Hybrid Control and Switched Systems

Lecture #14
Computational methods to construct multiple Lyapunov functions
& Applications

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Summary

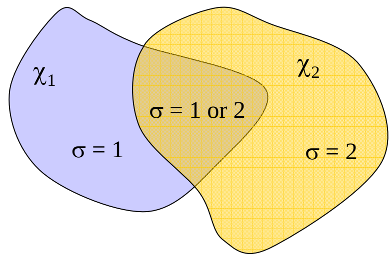
- Computational methods to construct multiple Lyapunov functions—Linear Matrix Inequalities (LMIs)
- Applications (vision-based control)

Current-state dependent switching

$\chi := \{\chi_q \in \mathbb{R}^n : q \in \mathcal{Q}\} \equiv$ (not necessarily disjoint) covering of $\mathbb{R}^n$, i.e., $\cup_{q\in\mathcal{Q}}\, \chi_q = \mathbb{R}^n$

Current-state dependent switching

$\mathcal{S}[\chi] \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous such that $\forall\, t$, $\sigma(t) = q$ is allowed only if $x(t) \in \chi_q$



Thus $(\sigma, x) \in \mathcal{S}[\chi]$ if and only if $x(t) \in \chi_{\sigma(t)}\ \forall\, t$

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem:
Suppose there exists $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{K}$ and a family of continuously differentiable, functions $V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ such that

$$V_q(z) \geq \alpha_1(\|z\|), \quad \frac{\partial V_q}{\partial x}(z) f_q(z) \leq -\alpha_2(\|z\|) \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$V_p(z) \geq V_q\big(\rho(q, p, z)\big)$$

Then the origin is (glob) uniformly asymptotically stable.

Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

Theorem:
Suppose there exist constants $\delta, \varepsilon \geq 0$ and symmetric matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$z' P_q z \geq \delta ||z||^2 \quad z'(A_q' P_q + P_q A_q) z \leq -\epsilon ||z||^2 \quad \forall q \in \mathcal{Q}, \; z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$z P_q z \leq z' P_p z$$

check!

Then the origin is uniformly asymptotically stable (actually exponentially).

Given the A_q, $q \in \mathcal{Q}$
can we find P_q, $q \in \mathcal{Q}$ and $\chi := \{ \chi_q : q \in \mathcal{Q} \}$ so that we have stability?

stabilization through choice
of switching regions

Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

Theorem:
Suppose there exist constants $\delta, \varepsilon \geq 0$ and symmetric matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

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and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$z P_q z \leq z' P_p z$$

check!

Then the origin is uniformly asymptotically stable (actually exponentially).

Given P_q, $q \in \mathcal{Q}$, suppose we choose:

$$\begin{aligned}\chi_q :=& \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \;\; \forall p \in \mathcal{Q}\} \\ =& \{z \in \mathbb{R}^n : z' P_q z = \max_{p \in \mathcal{Q}} z' P_p z\}\end{aligned}$$

sigma must be equal to (one of) the largest $V_q(z) := z' P_q z$ ⇒ for switching to be possible at $z \in \mathbb{R}^n$ from p to q, we must have $V_q(z) = V_p(z)$ ⇒ jump condition is always satisfied

Multiple Lyapunov functions (linear)

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

Theorem:
Suppose there exist constants $\delta, \varepsilon \geq 0$ and symmetric matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$z' P_q z \geq \delta \|z\|^2 \quad z'(A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2 \quad \forall q \in \mathcal{Q},\ z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$z P_q z \leq z' P_p z$$

check!

Then the origin is uniformly asymptotically stable (actually exponentially).

It is sufficient to find P_q, $q \in \mathcal{Q}$ such that

$$z' P_q z \geq \delta \|z\|^2 \quad z'(A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2 \quad \forall q \in \mathcal{Q},\ z \in \chi_q$$

$$\chi_q := \{ z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \ \ \forall p \in \mathcal{Q} \}$$

or equivalently $\forall\, q \in \mathcal{Q}$:

$$z' P_q z \geq z' P_p z \ \ \forall p \in \mathcal{Q} \quad \Rightarrow \quad z' P_q z \geq \delta \|z\|^2 \quad z'(A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2$$

