Hybrid Control and Switched Systems

Lecture #3
What can go wrong?
Trajectories of hybrid systems

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Summary

1. Trajectories of hybrid systems:
   - Solution to a hybrid system
   - Execution of a hybrid system
2. Degeneracies
   - Finite escape time
   - Chattering
   - Zeno trajectories
   - Non-continuous dependency on initial conditions

Hybrid Automaton (deterministic)

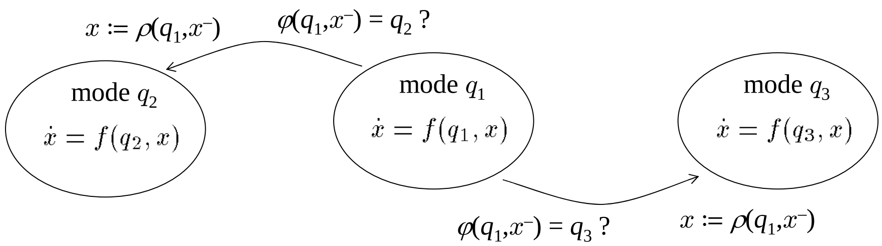
$\mathcal{Q}$ $\equiv$ set of discrete states

$\mathbb{R}^n$ $\equiv$ continuous state-space

$f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ $\equiv$ vector field

$\varphi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}$ $\equiv$ discrete transition

$\rho : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ $\equiv$ reset map



Hybrid Automaton

$\mathcal{Q}$ $\equiv$ set of discrete states

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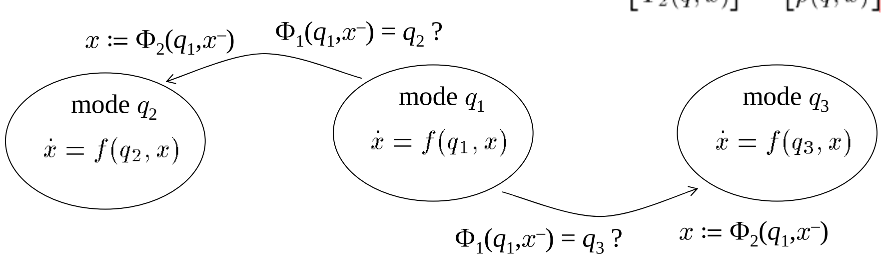
$f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ $\equiv$ vector field

$\Phi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q} \times \mathbb{R}^n$ $\equiv$ discrete transition (& reset map)

$$\Phi(q,x) = \begin{bmatrix} \Phi_1(q,x) \\ \Phi_2(q,x) \end{bmatrix} = \begin{bmatrix} \varphi(q,x) \\ \rho(q,x) \end{bmatrix}$$

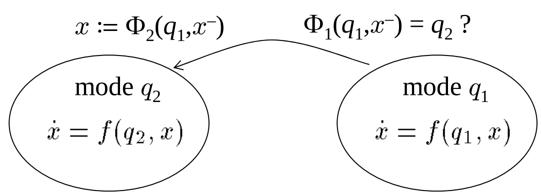


Compact representation of a hybrid automaton

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Solution to a hybrid automaton

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Definition: A ***solution*** to the hybrid automaton is a pair of right-continuous signals

$$x : [0,\infty) \to \mathbb{R}^n \qquad q : [0,\infty) \to \mathcal{Q}$$

such that

1. x is piecewise differentiable & q is piecewise constant
2. on any interval (t_1, t_2) on which q is constant and x continuous

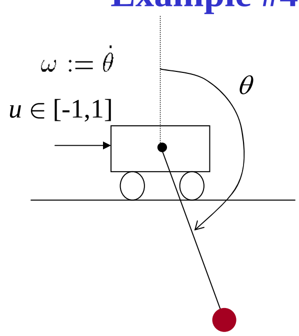
continuous evolution

$$x(t) = x(t_1) + \int_{t_1}^{t} f\big(q(t_1), x(\tau)\big)d\tau \qquad \forall t \in [t_1, t_2)$$

3. $\big(q(t), x(t)\big) = \Phi\big(q^-(t), x^-(t)\big) \quad \forall t \geq 0$

discrete transitions

Example #4: Inverted pendulum swing-up



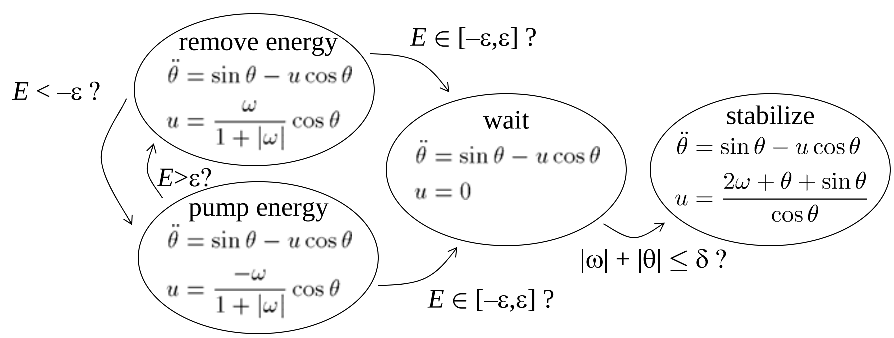
$$\ddot{\theta} = \sin\theta - u\cos\theta \qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$$

