

Hybrid Control and Switched Systems

Lecture #12 Controller realizations for stable switching

João P. Hespanha

University of California
at Santa Barbara



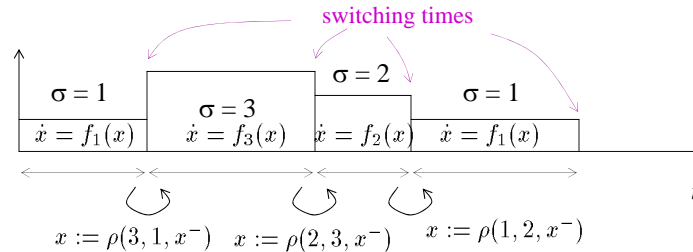
Summary

Controller realization for stable switching

Switched system

parameterized family of vector fields $\equiv f_p: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad p \in Q$
 switching signal \equiv piecewise constant signal $\sigma: [0, \infty) \rightarrow Q$ parameter set
 $S \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in \mathbb{R}^n

$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in S$$



A **solution** to the switched system is a pair $(x, \sigma) \in S$ for which

1. on every open interval on which σ is constant, x is a solution to

$$\dot{x} = f_{\sigma(t)}(x) \quad \text{time-varying ODE}$$
2. at every switching time t , $x(t) = \rho(\sigma(t), \sigma^-(t), x^-(t))$

Three notions of stability

Definition (class \mathcal{K} function definition): α is independent of $x(t_0)$ and σ
 The equilibrium point x_{eq} is **stable** if $\exists \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \quad \forall t \geq t_0 \geq 0, \|x(t_0) - x_{eq}\| \leq c$$
 along any solution $(x, \sigma) \in S$ to the switched system

Definition:
 The equilibrium point $x_{eq} \in \mathbb{R}^n$ is **asymptotically stable** if
 it is Lyapunov stable and for every solution that exists on $[0, \infty)$

$$x(t) \rightarrow x_{eq} \text{ as } t \rightarrow \infty.$$

Definition (class \mathcal{KL} function definition): β is independent of $x(t_0)$ and σ
 The equilibrium point $x_{eq} \in \mathbb{R}^n$ is **uniformly asymptotically stable** if $\exists \beta \in \mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall t \geq t_0 \geq 0$$
 along any solution $(x, \sigma) \in S$ to the switched system

exponential stability when $\beta(s, t) = c e^{-\lambda t} s$ with $c, \lambda > 0$

Stability under arbitrary switching

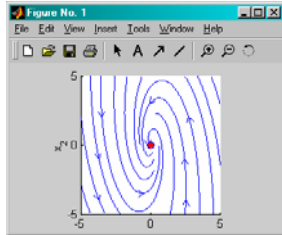
$$\dot{x} = f_\sigma(x) \quad x = \rho(\sigma, \sigma^-, x^-) \quad (\sigma, x) \in \mathcal{S}$$

$\mathcal{S}_{\text{all}} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

$$\rho(p, q, x) = x \quad \forall p, q \in \mathcal{Q}, x \in \mathbb{R}^n$$

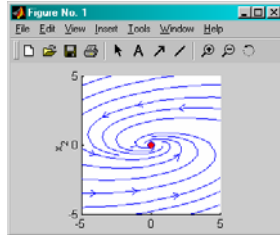
no resets

any switching signal is admissible



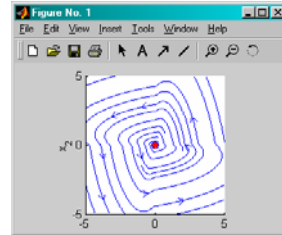
$$A_1 := \begin{bmatrix} -.5 & -.4 \\ 3 & -.5 \end{bmatrix}$$

$\dot{x} = A_1 z$ asymptot. stable



$$A_2 := \begin{bmatrix} -.5 & -.3 \\ .4 & -.5 \end{bmatrix}$$

$\dot{x} = A_2 z$ asymptot. stable

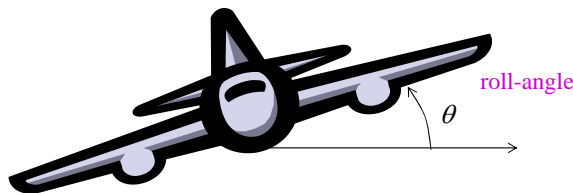


$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \geq 0 \\ 2 & x_1 x_2 < 0 \end{cases}$$

$\dot{x} = A_\sigma x$

Can we change the switching system to make it stable?