Finding multiple Lyapunov functions

We need to find P_q, $q \in \mathcal{Q}$ such that $\forall\, q \in \mathcal{Q}$

$$z' P_q z \geq z' P_p z \ \ \forall p \in \mathcal{Q} \quad \Rightarrow \quad z' P_q z \geq \delta \|z\|^2 \quad z'(A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2$$

Suppose we can find P_q, $q \in \mathcal{Q}$ and constants $\gamma_{p,q}$, $\mu_{p,q} \geq 0$, $p,q \in \mathcal{Q}$ such that

$$P_q \geq \delta I + \sum_{p \in \mathcal{Q}\setminus\{q\}} \mu_{p,q} (P_q - P_p)$$

$$A_q' P_q + P_q A_q \leq -\epsilon I - \sum_{p \in \mathcal{Q}\setminus\{q\}} \gamma_{p,q} (P_q - P_p)$$

Then

$$z' P_q z \geq \delta \|z\|^2 + \sum_{p \in \mathcal{Q}\setminus\{q\}} \mu_{p,q} z' (P_q - P_p) z$$

$$z'(A_q' P_q + P_q A_q) z \leq -\epsilon \|z\|^2 - \underbrace{\sum_{p \in \mathcal{Q}\setminus\{q\}} \gamma_{p,q} z' (P_q - P_p) z}_{\geq 0 \text{ when } z' P_q z \geq z' P_p z \ \forall p \in \mathcal{Q}}$$

Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

Theorem:
Suppose there exist constants δ, ε $\gamma_{p,q}$, $\mu_{p,q} \geq 0$, $p,q \in \mathcal{Q}$ and symmetric matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$P_q \geq \delta I + \sum_{p \in \mathcal{Q}\setminus\{q\}} \mu_{p,q}(P_q - P_p)$$

$$A_q' P_q + P_q A_q \leq -\epsilon I - \sum_{p \in \mathcal{Q}\setminus\{q\}} \gamma_{p,q}(P_q - P_p)$$

Then, for

$$\chi_q := \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \ \ \forall p \in \mathcal{Q}\}$$

the origin is uniformly asymptotically stable (actually exponentially).

Linear Matrix Inequality (LMI)

$F(x) \geq 0$ where $F : \mathbb{R}^m \to \mathbb{R}^{n\times n} \equiv$ affine function (i.e., $F(x)$–$F(0)$ is linear)

System of linear matrix inequalities:

$$\begin{cases} F_1(x) \geq 0 \\ F_2(x) \geq 0 \\ \vdots \\ F_k(x) \geq 0 \end{cases} \Leftrightarrow \quad \bar{F}(x) \geq 0 \quad \bar{F}(x) := \begin{bmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & F_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & F_k(x) \end{bmatrix}$$

single LMI

System of linear matrix equalities and inequalities

$$\begin{cases} F(x) \geq 0 \\ G(x) = 0 \end{cases} \Leftrightarrow \begin{cases} F(x) \geq 0 \\ G(x) \geq 0 \\ -G(x) \geq 0 \end{cases} \Leftrightarrow \begin{bmatrix} F(x) & 0 & 0 \\ 0 & G(x) & 0 \\ 0 & 0 & -G(x) \end{bmatrix} \geq 0$$

single LMI

(however most numerical solvers handle equalities separately more efficiently)

There are very efficient methods to solve LMIs (type 'help lmilab' in MATLAB)

Back to constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

no resets

$P_q - P_q' = 0$

Theorem:
Suppose there exist constants δ, ε $\gamma_{p,q}$, $\mu_{p,q} \geq 0$, $p,q \in \mathcal{Q}$ and symmetric matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$P_q \geq \delta I + \sum_{p\in\mathcal{Q}\setminus\{q\}} \mu_{p,q}(P_q - P_p)$$

$$A_q' P_q + P_q A_q \leq -\epsilon I - \sum_{p\in\mathcal{Q}\setminus\{q\}} \gamma_{p,q}(P_q - P_p)$$

Then, for

$$\chi_q := \{z \in \mathbb{R}^n : z' P_q z \geq z' P_p z \ \ \forall p \in \mathcal{Q}\}$$

the origin is uniformly asymptotically stable (actually exponentially).