Hybrid controller:

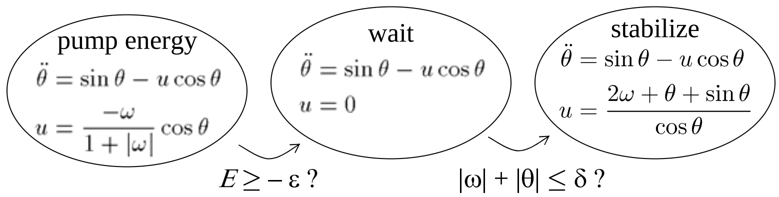
1st pump/remove energy into/from the system by applying maximum force, until E ≈ 0

2nd wait until pendulum is close to the upright position

3th next to upright position use feedback linearization controller



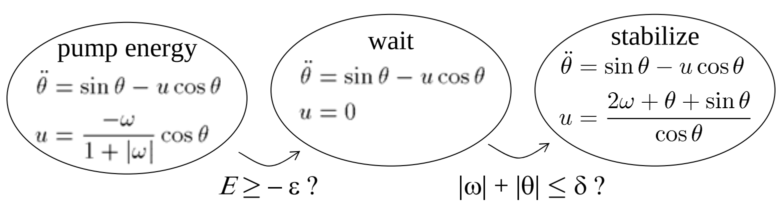
Example #4: Inverted pendulum swing-up



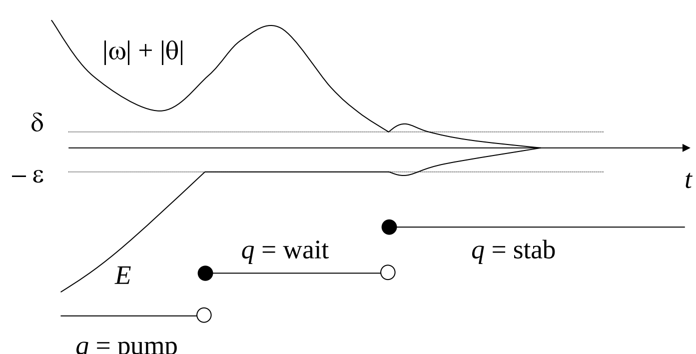
$\mathcal{Q}$ = { pump, wait, stab } $\mathbb{R}^2$ = continuous state-space

$$f(q,x) = \cdots \qquad \Phi(q,x) = \begin{cases} (\text{pump}, x) & q = \text{pump}, E < -\epsilon \\ (\text{wait}, x) & q = \text{pump}, E \geq -\epsilon \\ (\text{wait}, x) & q = \text{wait}, |\omega| + |\theta| > \delta \\ (\text{stab}, x) & q = \text{wait}, |\omega| + |\theta| \leq \delta \\ (\text{stab}, x) & q = \text{stab} \end{cases}$$

Example #4: Inverted pendulum swing-up

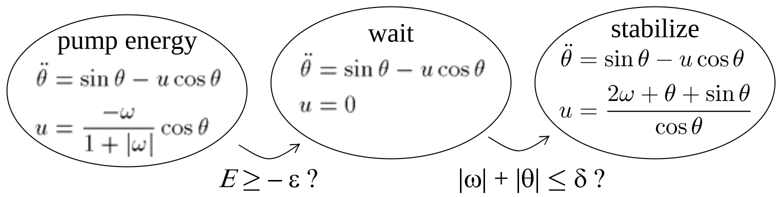


$\mathcal{Q}$ = { pump, wait, stab } $\mathbb{R}^2$ = continuous state-space



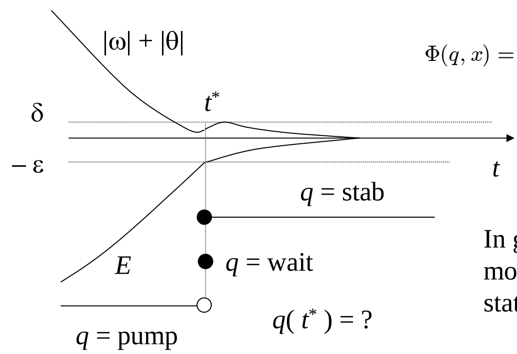
What if we already have $|\omega| + |\theta| \leq \delta$ when E reaches $-\varepsilon$?

