

Hybrid Control and Switched Systems

Lecture #4 Simulation of hybrid systems

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Summary

1. **Numerical simulation of hybrid automata**
 - simulations of ODEs
 - zero-crossing detection
2. Simulators
 - Simulink
 - Stateflow
 - SHIFT
 - Modelica

Numerical simulation of ODEs

Initial value problem (IVP) $\equiv \dot{x} = f(x) \quad x(0) = x_0$

Definition: A signal $x : [0, T] \rightarrow \mathbb{R}^n$ is a **solution** to the IVP if

$$x(t) = x_0 + \int_0^t f(x(\tau)) d\tau \quad \forall t \in [0, T]$$

Euler method (first order method):

1st partition interval into N subintervals of length $h := T/N$

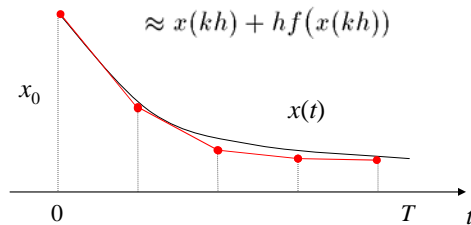
$$[kh, (k+1)h] \quad k \in \{0, 1, \dots, N-1\}$$

2nd assume derivative of x constant on each subinterval

$$x((k+1)h) = x(kh) + \int_{kh}^{(k+1)h} f(x(\tau)) d\tau$$

$$\approx x(kh) + hf(x(kh))$$

on each subinterval x
is assumed linear



Numerical simulation of ODEs

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Runge-Kutta methods (m -order method):

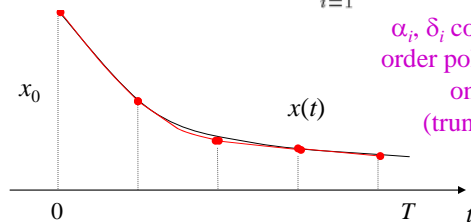
1st partition interval into N subintervals of length $h := T/N$

$$[kh, (k+1)h] \quad k \in \{0, 1, \dots, N-1\}$$

2nd assume derivative of x constant on each subinterval

$$x((k+1)h) \approx x(kh) + h \sum_{i=1}^m \alpha_i f(x(kh) + \delta_i)$$

α_i, δ_i computed assuming a m -
order polynomial approximation
on each subinterval
(truncated Taylor series)



Numerical simulation of ODEs

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Variable-step methods (e.g., Euler):

Pick tolerance ϵ and define $t_0 := 0$

$$\begin{aligned} x(t_{k+1}) &= x(t_k) + \int_{t_k}^{t_{k+1}} f(x(\tau)) d\tau \\ &\approx x(t_k) + (t_{k+1} - t_k) f(x(t_k)) \end{aligned}$$

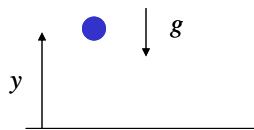
choose t_{k+1} sufficiently close to t_k so that

$$\|f(x(t_k)) - f(x(t_{k+1}))\| \leq \epsilon$$

Simulation can be both *fast* and *accurate*:

1. when f is “flat” one can advance time fast,
2. when f is “steep” one advances time slowly (to retain accuracy)

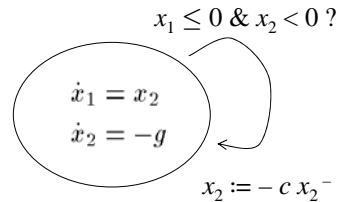
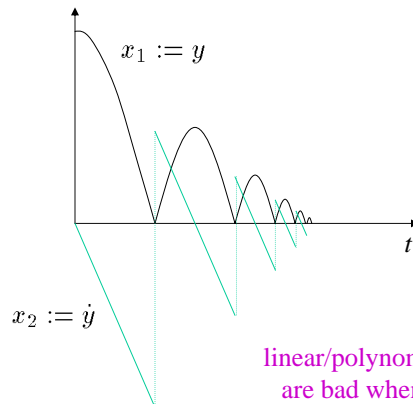
Example #1: Bouncing ball



