Hybrid Control and Switched Systems

Lecture #11
Stability of switched system:
Arbitrary switching

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Summary

Stability under arbitrary switching
- Instability caused by switching
- Common Lyapunov function
- Converse results
- Algebraic conditions

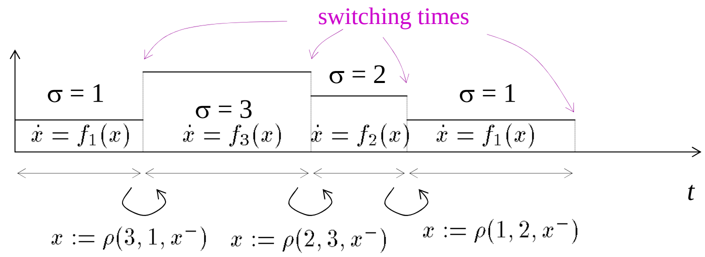
Switched system

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad p \in \mathcal{Q}$

switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$ $\qquad$ parameter set

$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair $(x, \sigma) \in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to
$$\dot{x} = f_{\sigma(t)}(x)$$
time-varying ODE
2. at every switching time t, $x(t) = \rho(\sigma(t), \sigma^-(t), x^-(t))$

Time-varying systems vs. Hybrid systems vs. Switched systems

Time-varying system $\equiv$ for each initial condition $x(0)$ there is only one solution

$$\dot{x} = f_{\sigma(t)}(x) \qquad \text{(all } f_p \text{ locally Lipschitz)}$$

Hybrid system $\equiv$ for each initial condition $q(0)$, $x(0)$ there is only one solution

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-)$$

Switched system $\equiv$ for each $x(0)$ there may be several solutions, one for each admissible σ

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

the notions of stability, convergence, etc.
must address "uniformity" over all solutions

Three notions of stability

Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***stable*** if $\exists\ \alpha \in \mathcal{K}$:

α is independent of $x(t_0)$ and σ

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \ \forall\, t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***asymptotically stable*** if it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t\to\infty.$$

Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\ \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \ \ \forall\, t \geq t_0 \geq 0$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s,t) = c\, e^{-\lambda t} s$ with $c,\lambda > 0$

Stability under arbitrary switching

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

$\mathcal{S}_{all} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

any switching signal is admissible

$\rho(p, q, x) = x\ \forall\, p,q \in \mathcal{Q},\ x \in \mathbb{R}^n$

no resets

If one of the vector fields f_q, $q \in \mathcal{Q}$ is unstable then the switched system is unstable

Why?

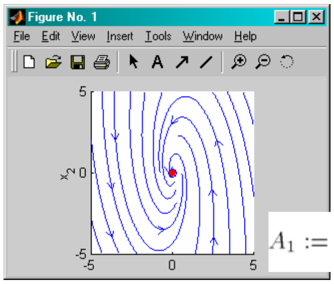
1. because the switching signals $\sigma(t) = q\ \forall\, t$ is admissible
2. for this σ we cannot find $\alpha\in\mathcal{K}$ such that

   $$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \ \forall\, t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

   (must less for all σ)

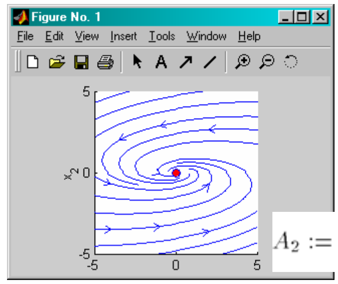
But even if all f_q, $q \in \mathcal{Q}$ are stable the switched system may still be unstable …

Stability under arbitrary switching



$A_1 := \begin{bmatrix} -.5 & -.4 \\ 3 & -.5 \end{bmatrix}$

$\dot{x} = A_1 z$ asympot. stable

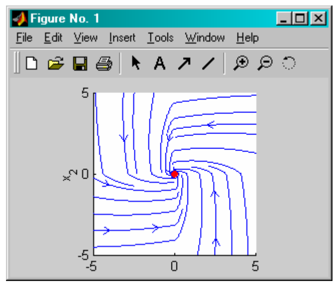


$A_2 := \begin{bmatrix} -.5 & -3 \\ .4 & -.5 \end{bmatrix}$

$\dot{x} = A_2 z$ asympot. stable

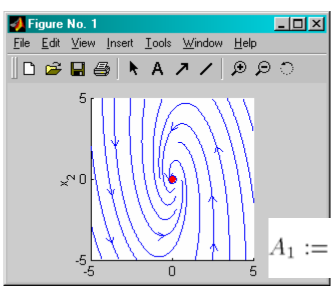
$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \le 0 \\ 2 & x_1 x_2 > 0 \end{cases}$$