Example #4: Inverted pendulum swing-up



$\mathcal{Q}$ = { pump, wait, stab } $\mathbb{R}^2$ = continuous state-space

$$\Phi(q,x) = \begin{cases} (\text{pump}, x) & q = \text{pump}, E < -\epsilon \\ (\text{wait}, x) & q = \text{pump}, E \geq -\epsilon \\ (\text{wait}, x) & q = \text{wait}, |\omega| + |\theta| > -\epsilon \\ (\text{stab}, x) & q = \text{wait}, |\omega| + |\theta| \leq -\epsilon \\ (\text{stab}, x) & q = \text{stab} \end{cases}$$



In general, we may need more than one value for the state at each instant of time

Hybrid signals

Definition: A ***hybrid time trajectory*** is a (finite or infinite) sequence of closed intervals

$$\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i,\ \tau'_i = \tau_{i+1},\ i = 1,2, \dots \}$$

(if τ is finite the last interval may by open on the right)

$\mathcal{T} \equiv$ set of hybrid time trajectories

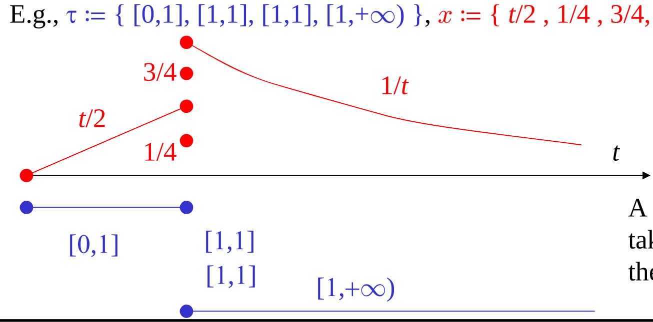
Definition: For a given $\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i,\ \tau_{i+1} = \tau'_i,\ i = 1,2, \dots \} \in \mathcal{T}$

a ***hybrid signal defined on τ*** with values on $\mathcal{X}$ is a sequence of functions

$$x = \{ x_i : [\tau_i, \tau'_i] \to \mathcal{X} \quad i = 1,2, \dots \}$$

$x : \tau \to \mathcal{X} \equiv$ hybrid signal defined on τ with values on $\mathcal{X}$

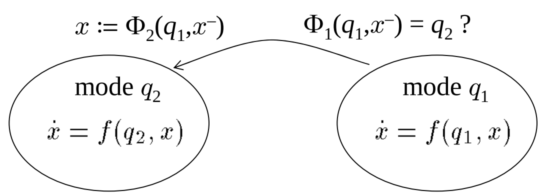
E.g., τ := { [0,1], [1,1], [1,1], [1,+∞) }, x := { t/2 , 1/4 , 3/4, 1/t }



A hybrid signal can take multiple values for the same time-instant

Execution of a hybrid automaton

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Definition: An ***execution*** of the hybrid automaton is a pair of hybrid signals

$$x : \tau \to \mathbb{R}^n \qquad q : \tau \to \mathcal{Q} \qquad \tau = \{ [\tau_i, \tau'_i] : i = 1, 2, \dots \} \in \mathcal{T}$$

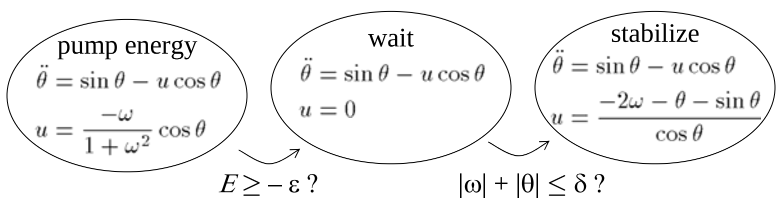
such that

1. on any $[\tau_i, \tau'_i] \in \tau$, q_i is constant and — continuous evolution

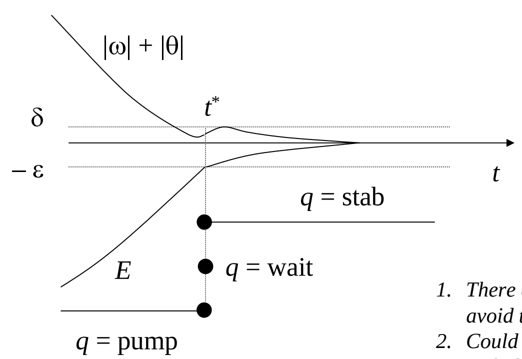
$$x_i(t) = x_i(t_1) + \int_{\tau_i}^{t} f\big(q_i(\tau_i), x_i(\tau)\big) d\tau \qquad \forall t \in [\tau_i, \tau'_i]$$