*

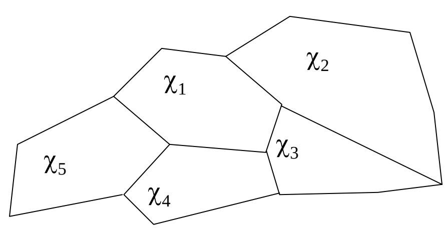
LMI on the unknowns δ, $\varepsilon > 0$, $\gamma_{p,q}$, $\mu_{p,q} \in \mathbb{R}$, $P_q \in \mathbb{R}^{n\times n}$, $p,q \in \mathcal{Q}$,

Only useful if we have the freedom to choose the switching region χ_q

Convex polyhedral regions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

χ_q, $q \in \mathcal{Q} \equiv$ convex polyhedral with disjoint interiors



cell $q \in \mathcal{Q}$:

component-wise

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad E_q \in \mathbb{R}^{m_q \times (n+1)}$$

boundary between regions p, $q \in \mathcal{Q}$:

$$\bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f_{pq}' \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\} \qquad f_{p,q} \in \mathbb{R}^{n+1}$$

Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

Theorem:

Suppose there exist

1. symmetric matrices $P_q \in \mathbb{R}^{(n+1)\times(n+1)}$, $q \in \mathcal{Q}$
2. vectors $k_{pq} \in \mathbb{R}^{n+1}$, $p,q \in \mathcal{Q}$
3. constants ϵ, $\delta > 0$
4. symmetric matrices with nonnegative entries U_q, W_q, $q \in \mathcal{Q}$ (not nec. pos. def)

could be generalized to affine systems, i.e., $\dot{x} = A_\sigma x + b_\sigma$ (homework!)

such that for every $p,q \in \mathcal{Q}$ for which χ_q and χ_p have a common boundary

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq}$$

and $\forall\, q \in \mathcal{Q}$

$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \qquad \Pi := \begin{bmatrix} I_{n\times n} & 0_{1\times n} \end{bmatrix}$$

$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi$$

the origin is uniformly asymptotically stable (actually exponentially).

LMI on the unknowns P_q, k_{pq}, δ, ε, U_q, W_q

Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq} \quad *$$

$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \qquad \Pi := \begin{bmatrix} I_{n\times n} & 0_{1\times n} \end{bmatrix}$$

$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi$$

Why?

Consider multiple Lyapunov functions: $V_q(z) := \begin{bmatrix} z^T & 1 \end{bmatrix} P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \quad q \in \mathcal{Q}$

1st On the boundary between χ_q and χ_p:

$$f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad V_q(z) = V_p(z)$$

Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq}$$
$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \; * \qquad \Pi := [I_{n \times n} \quad 0_{1 \times n}]$$
$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi$$

Why?

Consider multiple Lyapunov functions $V_q(z) := [z^T \quad 1] \, P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \qquad q \in \mathcal{Q},$

2nd On χ_q: $V_q(z) = [z^T \quad 1] \, P_q \begin{bmatrix} z \\ 1 \end{bmatrix}$

$$= [z^T \quad 1] \, (P_q - E_q^T W_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} + [z^T \quad 1] \, E_q^T W_q \underbrace{E_q \begin{bmatrix} z \\ 1 \end{bmatrix}}_{\geq 0}$$
$$\geq [z^T \quad 1] \, (P_q - E_q^T W_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix}$$
$$\geq \delta \, [z^T \quad 1] \, \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = \delta \|z\|^2$$

Constructing multiple Lyapunov functions

$$\dot{x} = A_\sigma x \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f'_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

$$P_q - P_p = k_{pq} f'_{pq} + f_{pq} k'_{pq}$$
$$P_q - E_q^T W_q E_q \geq \delta \Pi^T \Pi \qquad \Pi := [I_{n \times n} \quad 0_{1 \times n}]$$
$$\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q \leq -\epsilon \Pi^T \Pi \; *$$

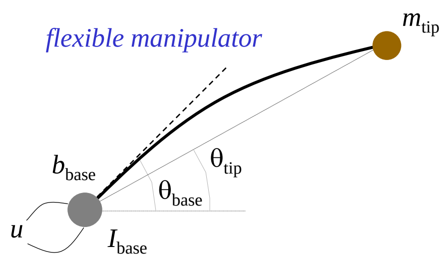
Why?