Free fall $\equiv \ddot{y} = -g$

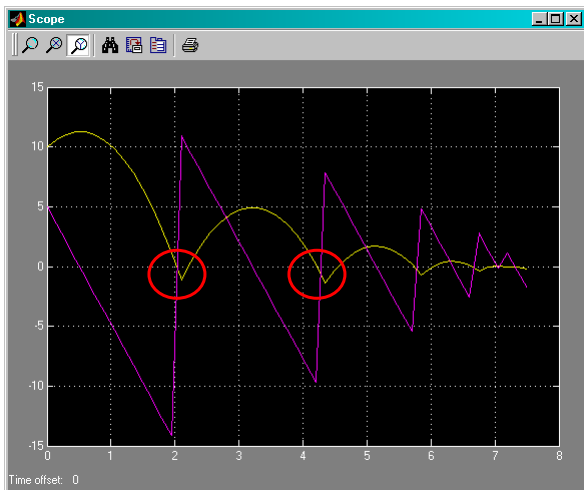
Collision $\equiv \begin{aligned} y(t) &= y^-(t) = 0 \\ \dot{y}(t) &= -c\dot{y}^-(t) \end{aligned}$

$c \in [0, 1) \equiv$ energy absorbed at impact

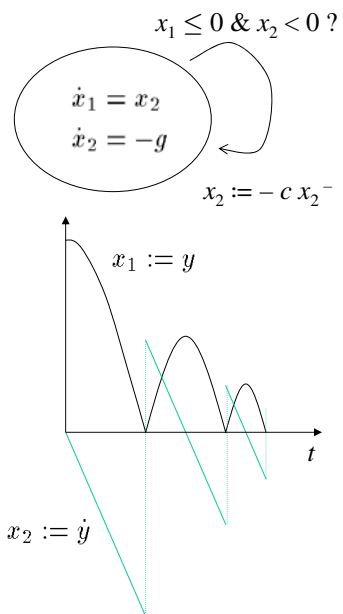


linear/polynomial approximations
are bad when transitions occur

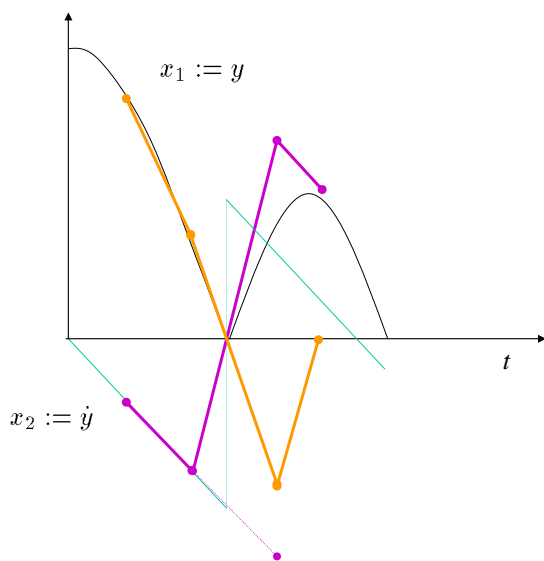
Example #1: Bouncing ball



ball_nozercross.mdl



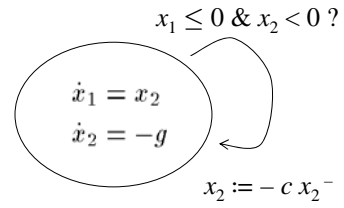
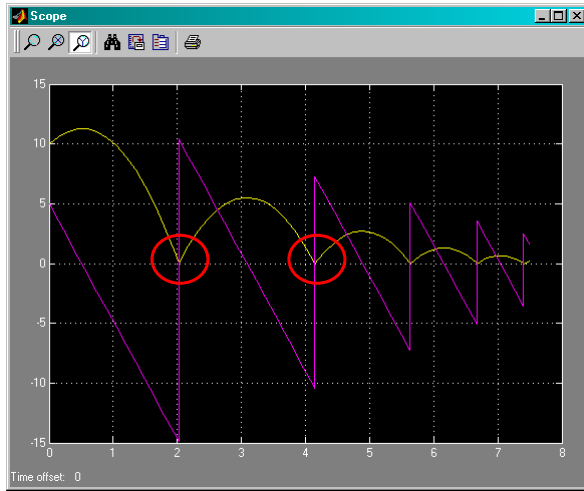
Example #1: Bouncing ball



before the transition the linear approximation seems very good so the integration algorithm is fooled into choosing a large integration step

Zero-crossing detection

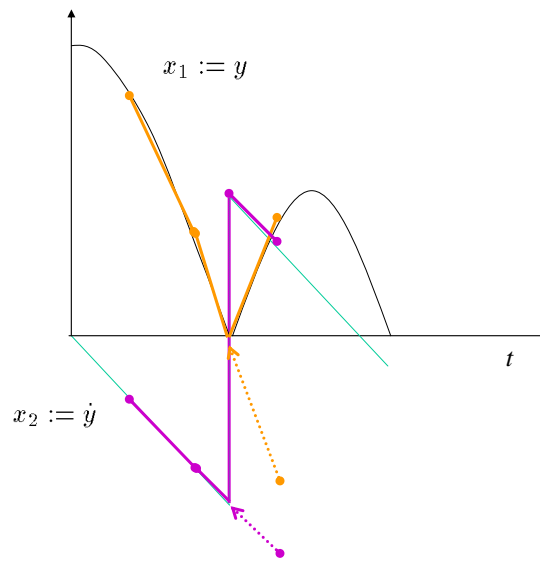
After a transition is detected, the integration algorithm “goes back in time” to determine where the transition occurred and starts a new integration step at that point.



ball_withzerocross.mdl

Zero-crossing detection

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Summary