2. $\big(q(\tau_{i+1}), x(\tau_{i+1})\big) = \Phi\big(q(\tau'_i), x(\tau'_i)\big)$ — discrete transitions

Example #4: Inverted pendulum swing-up



$\mathcal{Q}$ = { pump, wait, stab } $\mathbb{R}^2$ = continuous state-space



$\tau := \{ [0,t^*], [t^*, t^*], [t^*,+\infty) \}$

$q :=$ { pump , wait , stab }

$x := \{ \dots, \dots, \dots \}$

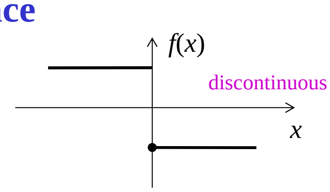
1. *There are other concepts of solution that also avoid this problem [Teel 2005]*
2. *Could one "fix" this hybrid system to still work with the usual notion of solution?*

What can go wrong?

1. Problems in the continuous evolution :
   - existence
   - uniqueness
   - finite escape
2. Problems in the hybrid execution:
   - Chattering
   - Zeno
3. Non-continuous dependency on initial conditions

Existence

$$\dot{x} = f(x) = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$$



There is no solution to this differential equation that starts with $x(0) = 0$

$$x(t) = \int_0^t f(x(\tau))d\tau \qquad \forall t \geq 0$$

Why? one any interval $[0,\varepsilon)$ x cannot: remain zero,
become positive, or
become negative.

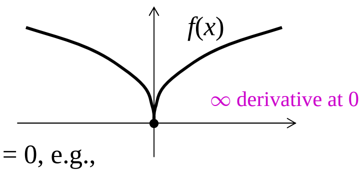
($x = 0$ would make some sense)

Theorem [Existence of solution]

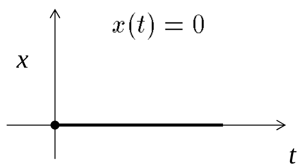
If $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***continuous***, then $\forall\, x_0 \in \mathbb{R}^n$ there exists at least one solution with $x(0) = x_0$, defined on some interval $[0,\varepsilon)$

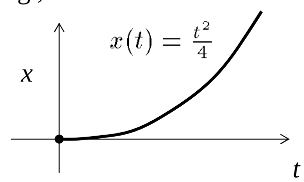
Uniqueness

$$\dot{x} = f(x) = \sqrt{|x|}$$



There are multiple solutions that start at $x(0) = 0$, e.g.,





Definitions: A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***Lipschitz continuous*** if in any bounded subset of $\mathcal{S}$ of $\mathbb{R}^n$ there exists a constant c such that

$$||f(x) - f(y)|| \leq c||x - y|| \quad \forall x, y \in \mathcal{S}$$

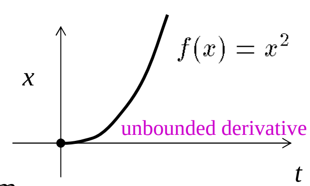
(f is differentiable almost everywhere and the derivative is bounded on any bounded set)

Theorem [Uniqueness of solution]

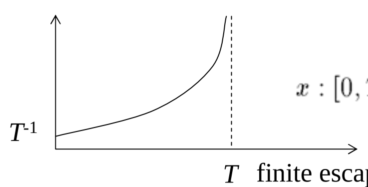
If $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***Lipschitz continuous***, then $\forall\, x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on some interval $[0,\varepsilon)$

Finite escape time

$$\dot{x} = x^2$$



Any solution that does not start at $x(0) = 0$ is of the form



$$x : [0, T) \to \mathbb{R} \qquad x(t) = \frac{1}{T - t} \qquad t \in [0, T)$$

finite escape ≡ solution x tends to ∞ in finite time

Definitions: A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***globally Lipschitz continuous*** if there exists a constant c such that

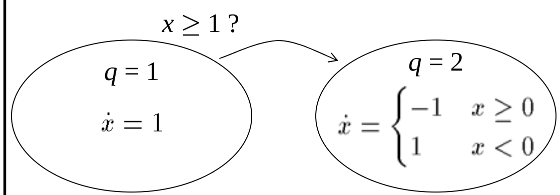
$$||f(x) - f(y)|| \leq c||x - y|| \quad \forall x, y \in \mathbb{R}^n$$

f grows no faster than linearly

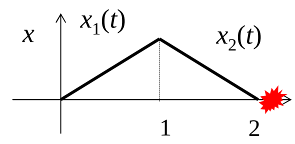
Theorem [Uniqueness of solution]

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***globally Lipschitz continuous***, then $\forall\, x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on $[0,\infty)$

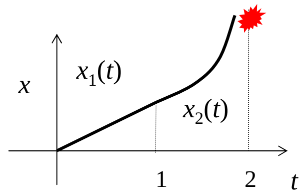
Degenerate executions due to problems in the continuous evolution