$$\dot{x} = A_\sigma x$$



unstable.m

Stability under arbitrary switching



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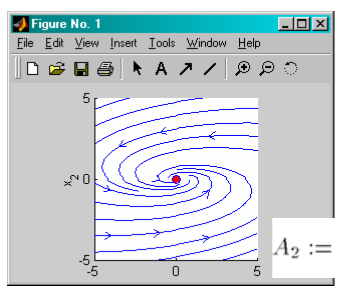
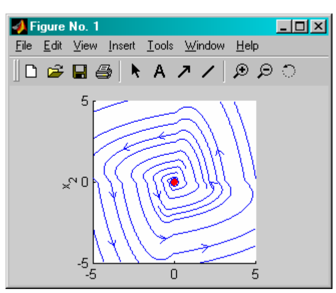


$A_2 := \begin{bmatrix} -.5 & -3 \\ .4 & -.5 \end{bmatrix}$

$\dot{x} = A_2 x$ asympot. stable

$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \ge 0 \\ 2 & x_1 x_2 < 0 \end{cases}$$

$$\dot{x} = A_\sigma x$$



for some admissible switching signals the trajectories grow to infinity ⇒ switched system is unstable

unstable.m

Lyapunov's stability theorem (ODEs)

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \ \forall\, t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

Theorem (Lyapunov):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{eq}) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if = 0 only for $z = x_{eq}$ then x_{eq} is a (globally) asymptotically stable equilibrium.

Why?
V can only stop decreasing when $x(t)$ reaches x_{eq}
but V must stop decreasing because it cannot become negative
Thus, $x(t)$ must converge to x_{eq}

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{eq}) f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

The same V could be used to prove stability for all the unswitched systems

$$\dot{x} = f_q(x)$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$)
1st Take an arbitrary solution (σ, x) and define $v(t) := V(x(t))\ \forall t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

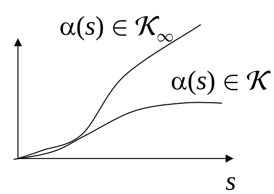
2nd Therefore

$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

$V(x(t))$ is always bounded

Some facts about functions of class $\mathcal{K}$, $\mathcal{KL}$

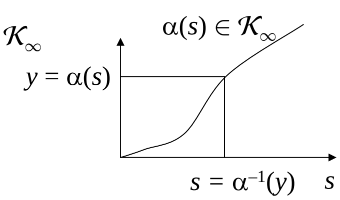
class $\mathcal{K} \equiv$ set of functions $\alpha : [0,\infty)\to[0,\infty)$ that are
1. continuous
2. strictly increasing
3. $\alpha(0)=0$



class $\mathcal{K}_\infty \equiv$ subset of $\mathcal{K}$ containing those functions that are unbounded

Lemma 1: If $\alpha_1,\alpha_2 \in \mathcal{K}$ then $\alpha(s) := \alpha_1(\alpha_2(s)) \in \mathcal{K}$ (same for $\mathcal{K}_\infty$)

Lemma 2: If $\alpha \in \mathcal{K}_\infty$ then α is invertible and $\alpha^{-1} \in \mathcal{K}_\infty$



Lemma 3: If $V: \mathbb{R}^n \to \mathbb{R}$ is positive definite and radially unbounded function then $\exists\ \alpha_1,\alpha_2 \in \mathcal{K}_\infty$: $\quad \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall\, x \in \mathbb{R}^n$

$\|x\| \leq \alpha_1^{-1}(V(x))$ $\qquad$ $\alpha_2^{-1}\big(V(x)\big) \leq \|x\|$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$)
2nd Therefore

$$v(t) := V\big(x(t)\big) \le v(0) := V\big(x(0)\big) \qquad \forall t \ge 0$$

$\|x\| \le \alpha_1^{-1}(\,V(x)\,)$

3rd Since $\exists\, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$

$V(\,x(t)\,)$ is always bounded

$$\|x(t)\| \le \alpha_1^{-1}\big(V(x(t))\big) \le \alpha_1^{-1}\big(V(x(0))\big) \le \alpha_1^{-1}\big(\alpha_2(\|x(0)\|)\big) \qquad \forall t \ge 0$$