1. Numerical simulation of hybrid automata

- simulations of ODEs
- zero-crossing detection

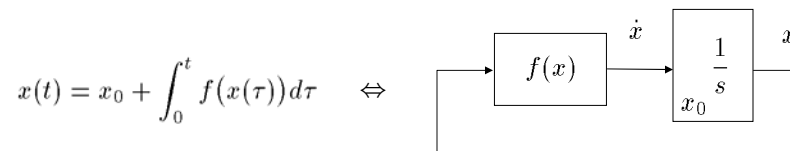
2. Simulators

- Simulink { suitable for a small number of discrete modes
difficult to recover hybrid automaton from Simulink file
- Stateflow { good for large numbers of discrete modes and complex transitions
poor integration between continuous and discrete
- SHIFT { very good semantics (easily understandable)
poor numerical algorithms
- Modelica { very good numerical algorithms
very convenient for large models with interconnected components
difficult to recover hybrid automaton from simulink file

MATLAB's Simulink

$$\dot{x} = f(x) \quad x(0) = x_0$$

1. What you see: graphical user interface to build models of dynamical systems



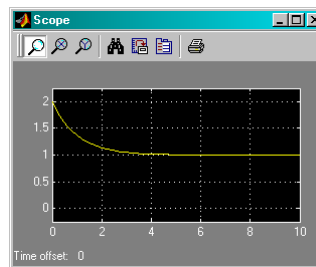
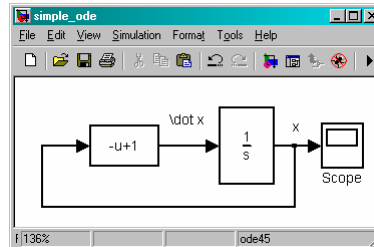
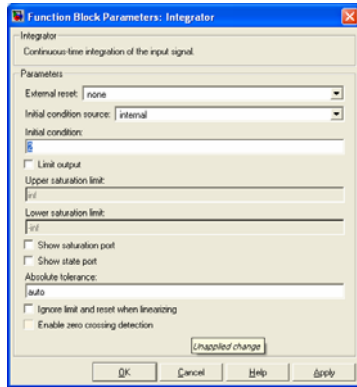
2. What's behind: numerical solver of ODEs with zero-crossing detection

A little history...

