: Server system with congestion control



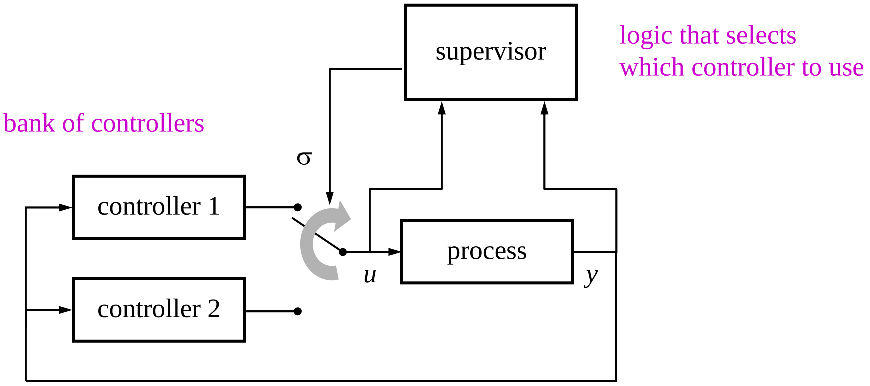
Additive increase/multiplicative decrease congestion control (AIMD):

- while $q < q_{max}$ increase r linearly
- when q reaches q_{max} instantaneously multiply r by $\gamma \in (0,1)$

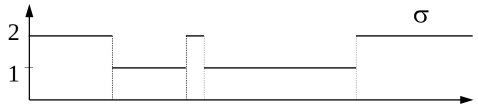


no longer an initialized rectangular hybrid automata …

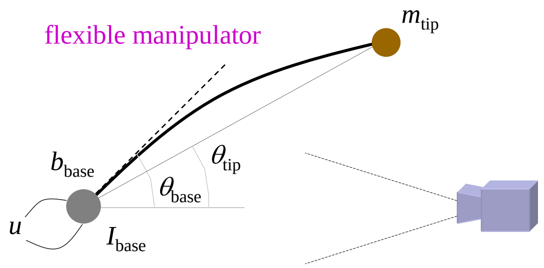
Example #8: Supervisory control



σ ≡ switching signal taking values in the set {1,2}



E.g. #8 a): Vision-based control of a flexible manipulator



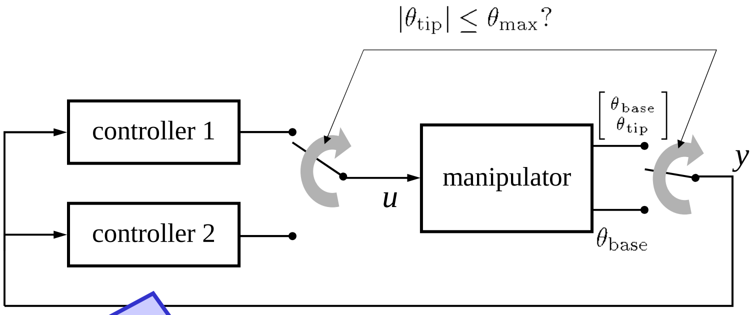
goal ≡ drive θ_{tip} to zero, using feedback from

θ_{base} → encoder at the base
θ_{tip} → machine vision (needed to increase the damping of the flexible modes in the presence of noise)
To achieve high accuracy in the measurement of θ_{tip} the camera must have a **small field of view**

output feedback output:

$$y := \begin{cases} \begin{bmatrix} \theta_{\text{base}} \\ \theta_{\text{tip}} \end{bmatrix} & |\theta_{\text{tip}}| \leq \theta_{\max} \\ \theta_{\text{base}} & |\theta_{\text{tip}}| > \theta_{\max} \end{cases}$$

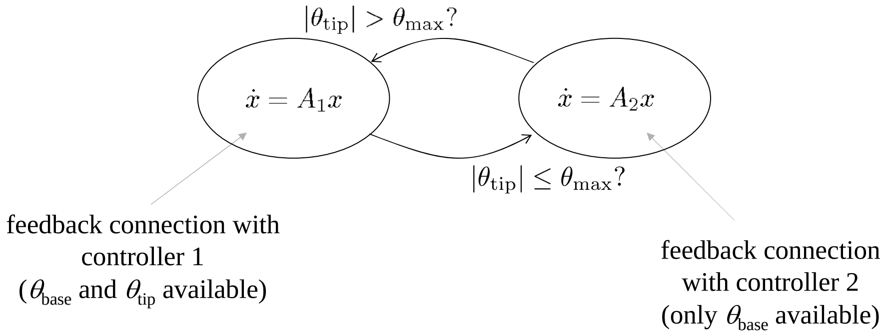
E.g. #8 a): Vision-based control of a flexible manipulator



controller 1 optimized for feedback from θ_{base} and θ_{tip}
and
controller 2 optimized for feedback only from θ_{base}