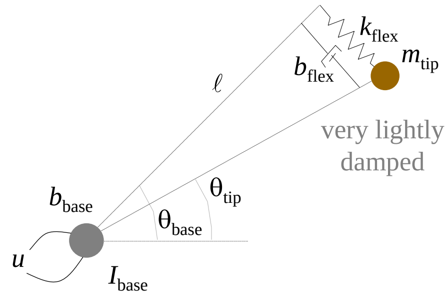
Consider multiple Lyapunov functions $V_q(z) := [z^T \quad 1] \, P_q \begin{bmatrix} z \\ 1 \end{bmatrix} \qquad q \in \mathcal{Q},$

3rd On χ_q:

$$\frac{\partial V_q}{\partial z}(z) A_q z = [z^T \quad 1] \, (\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi) \begin{bmatrix} z \\ 1 \end{bmatrix}$$
$$= [z^T \quad 1] \, (\Pi^T A_q^T \Pi P_q + P_q \Pi^T A_q \Pi + E_q^T U_q E_q) \begin{bmatrix} z \\ 1 \end{bmatrix} - [z^T \quad 1] \, E_q^T U_q E_q \begin{bmatrix} z \\ 1 \end{bmatrix}$$
$$\leq -\epsilon \, [z^T \quad 1] \, \Pi^T \Pi \begin{bmatrix} z \\ 1 \end{bmatrix} = -\epsilon \|z\|^2.$$

Example: Vision-based control of a flexible manipulator





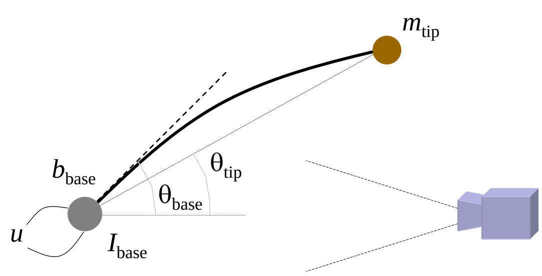
4th dimensional small bending approximation

$$m_{\text{tip}}\ell^2\ddot{\theta}_{\text{tip}} = \ell k_{\text{flex}}(\theta_{\text{base}} - \theta_{\text{tip}}) + \ell b_{\text{flex}}(\dot{\theta}_{\text{base}} - \dot{\theta}_{\text{tip}})$$
$$I_{\text{base}}\ddot{\theta}_{\text{base}} = -b_{\text{base}}\dot{\theta}_{\text{base}} + \ell k_{\text{flex}}(\theta_{\text{tip}} - \theta_{\text{base}}) + \ell b_{\text{flex}}(\dot{\theta}_{\text{tip}} - \dot{\theta}_{\text{base}}) + k_{\text{motor}}u$$

Control objective: drive θ_{tip} to zero, using feedback from

$\theta_{\text{base}} \rightarrow$ encoder at the base
$\theta_{\text{tip}} \rightarrow$ machine vision (essential to increase the damping of the flexible modes in the presence of noise)

Example: Vision-based control of a flexible manipulator



To achieve high accuracy in the measurement of θ_{tip} the camera must have a small field of view

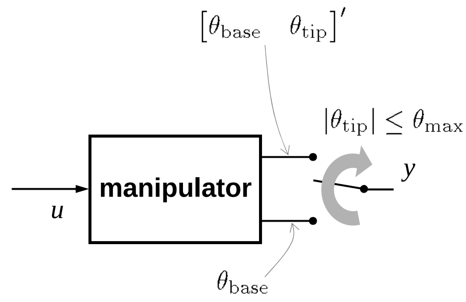
feedback output:

$$y := \begin{cases} \begin{bmatrix} \theta_{\text{base}} & \theta_{\text{tip}} \end{bmatrix}' & |\theta_{\text{tip}}| \leq \theta_{\text{max}} \\ \theta_{\text{base}} & |\theta_{\text{tip}}| > \theta_{\text{max}} \end{cases}$$