- Commercial product developed by MathWorks (founded 1984, flag product is MATLAB/Simulink)
- MATLAB's Simulink was inspired by MATRIXx's SystemBuild (in 2001 MathWorks bought MATRIXx)

Simulation of ODEs

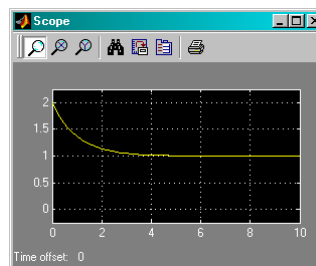
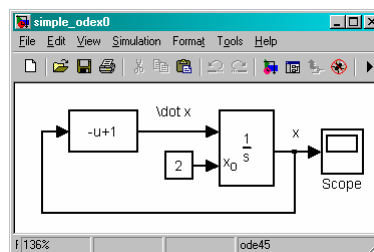
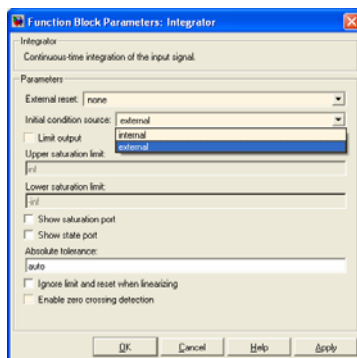
$$\dot{x} = -x + 1 \quad x(0) = 2$$



simple_ode.mdl

Simulation of ODEs

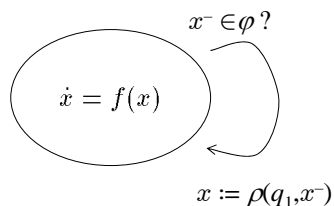
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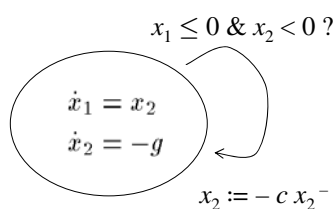
simple_ode0.mdl

ODEs with resets (or impulse systems)

- Q \equiv set of discrete states
- \mathbb{R}^n \equiv continuous state-space
- $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv vector field
- $\varphi \subset \mathbb{R}^n$ \equiv transition set
- $\rho: \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv reset map

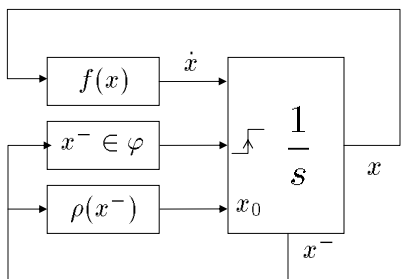
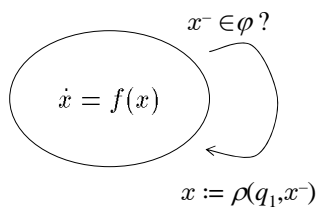


E.g., bouncing ball

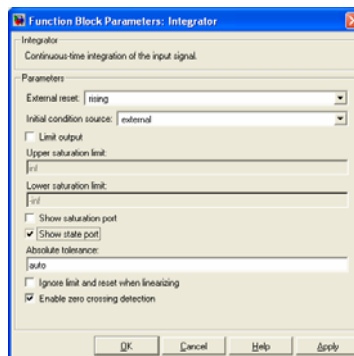


Simulation of ODEs with resets

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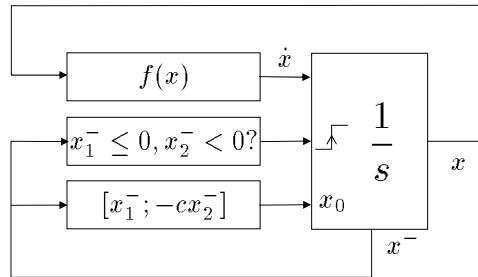
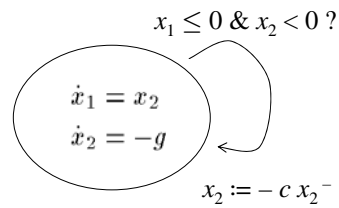