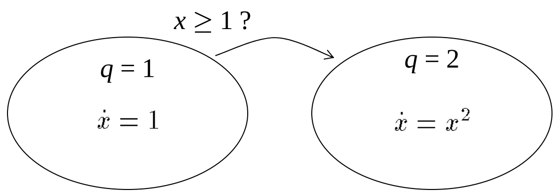


$\tau = \{ [0,1] , [1,2] \}$
$q = \{q_1(t) = 1 , q_2(t) = 2 \}$
$x = \{x_1(t) = t , x_2(t) = 2 - t \}$



$\tau = \{ [0,1] , [1,2) \}$
$q = \{q_1(t) = 1 , q_2(t) = 2 \}$
$x = \{x_1(t) = t , x_2(t) = 1/(2 - t) \}$

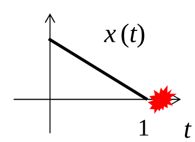


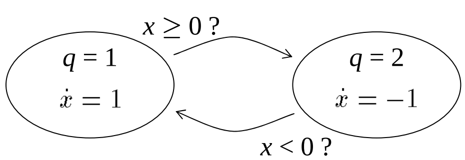


Chattering

$$\dot{x} = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

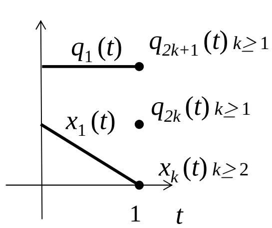
There is no solution past $x = 0$





but there is an execution

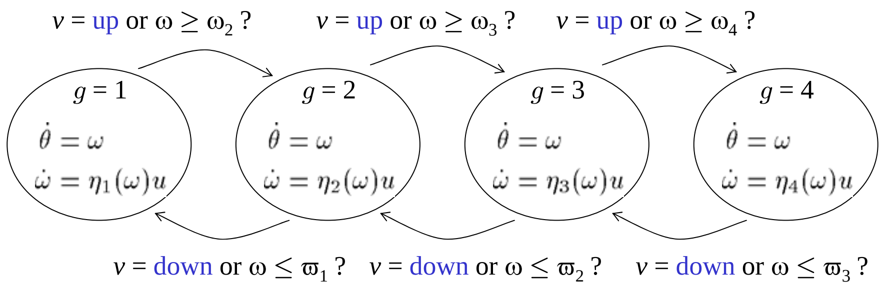
$\tau = \{ [0,1] , [1], [1], [1], \ldots \}$
$q = \{q_1(t) = 2 , q_2(t) = 1, q_1(t) = 2, \ldots\}$
$x = \{x_1(t) = 1 - t , x_2(t) = 0, x_2(t) = 0, \ldots\}$

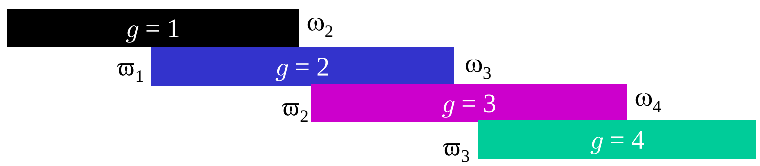


Chattering execution ≡ τ is infinite but after some time, all intervals are singletons

Example #3: Semi-automatic transmission

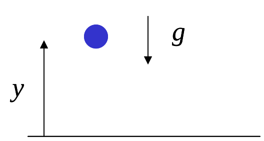
$v(t) \in \{$ up, down, keep $\} \equiv$ drivers input (discrete)





If the driver sets $v(t)$ = up $\forall\, t \geq t^*$ and $\omega(t^*) \leq \varpi_1$ one gets chattering. *For ever*?

Example #1: Bouncing ball (Zeno execution)