Example #11: Roll-angle control

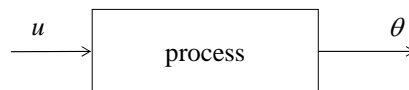


roll-angle

$$\ddot{\theta} + 50.875\dot{\theta} + 43.75\theta = -1000u$$

θ is uniquely determined by u and the initial conditions

input-output model



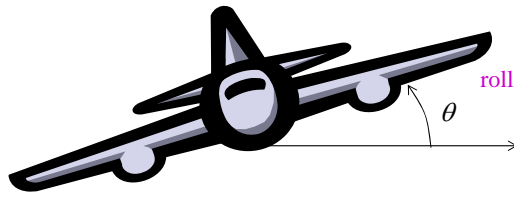
state-space realization

$$x_P := \begin{bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \Rightarrow \dot{x}_P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -50.875 & -43.75 & 0 \end{bmatrix} x_P + \begin{bmatrix} 0 \\ 0 \\ -1000 \end{bmatrix} u$$

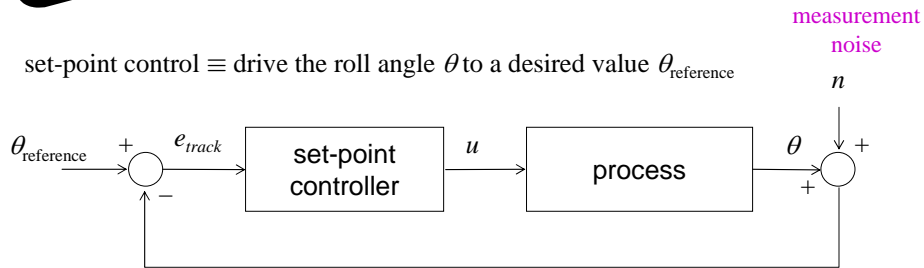
$$\theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_P$$

c_P

Example #11: Roll-angle control

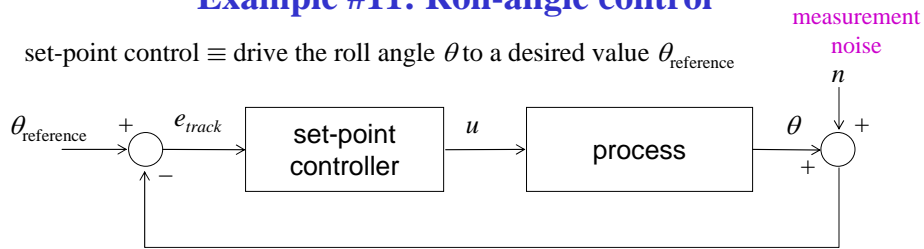


set-point control \equiv drive the roll angle θ to a desired value $\theta_{\text{reference}}$



Example #11: Roll-angle control

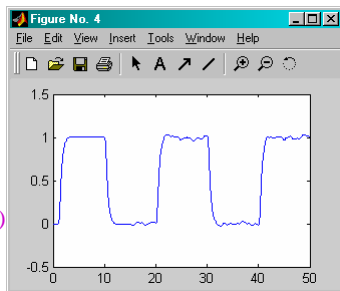
set-point control \equiv drive the roll angle θ to a desired value $\theta_{\text{reference}}$



controller 1

$$\ddot{u} + 63\dot{u} + 751u + 4471e$$

$$= 6.7\ddot{e} + 340\dot{e} + 316e$$

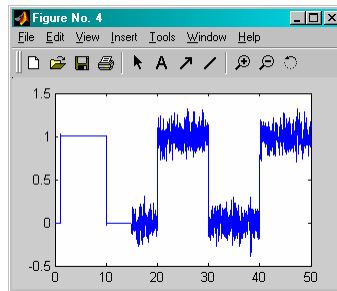


slow but
not very
sensitive to noise
(low-gain)

controller 2

$$\ddot{u} + 974\dot{u} + 4.7 \times 10^5 u + 1.2 \times 10^8 e$$

$$= 10^6 (48\ddot{e} + 322\dot{e} + 316e)$$



fast but
very
sensitive to noise
(high-gain)