α_1^{-1} monotone

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$)
3rd Since $\exists\, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$

$$\|x(t)\| \le \alpha_1^{-1}\big(V(x(t))\big) \le \alpha_1^{-1}\big(V(x(0))\big) \le \alpha_1^{-1}\big(\alpha_2(\|x(0)\|)\big) \qquad \forall t \ge 0$$

4th Defining $\alpha(s) := \alpha_1^{-1}(\alpha_2(s)) \in \mathcal{K}$ then

$$\|x(t)\| \le \alpha(\|x(0)\|) \qquad \forall t \ge 0$$

stability ! (1. is proved)

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$, $W(z) \nrightarrow 0$ as $z \to \infty$) $\alpha_2^{-1}(V(x)) \le \|x\|$

1st As long as $W \nrightarrow 0$ as $z \to \infty$, $\exists\, \alpha_3 \in \mathcal{K}$

$$W(x) \le -\alpha_3(\|x\|) \le -\alpha_3(\alpha_2^{-1}(V(x)))$$

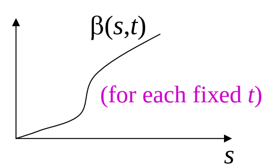
2nd Take an arbitrary solution (σ, x) and define $v(t) := V(x(t))\ \forall\, t \ge 0$

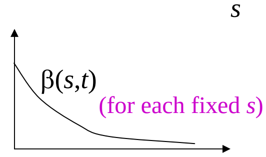
$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \le W(x(t)) \le -\alpha(v)$$

$$\alpha(s) := \alpha_3(\alpha_2^{-1}(s)) \in \mathcal{K}$$

Some more facts about functions of class $\mathcal{K}$, $\mathcal{KL}$

class $\mathcal{KL} \equiv$ set of functions $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ s.t.
1. for each fixed t, $\beta(\cdot,t) \in \mathcal{K}$
2. for each fixed s, $\beta(s,\cdot)$ is monotone decreasing and $\beta(s,t) \to 0$ as $t\to\infty$





Lemma 3: Given $\alpha \in \mathcal{K}$, in case

$$\dot{v} \le -\alpha(v)$$

then $\exists\, \beta \in \mathcal{KL}$ such that

$$v(t) \le \beta(v(t_0), t - t_0) \qquad t \ge t_0$$

After some work …

$$\beta(s,t) = \begin{cases} \eta^{-1}(\eta(s) + t) & s > 0 \\ 0 & s = 0 \end{cases} \qquad \eta(s) := -\int_1^s \frac{dr}{\alpha(r)}$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$, $W(z) \not\to 0$ as $z \to \infty$)

2nd Take an arbitrary solution (σ, x) and define $v(t) := V(\ x(t)\ )\ \forall\, t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq -\alpha(v)$$

3rd Then

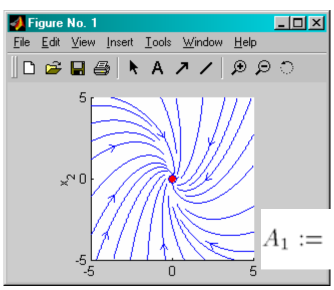
$$v(t) \leq \beta(v(0), t) \qquad t \geq 0$$

class $\mathcal{KL}$ function
(check !!!)
independent of σ

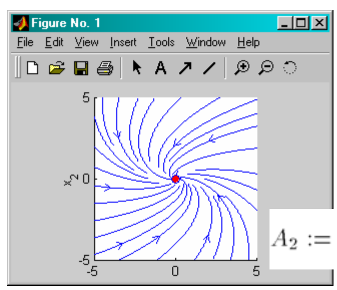
4th Going back to x…

$$\|x(t)\| \leq \alpha_1^{-1}\big(\beta(\alpha_2(\|x(0)\|), t)\big) \qquad t \geq 0$$

Example



$$A_1 := \begin{bmatrix} -1 & .25 \\ -1 & -1 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} -1 & -1 \\ .25 & -1 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $\quad V(x_1, x_2) := x_1^2 + x_2^2$

common Lyapunov function

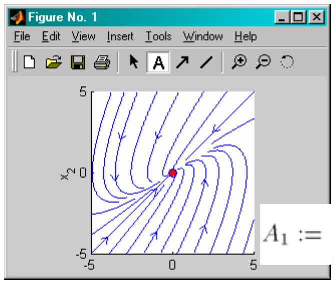
$$\frac{\partial V}{\partial x} A_1 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + .75 x_2)^2 < 0$$

$$\frac{\partial V}{\partial x} A_2 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + .75 x_2)^2 < 0$$