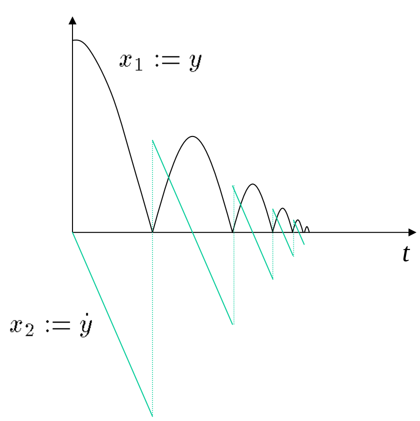


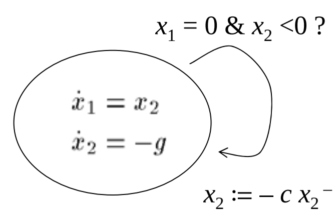
Free fall $\equiv$ $\ddot y = -g$

Collision $\equiv$ $y(t) = y^-(t) = 0$

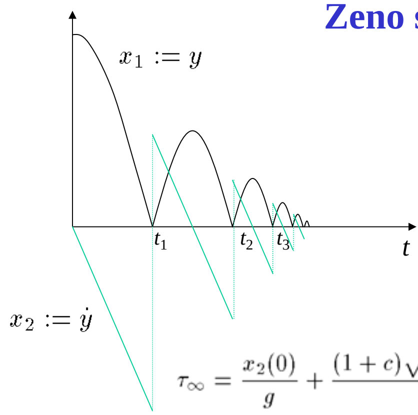
$\dot y(t) = -c\dot y^-(t)$

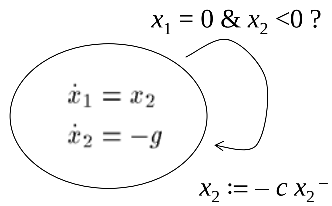
$c \in [0,1) \equiv$ energy absorbed at impact





Zeno solution





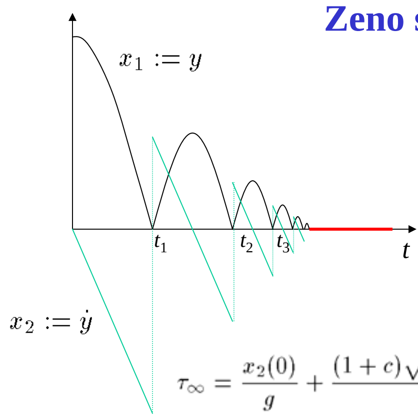
$$\tau_\infty = \frac{x_2(0)}{g} + \frac{(1+c)\sqrt{x_2(0)^2 + 2gx_1(0)}}{(1-c)g}$$

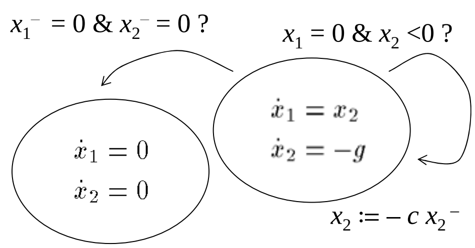
$$q : [0, \tau_\infty) \to \{1\} \qquad q(t) = 1$$

$$x : [0, \tau_\infty) \to \mathbb{R}^2 \qquad x(t) = \begin{cases} \begin{bmatrix} y_0 + \dot{y}_0 t - gt^2/2 \\ \dot{y}_0 - gt \end{bmatrix} & t \in [0, t_1] \\ \dots & t \in [t_1, t_2] \\ \dots & \end{cases}$$

$$x(t) = x(t_1) + \int_{t_1}^{t} f(q(t_1), x(\tau))d\tau \qquad \forall t \in [t_1, t_2) \qquad (q(t), x(t)) = \Phi(q^-(t), x^-(t)) \quad \forall t \geq 0$$

Zeno solution





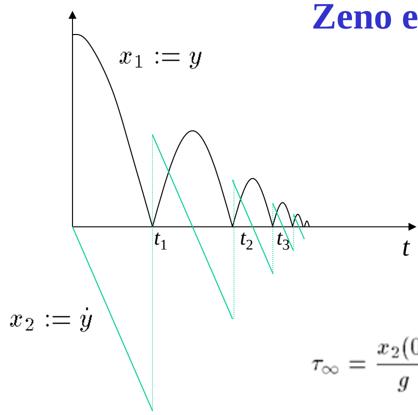
$$\tau_\infty = \frac{x_2(0)}{g} + \frac{(1+c)\sqrt{x_2(0)^2 + 2gx_1(0)}}{(1-c)g}$$

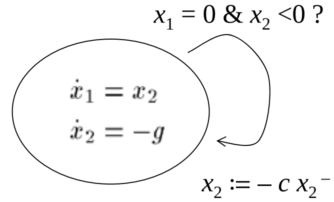
$$q : [0, \infty) \to \{1\} \qquad q(t) = 1$$

$$x : [0, \infty) \to \mathbb{R}^2 \qquad x(t) = \begin{cases} \begin{bmatrix} y_0 + \dot{y}_0 t - gt^2/2 \\ \dot{y}_0 - gt \end{bmatrix} & t \in [0, t_1] \\ \dots & t \in [t_1, t_2] \\ \dots & \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & t \geq \tau_\infty \end{cases}$$

$$x(t) = x(t_1) + \int_{t_1}^{t} f(q(t_1), x(\tau))d\tau \qquad \forall t \in [t_1, t_2) \qquad (q(t), x(t)) = \Phi(q^-(t), x^-(t)) \quad \forall t \geq 0$$