Realization theory (SISO)

n th order input-output model $u, y \in \mathbb{R}$

$$y^{(n)} + \alpha_1 y^{(n-1)} + \dots + \alpha_{n-1} \dot{y} + \alpha_n y = \beta_1 u^{(n-1)} + \dots + \beta_{n-1} \dot{u} + \beta_n u$$

state-space realization of the IO model for short $\alpha(y) = \beta(u)$

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \quad \left| \quad x \in \mathbb{R}^m \equiv \text{state} \right.$$

Suppose A is asymptotically stable: $\exists P > 0, PA + A'P = -I$

$(P^{1/2}AP^{-1/2}, P^{1/2}b, cP^{-1/2})$ is also a realization of the IO model

$$\bar{A} \quad \bar{b} \quad \bar{c}$$

$$\underbrace{P^{-1/2}(A'P + PA)P^{-1/2}}_{\bar{A} + \bar{A}'} = -P^{-1} < 0$$

Theorem:

Given any n th order input-output model for which all roots of $\alpha(s)$ have negative real parts, it is always possible to find a realization for it, for which

$$A + A' = Q < 0$$

Switching between input-output models

$\mathcal{M} := \{ \alpha_q(y) = \beta_q(u) : q \in \mathcal{Q} \} \equiv$ finite family of n th order input-output models

Theorem:

There exists a family of realizations for \mathcal{M}

$$\mathcal{R} := \{ (A_q, b_q, c_q) : q \in \mathcal{Q} \}$$

such that the switched system

$$\dot{x} = A_\sigma x$$

is exponentially stable for arbitrary switching

Why?

1st Choose realizations such that $A_q + A_q' = -Q_q < 0 \forall q \in \mathcal{Q}$

2nd The function $V(z) = z'z$ is a common Lyapunov function for the switched system: continuously differentiable, positive definite, radially unbounded,

$$\frac{\partial V}{\partial x} A_q x = x'(A_q + A_q')x = -x'Q_q x < 0 \quad \forall x \neq 0$$

system is uniformly asymptotically stable \Rightarrow exponentially stable

Back to switching controllers...

controller 1

$$\ddot{u} + 63\dot{u} + 751u + 4471u = 6.7\ddot{e} + 340\dot{e} + 316e$$

realization: $\dot{z} = F_1 z + g_1 e$
 $u = h_1 z$

$$F_1 = \begin{bmatrix} -63 & -23 & -17 \\ 32 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$h_1 = [1.7 \quad 2.7 \quad .31]$$

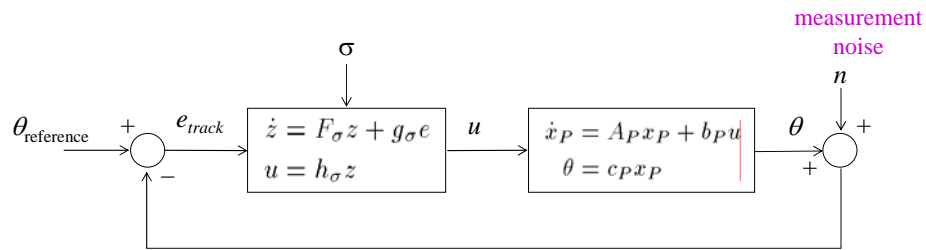
controller 2

$$\ddot{u} + 974\dot{u} + 4.7 \times 10^5 u + 1.2 \times 10^8 u = 10^6(48\ddot{e} + 322\dot{e} + 316e)$$