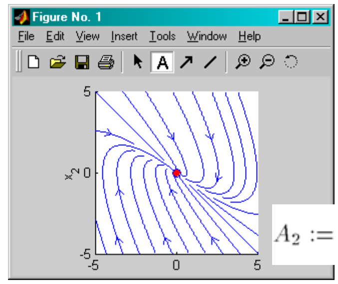
uniform asymptotic stability

stable.m

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $V(x_1,x_2) := x_1^2 + x_2^2$

common Lyapunov function

$$\frac{\partial V}{\partial x} A_1 x = -4x_2^2 \leq 0$$
$$\frac{\partial V}{\partial x} A_2 x = -4x_2^2 \leq 0$$

stability (not asymptotic)

(problems, e.g., close to the $x_2=0$ axis)

stable.m

Converse result

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

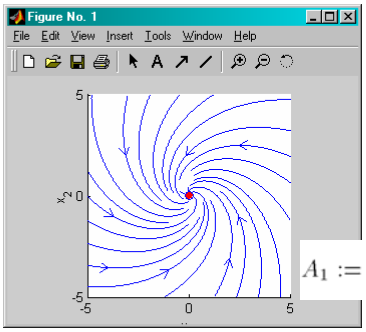
Theorem:
Assume Q is finite. The switched system is uniformly asymptotically stable (on $\mathcal{S}_{\text{all}}$) if and only if there exists a common Lyapunov function, i.e., continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) < 0 \qquad \forall z \in \mathbb{R}^n \setminus \{0\},\ q \in \mathcal{Q}$$

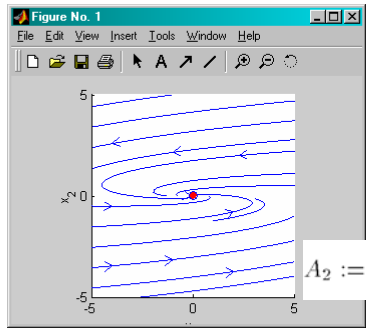
Note that…

1. This result generalized for infinite Q but one needs extra technical assumptions
2. The sufficiency was already established. It turns out that the existence of a common Lyapunov function is also necessary.
3. Finding a common Lyapunov function may be difficult. E.g., even for linear systems V may not be quadratic

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

The switched system is uniformly exponentially stable for arbitrary switching but there is no common quadratic Lyapunov function

stable.m

Algebraic conditions for stability under arbitrary switching

$$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$ linear switched system

Suppose $\exists\, m \geq n$, $M \in \mathbb{R}^{m\times n}$ full rank & $\{ B_q \in \mathbb{R}^{m\times m} : q \in \mathcal{Q} \}$:

$$M A_q = B_q M \qquad \forall q \in \mathcal{Q}$$

Defining $z := M x$

$$\dot{z} = M\dot{x} = M A_\sigma x = B_\sigma M x = B_\sigma z$$

Theorem: If $V(z) = z' z$ is a common Lyapunov function for

$$\dot{z} = B_\sigma z$$

i.e.,

$$B_q' + B_q < 0 \qquad \forall q \in \mathcal{Q}$$

$$\left(\frac{\partial V}{\partial z} B_q z = z'(B_q + B_q') z \right)$$

then the original switched system is uniformly (exponentially) asymptotically stable

Why?

1. If $V(z) = z' z$ is a common Lyapunov function then z converges to zero exponentially fast
2. $z := M x \Rightarrow M' z = M' M x \Rightarrow (M' M)^{-1} M' z = x \Rightarrow x$ converges to zero exponentially fast

Algebraic conditions for stability under arbitrary switching

$$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$ linear switched system

Suppose $\exists\, m \geq n$, $M \in \mathbb{R}^{m\times n}$ full rank & $\{ B_q \in \mathbb{R}^{m\times m} : q \in \mathcal{Q} \}$:

$$M A_q = B_q M \qquad \forall\, q \in \mathcal{Q}$$

Defining $z := M x$

$$\dot{z} = M\dot{x} = M A_\sigma x = B_\sigma M x = B_\sigma z$$

Theorem: If $V(z) = z' z$ is a common Lyapunov function for $\dot{z} = B_\sigma z$

i.e.,

$$B_q' + B_q < 0 \qquad \forall q \in \mathcal{Q}$$

$$\left(\frac{\partial V}{\partial z} B_q z = z'(B_q + B_q')z\right)$$

then the original switched system is uniformly (exponentially) asymptotically stable

It turns out that …

If the original switched system is uniformly asymptotically stable then such an M always exists (for some $m \geq n$) but may be difficult to find…