Zeno execution





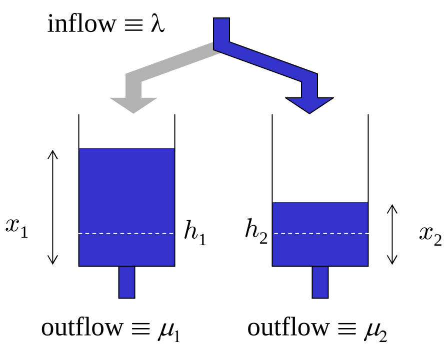
$$\tau_\infty = \frac{x_2(0)}{g} + \frac{(1+c)\sqrt{x_2(0)^2 + 2gx_1(0)}}{(1-c)g}$$

$$\tau = \{[0,t_1],[t_1,t_2],[t_2,t_3],\dots\}$$
$$q = \{q_1(t)=1, q_2(t)=1, q_3(t)=1,\dots\}$$
$$x = \left\{x_1(t) = \begin{bmatrix} y_0+\dot{y}_0 t - gt^2/2 \\ \dot{y}_0 - gt \end{bmatrix}, x_2(t) = \dots, \dots\right\}$$

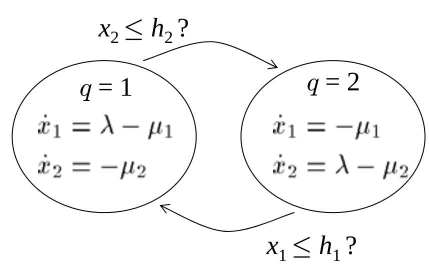
Zeno execution $\equiv$ τ is infinite but the execution does not extend to $t = +\infty$

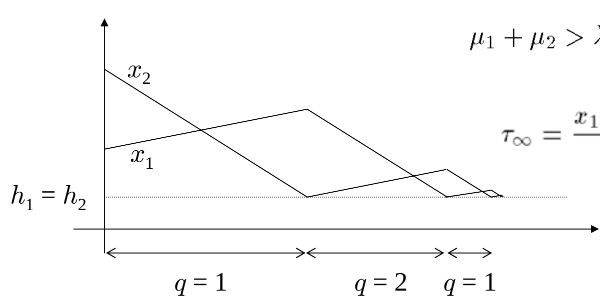
Zeno time $\equiv \tau_\infty := \sup_{s \in \tau} s$

Example #9: Water tank system (regularization)



goal ≡ prevent both tanks from becoming emptying



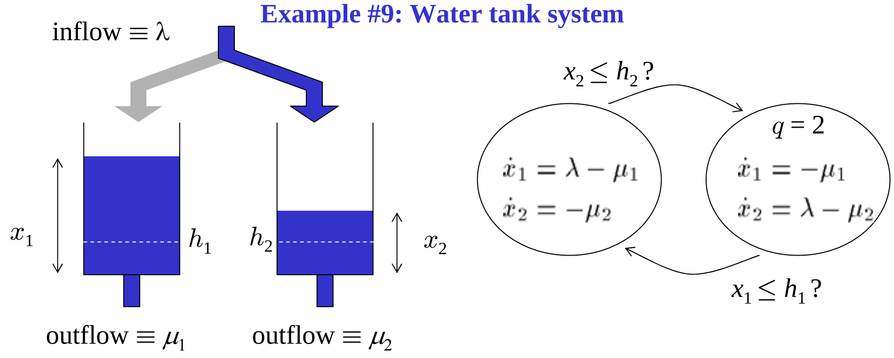


$$\mu_1 + \mu_2 > \lambda$$

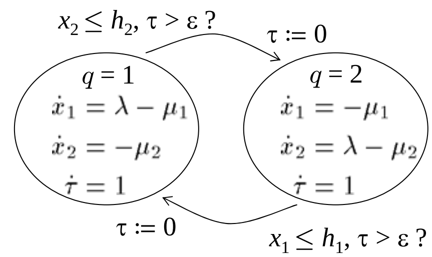
$$\tau_\infty = \frac{x_1(0) - h_1 + x_2(0) - h_2}{\mu_1 + \mu_2 - \lambda}$$