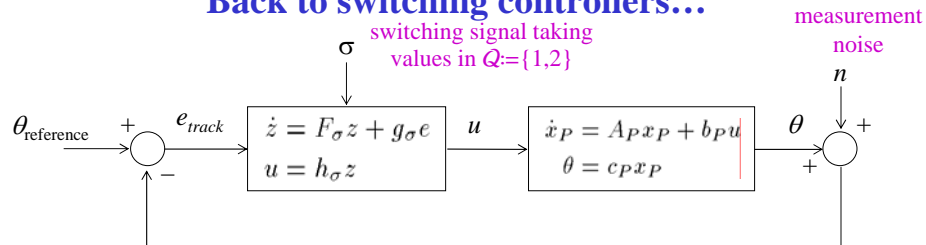
realization: $\dot{z} = F_2 z + g_2 e$
 $u = h_2 z$

$$F_2 = \begin{bmatrix} -974 & -459 & -229 \\ 1024 & 0 & 0 \\ 0 & 512 & 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 8192 \\ 0 \\ 0 \end{bmatrix}$$

$$h_2 = [5859 \quad 38 \quad 0.074]$$



Back to switching controllers...



overall system:

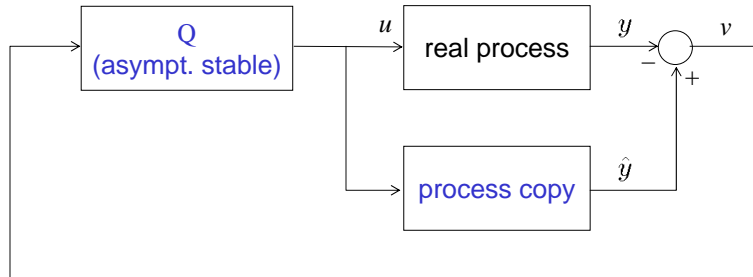
$$\begin{bmatrix} \dot{x}_P \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_P & b_P h_\sigma \\ -g_\sigma c_P & F_\sigma \end{bmatrix} \begin{bmatrix} x_P \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ g_\sigma \end{bmatrix} (\theta_{\text{ref}} - n)$$

$$\dot{x} = A_\sigma x + b_\sigma (\theta_{\text{ref}} - n) \quad A_q := \begin{bmatrix} A_P & b_P h_q \\ -g_q c_P & F_q \end{bmatrix} \quad q \in Q := \{1, 2\}$$

Assuming each controller was properly designed, each A_q is asymptotically stable but the overall switched systems could still be unstable
 This can be avoided by proper choice of the controller realizations

Youla parameterization (non-switched systems)

Assume process is asymptotically stable

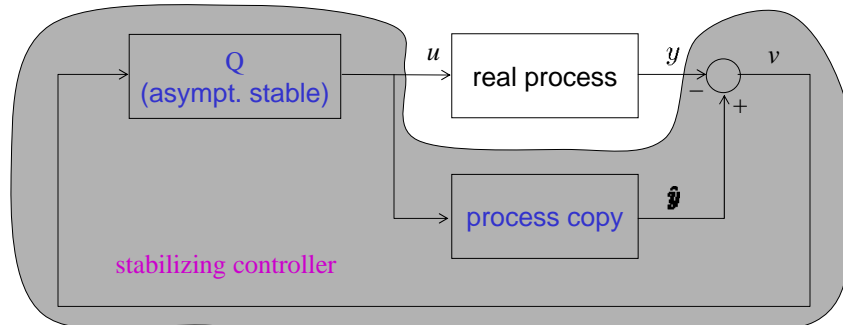


1. If the real process and its copy have the same initial conditions $\Rightarrow v = 0 \forall t$
otherwise v converges to zero exponentially fast
2. Since the Q system is asymptotically stable, u converges to zero exponentially fast

No matter what we choose for Q , as long as it is asymptotically stable, the overall system is asymptotically stable

Youla parameterization (non-switched systems)