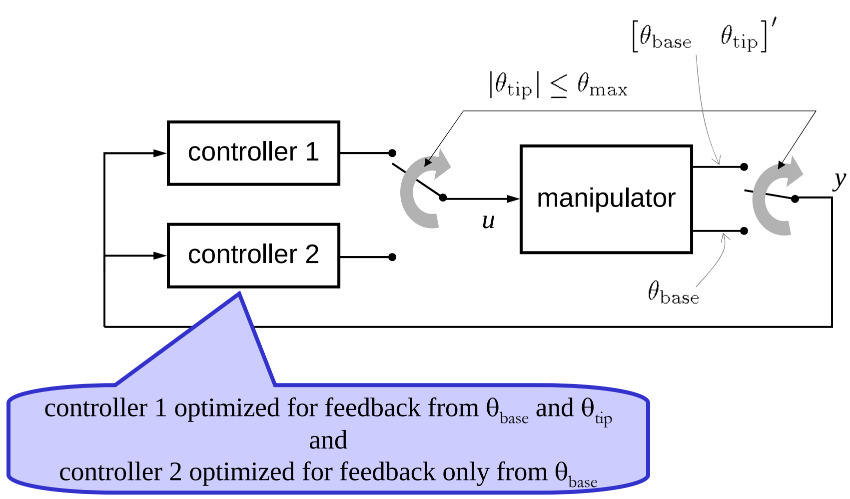
Control objective: drive θ_{tip} to zero, using feedback from

$\theta_{\text{base}} \rightarrow$ encoder at the base
$\theta_{\text{tip}} \rightarrow$ machine vision (essential to increase the damping of the flexible modes in the presence of noise)

Switched process

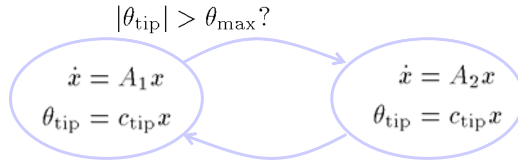


Switched process



E.g., LQG controllers that minimize $\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \theta_{\text{tip}}^2 + \dot{\theta}_{\text{tip}}^2 + \rho u^2 dt\right]$

Switched system



feedback connection with controller 1 (θ_{base} and θ_{tip} available)

feedback connection with controller 2 (only θ_{base} available)

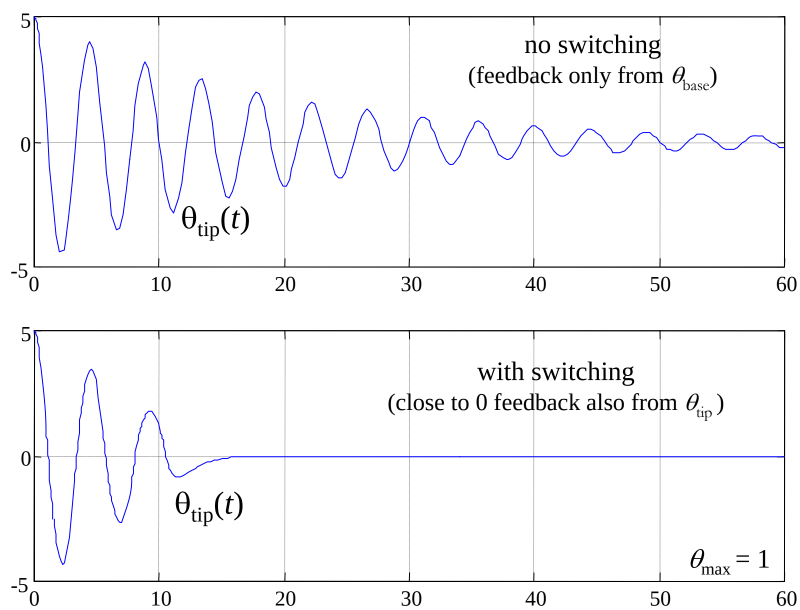
$$f_{13} := \begin{bmatrix} c_{\text{tip}} & \theta_{\max} \end{bmatrix} \qquad f_{12} := \begin{bmatrix} c_{\text{tip}} & -\theta_{\max} \end{bmatrix}$$

$$E_3 := \begin{bmatrix} -c_{\text{tip}} & -\theta_{\max} \end{bmatrix} \quad \Big| \quad E_1 := \begin{bmatrix} -c_{\text{tip}} & \theta_{\max} \\ c_{\text{tip}} & \theta_{\max} \end{bmatrix} \quad \Big| \quad E_2 := \begin{bmatrix} c_{\text{tip}} & -\theta_{\max} \end{bmatrix}$$