integrator with reset
(by default does zero-crossing
detection on reset input)

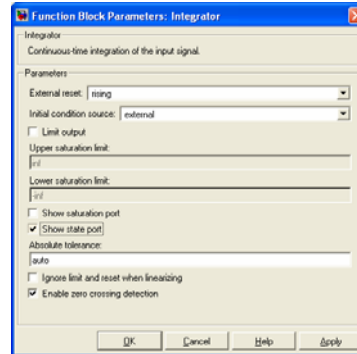


Example #1: Bouncing ball

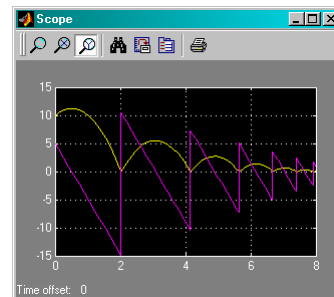
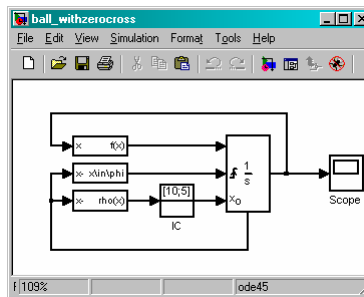
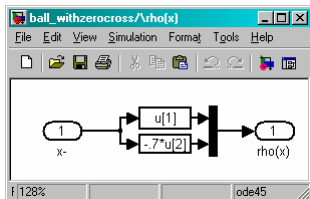
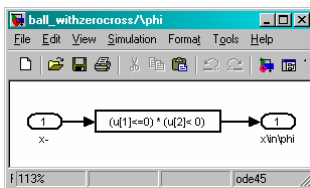
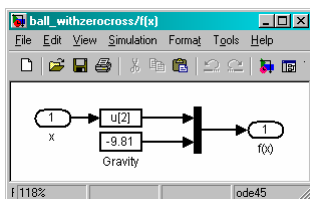
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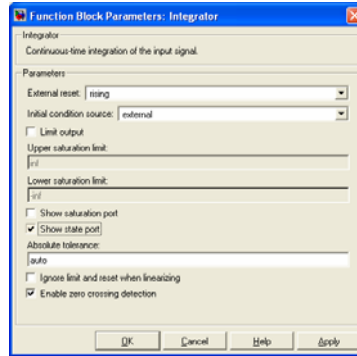
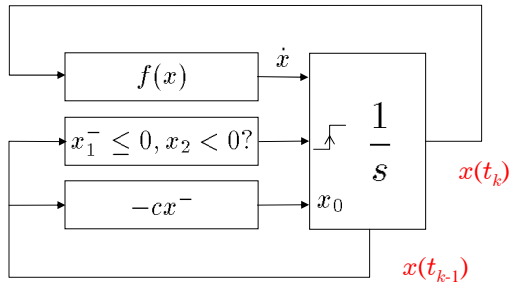
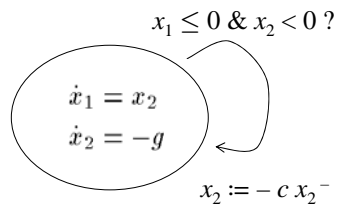
Example #1: Bouncing ball