Assume process is asymptotically stable

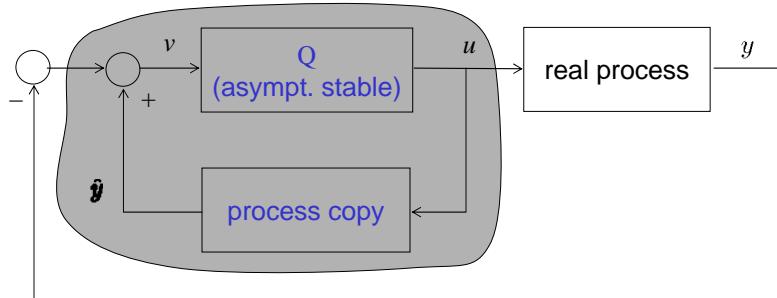


1. If the real process and its copy have the same initial conditions $\Rightarrow v = 0 \forall t$
otherwise v converges to zero exponentially fast
2. Since the Q system is asymptotically stable, u converges to zero exponentially fast

No matter what we choose for Q , as long as it is asymptotically stable, the overall system is asymptotically stable

Youla parameterization (non-switched)

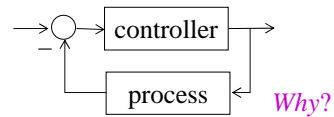
Assume process is asymptotically stable



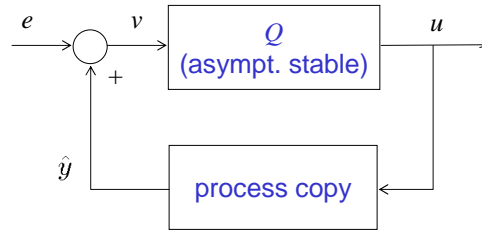
Theorem [Youla-Bongiorno]:

1. For any asymptotically stable Q , this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is of this form, for an asymptotically stable Q with the same IO model as:

a similar parameterization also exists when the process is not asymptotically stable...



“Youla” realizations



realization for Q

$$\begin{array}{l} \dot{x}_Q = \bar{A}x_Q + \bar{b}v \\ u = \bar{c}x_Q \end{array}$$

realization for the process copy

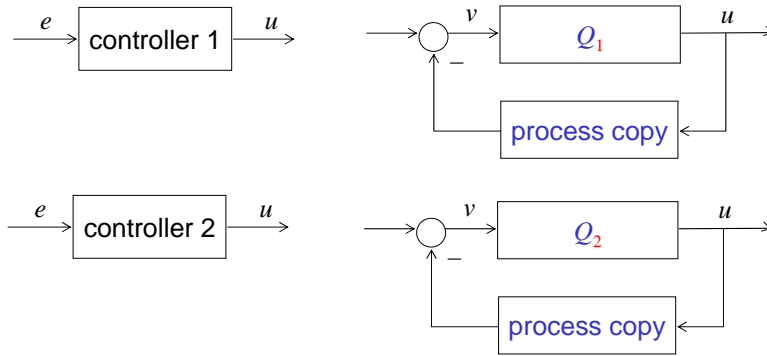
$$\begin{array}{l} \dot{x}_C = A_P x_C + b_P u \\ \hat{y} = c_P x_C \end{array}$$

realization for the controller

$$\begin{array}{l} \begin{bmatrix} \dot{x}_C \\ \dot{x}_Q \end{bmatrix} = \begin{bmatrix} A_P & b_P \bar{c} \\ \bar{b} c_P & \bar{A} \end{bmatrix} \begin{bmatrix} x_C \\ x_Q \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix} e \\ y = \begin{bmatrix} 0 & \bar{c} \end{bmatrix} \begin{bmatrix} x_C \\ x_Q \end{bmatrix} \end{array}$$

In general these realizations are not minimal

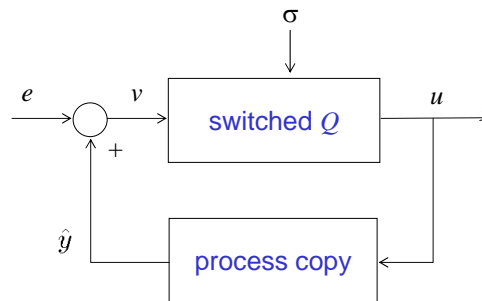
and back again to multiple controllers...