$-\theta_{\max}$ $\qquad$ $\theta_{\max}$ $\qquad$ θ_{tip}

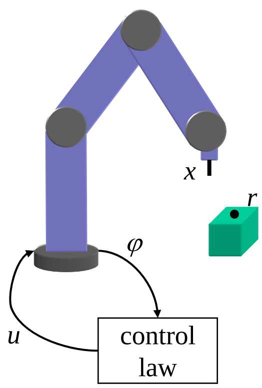
$$\chi_q := \left\{ z \in \mathbb{R}^n : E_q \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \right\} \qquad \bar{\chi}_q \cap \bar{\chi}_p \subset \left\{ z \in \mathbb{R}^n : f_{pq} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0 \right\}$$

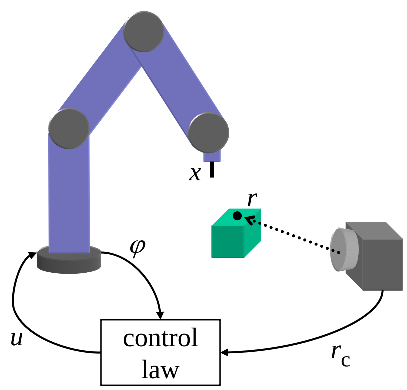
Closed-loop response



Visual servoing

Goal: Drive the position *x* of the end-effector to a reference point *r*





encoder-based control

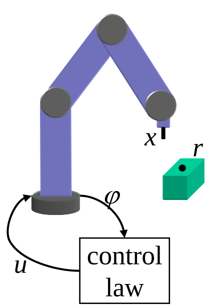
- reference *r* in pre-specified position
- end-effector position *x* determined from the joint angles

mixed vision/encoder-based control

- reference *r* determined from the camera measurement r_c
- end-effector position *x* determined from the joint angles

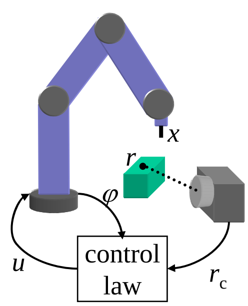
Visual servoing

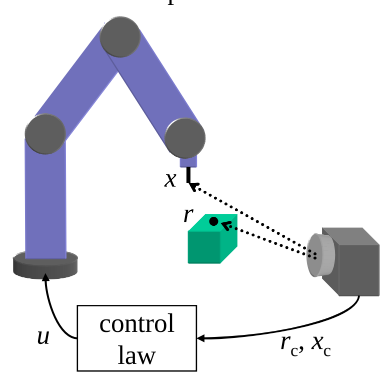
Goal: Drive the position *x* of the end-effector to a reference point *r*



encoder-based control

mixed vision/encoder-based control





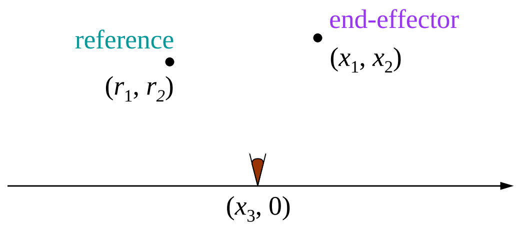
vision-based control

- reference *r* determined from the camera measurement r_c
- end-effector position *x* determined from the camera measurement x_c

Typically guarantees zero error, even in the presence of manipulator and camera modeling errors ↗

Prototype problem

Goal: Drive the position (x_1, x_2) of the end-effector to the reference point (r_1, r_2), using visual feedback



For simplicity:

1. Normalized pin-hole camera moves along a straight line perpendicular to its optical axis

$$y_1 = \frac{x_1 - x_3}{x_2} \qquad y_2 = \frac{r_1 - x_3}{r_2} \quad \Bigg\} \text{ camera measurements (robot \& reference)}$$

2. Feedback linearized kinematic model for the motions of robots and camera

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = u_3$$

Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = u_3 \qquad y_1 = \frac{x_1 - x_3}{x_2} \qquad y_2 = \frac{r_1 - x_3}{r_2}$$

Can we design an output-feedback controller that asymptotically stabilizes the equilibrium point

$$x_1 = r_1 \qquad x_2 = r_2 \qquad x_3 = \alpha$$

for some $\alpha \in \mathbb{R}$?