ball_withzerocross.mdl

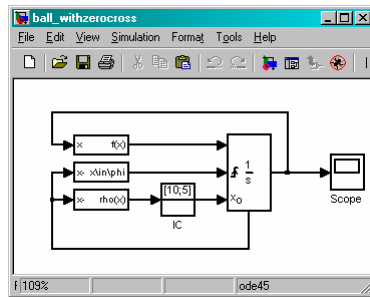
Simulation of ODEs with resets

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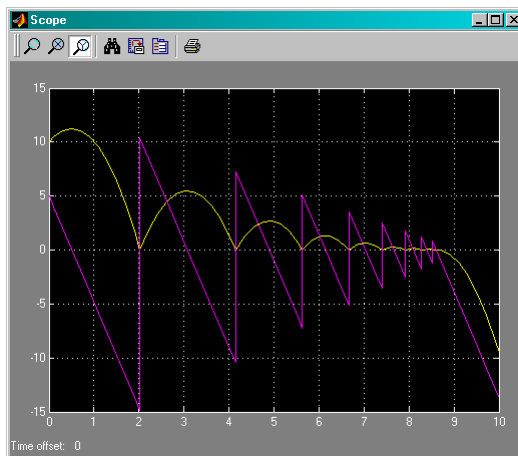


The “reset” pulse is not really instantaneous (may lead to problems for “Zeno” systems)

Example #1: Bouncing ball



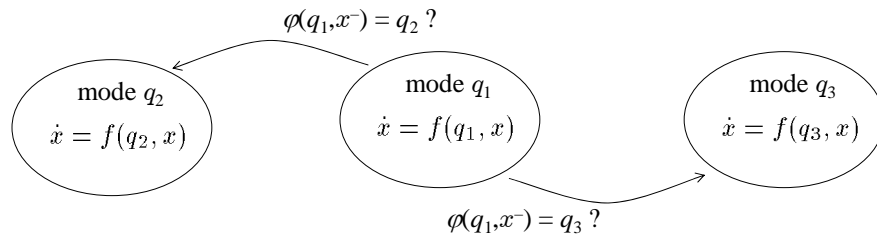
fails to catch the “rising edge” of the reset trigger and the ball falls



ball_withzerocross.mdl

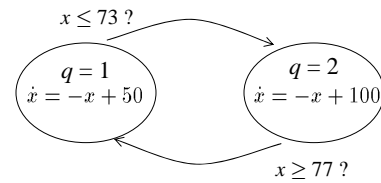
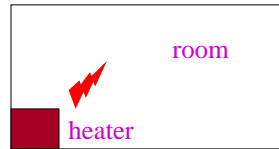
Simulation of hybrid automaton (no resets)

\mathcal{Q} \equiv set of discrete states
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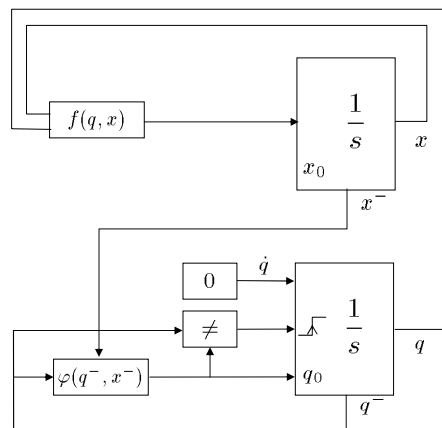
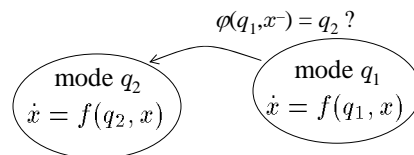
E.g., thermostat