$$\dot{z} = \begin{bmatrix} A_P & b_P \bar{c}_q \\ \bar{b}_q c_P & \bar{A}_q \end{bmatrix} z + \begin{bmatrix} 0 \\ \bar{b}_q \end{bmatrix} e \quad (\bar{A}_q, \bar{b}_q, \bar{c}_q) \text{ realization for } Q_q$$

$$y = \begin{bmatrix} 0 & \bar{c}_q \end{bmatrix} z$$

Switching controller



Switched Q realization for the process copy

$$\dot{x}_Q = \bar{A}_\sigma x_Q + \bar{b}_\sigma v$$

$$u = \bar{c}_\sigma x_Q$$

switched controller

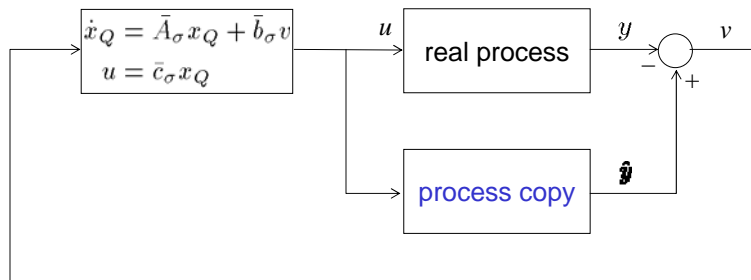
$$\dot{z} = F_\sigma z + g_\sigma e$$

$$u = h_\sigma z$$

$$A_q := \begin{bmatrix} A_P & b_P \bar{c}_q \\ \bar{b}_q c_P & \bar{A}_q \end{bmatrix} \quad b_q := \begin{bmatrix} 0 \\ \bar{b}_q \end{bmatrix}$$

$$c_q := \begin{bmatrix} 0 & \bar{c}_q \end{bmatrix}$$

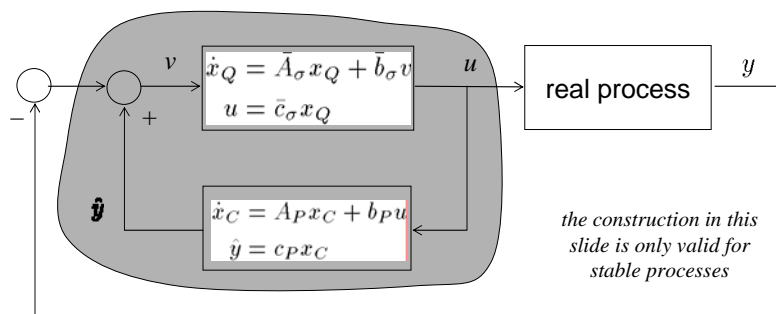
Switched closed-loop



1. If the real process and its copy have the same initial conditions $\Rightarrow v = 0 \forall t$
otherwise v converges to zero exponentially fast
2. If the switched Q system is asymptotically stable, u converges to zero exponentially fast and the overall system is asymptotically stable

Always possible by appropriate choice of realizations for each Q_q
(e.g., by choosing realizations so that $V(z) = z' z$ is a common Lyapunov function)

Switched closed-loop



Theorem:

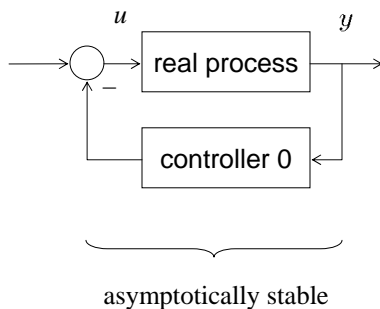
For every family of input-output controller models, there always exist a family a controller realizations such that the switched closed-loop systems is exponentially stable for arbitrary switching.

One can actually show that there exists a common quadratic Lyapunov function for the closed-loop.