Local linearization around equilibrium yields:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \frac{1}{r_2} \begin{bmatrix} 1 & \frac{\alpha - r_1}{r_2} & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

not detectable

Manifestation of a fundamental limitation on the class of controllers that asymptotically stabilize the system…

Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = u_3 \qquad y_1 = \frac{x_1 - x_3}{x_2} \qquad y_2 = \frac{r_1 - x_3}{r_2} \qquad (1)$$

Can we design an output-feedback controller that asymptotically stabilizes the equilibrium point

$$x_1 = r_1 \qquad x_2 = r_2 \qquad x_3 = \alpha$$

for some $\alpha \in \mathbb{R}$?

Lemma: There exist no locally Lipschitz functions

$$F: \mathbb{R}^n \times \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^n \qquad G: \mathbb{R}^n \times \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^3$$

for which the closed-loop system defined by (1) and

$$\dot{z} = F(z, y_1, y_2, t) \qquad \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = G(z, y_1, y_2, t)$$

has an asymptotically stable equilibrium point with

$$x_1 = r_1 \qquad x_2 = r_2$$

Stabilizability by output-feedback

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = u_3 \qquad y_1 = \frac{x_1 - x_3}{x_2} \qquad y_2 = \frac{r_1 - x_3}{r_2} \qquad (1)$$

This limitation is not caused by lack of controllability/observability

Lemma 2: For every $r_1, r_2 \in \mathbb{R}$, the system (1) is observable. Moreover, there exists a constant input which allows one to uniquely determine the initial state by observing the output over any finite interval (t_0, t_0+T). Indeed, for

$u_1 = u_2 = 0$ ➞ robot stopped

$u_3 = 1$ ➞ camera moving at constant speed

there exists a state reconstruction function h_T for which

$$\begin{bmatrix} x_1(t_0+T) \\ x_2(t_0+T) \\ x_3(t_0+T) \end{bmatrix} = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0)+T \end{bmatrix} = h_T\left(\int_{t_0}^{t_0+T} \begin{bmatrix} y_1 & ty_1 & y_2 & ty_2 \end{bmatrix}' dt \right) + \begin{bmatrix} r_1 \\ r_2 \\ r_1 \end{bmatrix}$$

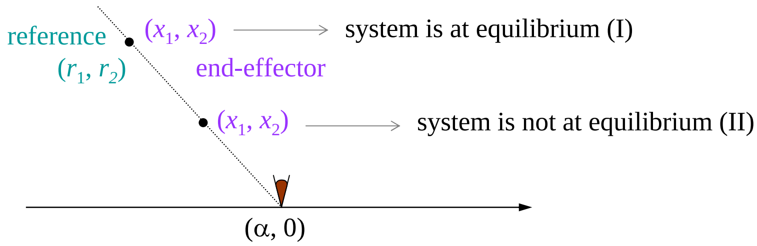
final state

measured outputs

relative position with respect to reference

Limitation on asymptotic stabilization

Goal: Drive the position (x_1, x_2) of the end-effector to the reference point (r_1, r_2), using visual feedback

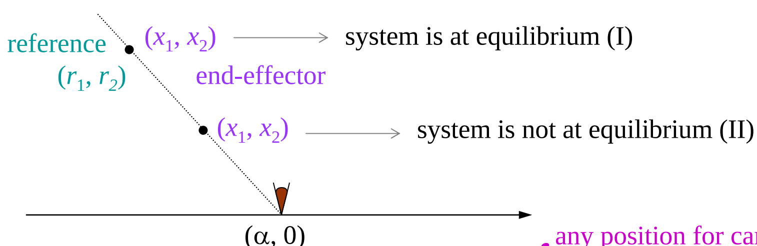


To disambiguate between cases I and II, the camera must move away from $(\alpha, 0)$

This would violate the stability of the equilibrium point in case I

Limitation on asymptotic stabilization

Goal: Drive the position (x_1, x_2) of the end-effector to the reference point (r_1, r_2), using visual feedback



any position for camera

There is no need to stabilize the camera and the state of the controller to a fixed equilibrium. It is sufficient to make the set

$$\mathcal{G} := \{\, [r_1 \;\; r_2 \;\; \alpha \;\; z']' \in \mathbb{R}^{n+3} : \; \alpha \in \mathbb{R} \,, z \in \mathbb{R}^n \,\}$$

asymptotically stable

any state for controller

The set $\mathcal{G}$ is asymptotically stable if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall t_0 \geq 0$

$$d(x_0, \mathcal{G}) < \delta \quad \Rightarrow \quad d(\, x(t), \mathcal{G}\,) < \varepsilon$$

and $d(x(t), \mathcal{G}\,) \to 0$ as $t \to \infty$

solution to closed-loop (required to exist globally)

Hybrid controller

look mode:

$u_1 = u_2 = 0, \quad u_3 = 1$ ➞ robot stopped, camera moving

$\dot{z} = [y_1 \quad ty_1 \quad y_2 \quad ty_2]$ ➞ collect information for reconstruction

$\dot{\tau} = 1$ ➞ timer

Relies on the camera motion to estimate the robot's position w.r.t. the target.