E.g., LQG controllers that minimize $\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \theta_{\text{tip}}^2 + \dot{\theta}_{\text{tip}}^2 + \rho u^2 dt\right]$

E.g. #8 a): Vision-based control of a flexible manipulator



$|\theta_{\text{tip}}| > \theta_{\text{max}}?$

$\dot{x} = A_1 x$

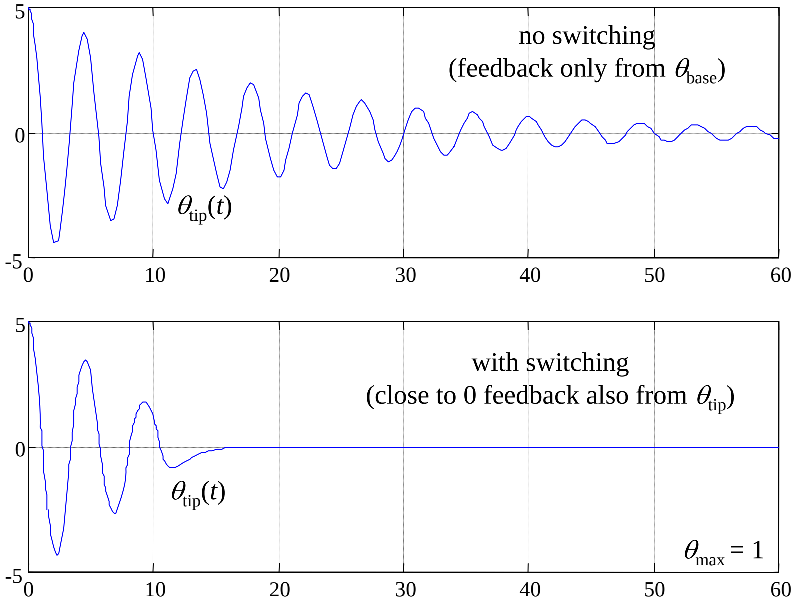
$\dot{x} = A_2 x$

$|\theta_{\text{tip}}| \leq \theta_{\text{max}}?$

feedback connection with
controller 1
(θ_{base} and θ_{tip} available)

feedback connection
with controller 2
(only θ_{base} available)

E.g. #8 a): Vision-based control of a flexible manipulator



no switching
(feedback only from θ_{base})

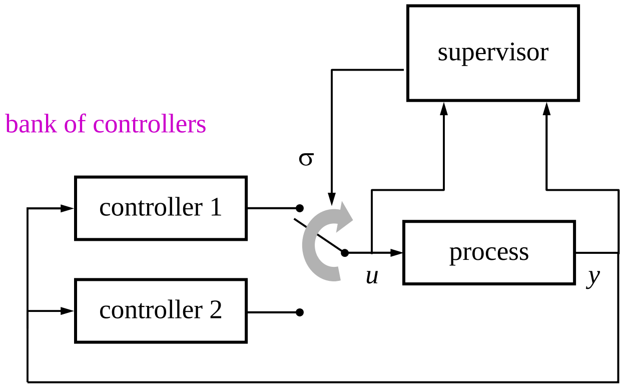
$\theta_{\text{tip}}(t)$

with switching
(close to 0 feedback also from θ_{tip})

$\theta_{\text{tip}}(t)$

$\theta_{\text{max}} = 1$

Example #8 b): Adaptive supervisory control



process can either be:

$$\dot{x} = A_1 x + b_1 u$$
$$y = c_1 x$$

or

$$\dot{x} = A_2 x + b_2 u$$
$$y = c_2 x$$

Goal: stabilize process, regardless of which is the actual process model

Supervisor must
- try to determine which is the correct process model by observing *u* and *y*
- select the appropriate controller

Next class…

1. **Formal models for hybrid systems**:
   - Finite automata
   - Differential equations
   - Hybrid automata
   - Open hybrid automaton
2. Nondeterministic vs. stochastic systems
   - Non-deterministic automata and differential inclusions
   - Markov chains and stochastic processes