Temporal regularization

Example #9: Water tank system

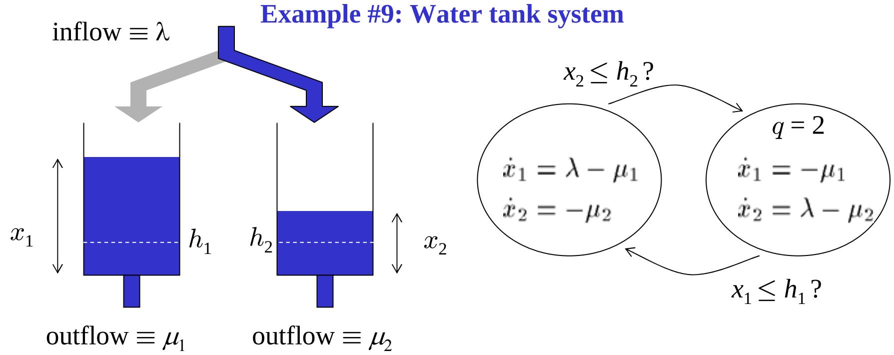


only jump after a minimum time has elapsed

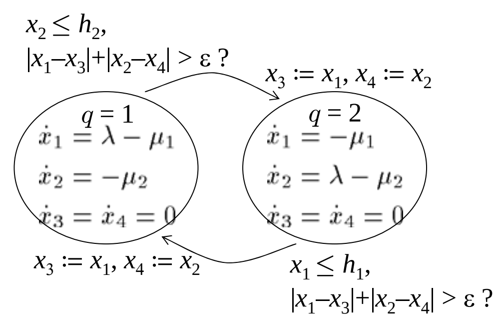


Spatial regularization

Example #9: Water tank system



only jump after a minimum change in the continuous state has occurred



Continuity with respect to initial conditions

$$\dot{x} = f(x)$$

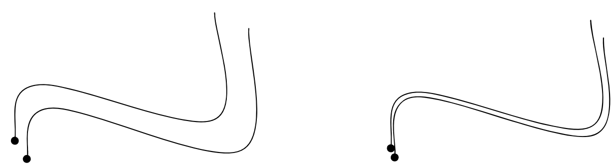
Theorem [Uniqueness & continuity of solution]

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***Lipschitz continuous***, then $\forall\, x_0 \in \mathbb{R}^n$ there a single solution with $x(0) = x_0$, defined on some interval $[0,\varepsilon)$

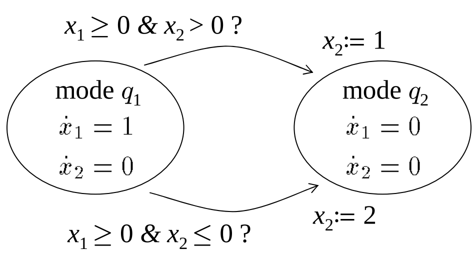
Moreover, given any $T < \infty$, and two solutions x_1, x_2 that exist on $[0,T]$:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \|x_1(0) - x_2(0)\| \leq \delta \;\; \Rightarrow \;\; \|x_1(t) - x_2(t)\| \leq \varepsilon \quad \forall\, t \in [0,T]$$

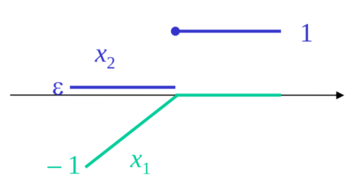
value of the solution on the interval $[0,T]$ is continuous with respect to the initial conditions



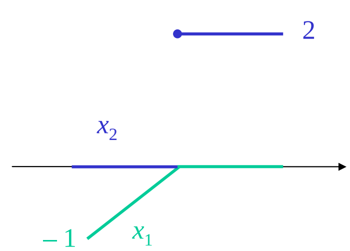
Discontinuity with respect to initial conditions



discontinuity of the reset map

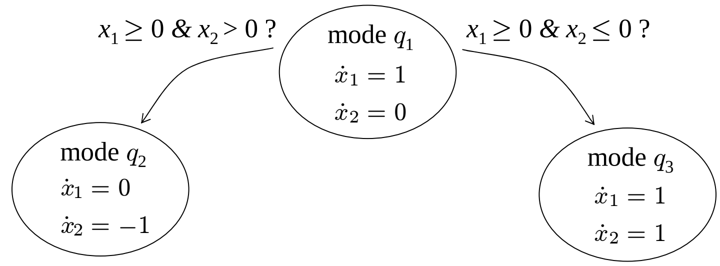


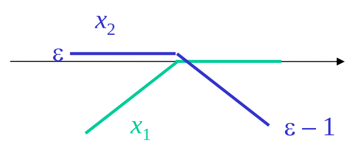
no matter how close to zero $x_2(0) = \varepsilon > 0$ is, $x_2(1) = 1$

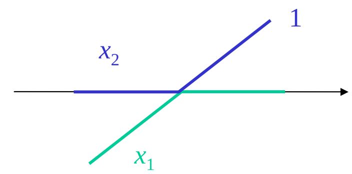


if $x_2(0) = 0$ then $x_2(1) = 2$

Discontinuity with respect to initial conditions







no matter how close to zero
$x_2(0) = \varepsilon > 0$ is, $x_2(2) = \varepsilon - 1$

if $x_2(0) = 0$ then $x_2(2) = 1$

problem arises from discontinuity of the transition function

Next class…

1. **Numerical simulation of hybrid automata**
   - simulations of ODEs
   - zero-crossing detection
2. Simulators
   - Simulink
   - Stateflow
   - SHIFT
   - Modelica

Follow-up homework
- Find conditions for the existence of solution to a hybrid system