Commuting matrices

$$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$ linear switched system

$$x(t) = \Phi_\sigma(t,\tau)x(\tau)$$ state-transition matrix (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \cdots e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \ldots, t_k \equiv$ switching times of σ in the interval $[t,\tau)$

Recall: in general $e^M e^N \neq e^{M+N} \neq e^N e^M$ unless $M N = N M$

Suppose that for all $p,q \in \mathcal{Q}$, $A_p A_q = A_q A_p$

$$\Phi_\sigma(t,\tau) = e^{A_{\sigma(t_k)}(t-t_k) + A_{\sigma(t_{k-1})}(t_k-t_{k-1}) + \cdots + A_{\sigma(\tau)}(t_1-\tau)}$$
$$= e^{\sum_{q\in\mathcal{Q}} A_q T_q} = \prod_{q\in\mathcal{Q}} e^{A_q T_q}$$

$T_q \equiv$ total time $\sigma = q$ on (t, τ)

Assuming all A_q are asymptotically stable: $\exists\, c,\lambda_0 > 0$ $\|e^{A_q t}\| \leq c\, e^{-\lambda_0 t}$

$$\|\Phi_\sigma(t,\tau)\| \leq c^{|\mathcal{Q}|} e^{-\lambda \sum_q T_q} = c^{|\mathcal{Q}|} e^{-\lambda(t-\tau)}$$

$|\mathcal{Q}| \equiv$ # elements in $\mathcal{Q}$

Commuting matrices

$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$ linear switched system

$x(t) = \Phi_\sigma(t,\tau)x(\tau)$ state-transition matrix (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \cdots e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \ldots, t_k \equiv$ switching times of σ in the interval $[t, \tau)$

Recall: in general $e^M e^N \neq e^{M+N} \neq e^N e^M$ unless $M N = N M$

Theorem:
If Q is finite all A_q, $q \in Q$ are asymptotically stable and

$$A_p A_q = A_q A_p \qquad \forall\, p,q \in Q$$

then the switched system is uniformly (exponentially) asymptotically stable

Triangular structures

$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$ linear switched system

Theorem:
If all the matrices A_q, $q \in Q$ are asymptotically stable and upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Why?
One can find a common quadratic Lyapunov function of the form $V(x) = x' P x$ with P diagonal… check!

Triangular structures

$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$ linear switched system

Theorem:
If all the matrices A_q, $q \in \mathcal{Q}$ are asymptotically stable and upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Theorem:
If there is a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that all the matrices

$$B_q = T A_q T^{-1} \qquad (T^{-1} B_q T = A_q)$$

common similarity transformation

are upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Why?
1st The B_q have a common quadratic Lyapunov function $V(x) = x' P x$, i.e.,

$$P B_q + B_q' P < 0$$

2nd Therefore

$$P T T^{-1} B_q + B_q' T^{-1'} T' P < 0 \;\Rightarrow\; \underbrace{T' P T}_{Q} \underbrace{T^{-1} B_q T}_{A_q} + \underbrace{T' B_q' T^{-1'}}_{A_q'} \underbrace{T' P T}_{Q} < 0$$

$Q := T'PT$ is a common Lyapunov function

Commuting matrices

$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$ linear switched system

Theorem:
If $\mathcal{Q}$ is finite all A_q, $q \in \mathcal{Q}$ are asymptotically stable and

$$A_p A_q = A_q A_p \qquad \forall\, p,q \in \mathcal{Q}$$

then the switched system is uniformly (exponentially) asymptotically stable

Another way of proving this result…
From Lie Theorem if a set of matrices commute then there exists a common similarity transformation that upper triangularizes all of them

There are weaker conditions for simultaneous triangularization
(Lie Theorem actually provides the necessary and sufficient condition
≡ Lie algebra generated by the matrices must be solvable)

Commuting vector fields

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$ nonlinear switched system

Theorem: If all unswitched systems

$$\dot{x} = f_q(x) \qquad q \in \mathcal{Q}$$

are asymptotically stable and

$$\frac{\partial f_p}{\partial x} f_q = \frac{\partial f_q}{\partial x} f_p \qquad \forall p, q \in \mathcal{Q}$$

then the switched system is uniformly asymptotically stable

For linear vector fields becomes: $A_p A_q x = A_q A_p x \quad \forall x \in \mathbb{R}^n, p,q \in \mathcal{Q}$

Next lecture…

Controller realization for stable switching
Stability under slow switching

- Dwell-time switching
- Average dwell-time
- Stability under brief instabilities