In general the realizations are not minimal

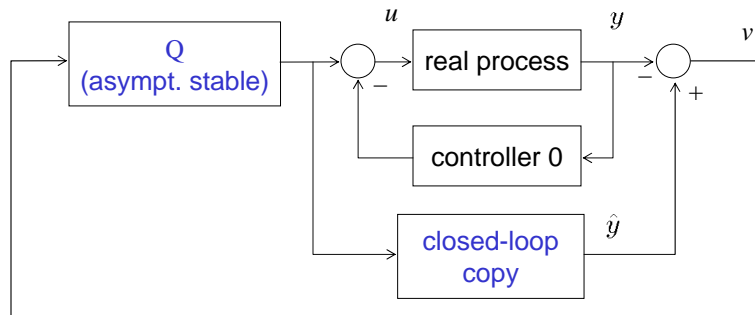
Non-asymptotically stable processes

1st Pick one stabilizing “nominal” controller



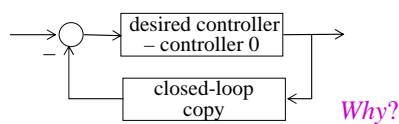
Non-asymptotically stable processes

2nd repeat previous construction



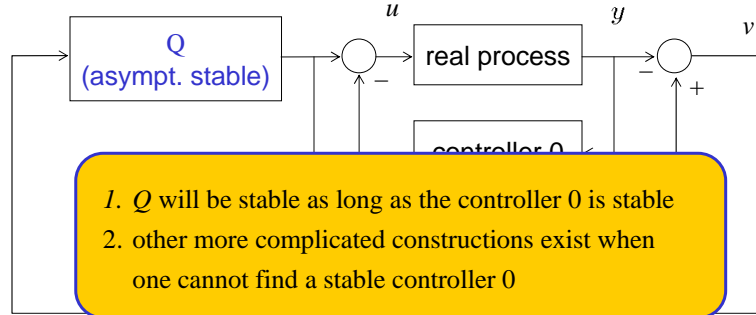
Theorem [Youla-Bongiorno]:

1. For any asymptotically stable Q , this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is this form, for an appropriately chosen Q :



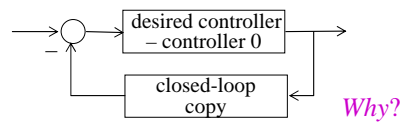
Non-asymptotically stable processes

2nd repeat previous construction

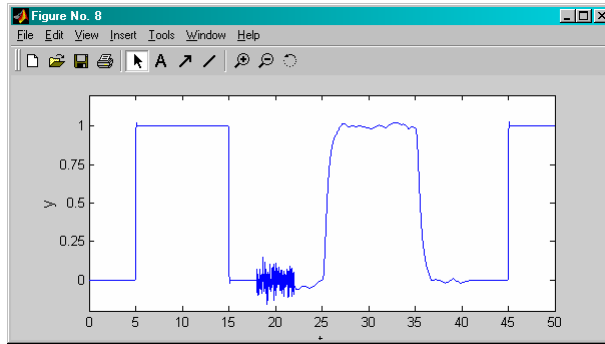
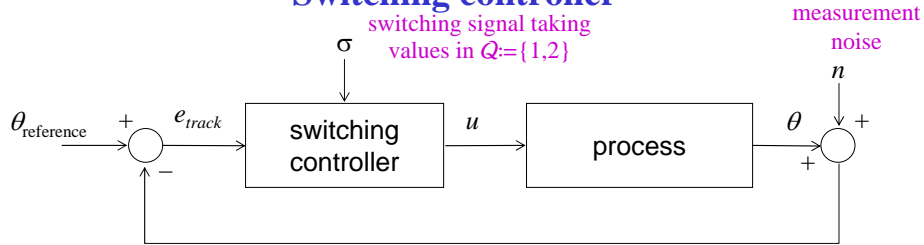


Theorem [Youla-Bongiorno]:

1. For any asymptotically stable Q , this controller asymptotically stabilizes the overall system
2. Any controller that asymptotically stabilizes the overall system is this form, for an appropriately chosen Q :



Switching controller



By proper choice of the controllers realization we can have stability for arbitrary switching.

Next lecture...

Stability under slow switching

- Dwell-time switching
- Average dwell-time
- Stability under brief instabilities