$\tau > T_{\text{move}}$
$\tau := 0$

$\tau > T_{\text{look}}$
$\tau := 0$

move mode:

$[u_1 \quad u_2 \quad u_3]' = \dfrac{\mu h_{T_{\text{look}}(z)}}{T_{\text{move}}}$ ➞ move towards reference at constant speed ($\mu \in (0,1]$)

$\dot{z} = 0 \qquad \dot{\tau} = 1$

Hybrid controller

look mode:

$u_1 = u_2 = 0, \quad u_3 = 1$ ➞ robot stopped, camera moving

$\dot{z} = [y_1 \quad ty_1 \quad y_2 \quad ty_2]$ ➞ collect information for reconstruction

$\dot{\tau} = 1$ ➞ timer

$\tau > T_{\text{move}}$
$\tau := 0$

$\tau > T_{\text{look}}$
$\tau := 0$

move mode:

$[u_1 \quad u_2 \quad u_3]' = \dfrac{\mu h_{T_{\text{look}}(z)}}{T_{\text{move}}}$ ➞ move towards reference at constant speed ($\mu \in (0,1]$)

$\dot{z} = 0 \qquad \dot{\tau} = 1$

Theorem:

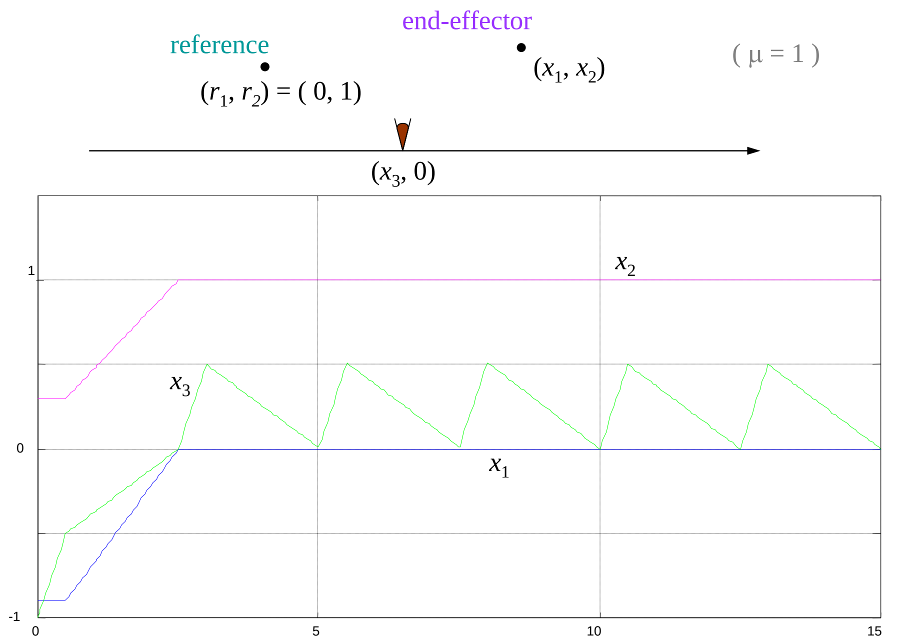
For any $T_{\text{look}}, T_{\text{move}} > 0$, $\mu \in (0,1]$, the hybrid controller renders the set

$$\mathcal{G} := \{ [r_1 \ r_2 \ \alpha \ z']' \in \mathbb{R}^{n+3} : \alpha \in \mathbb{R}, z \in \mathbb{R}^n \}$$

asymptotically stable for the closed-loop system.

For $\mu = 1$, the the set $\mathcal{G}$ is reached in finite time.

Stateflow simulation



Next lecture…

Applications: Supervisory control