$x \equiv$ mean temperature



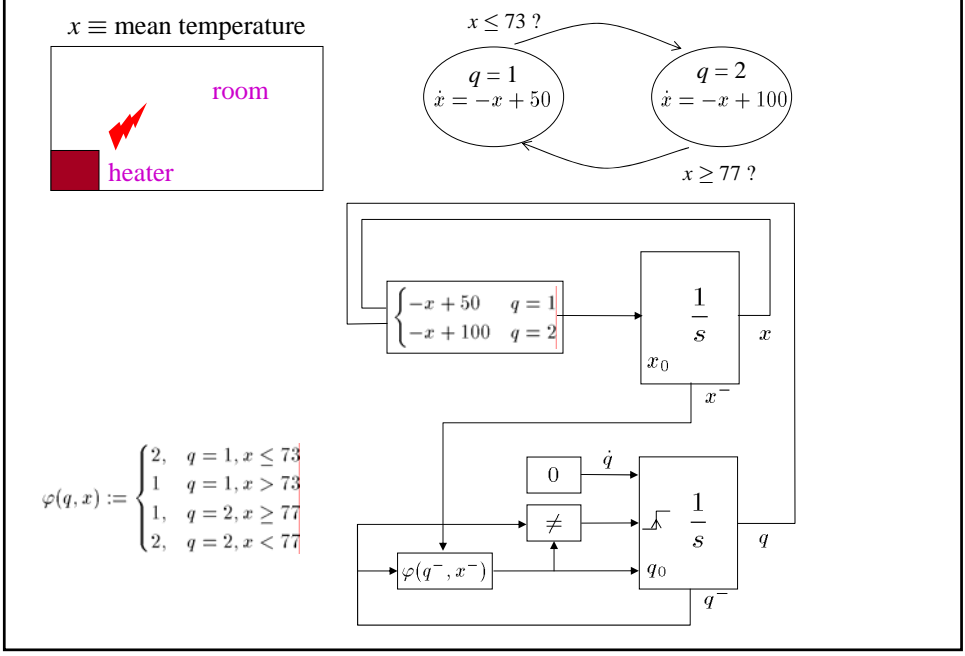
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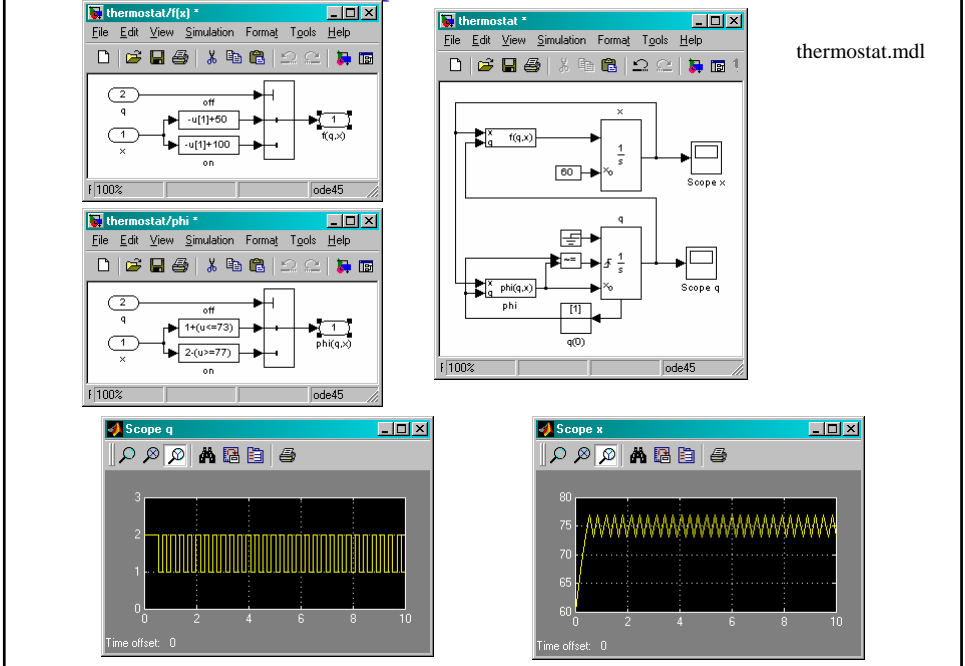


$\mathcal{Q} \subset \mathbb{R}$

Example #2: Thermostat

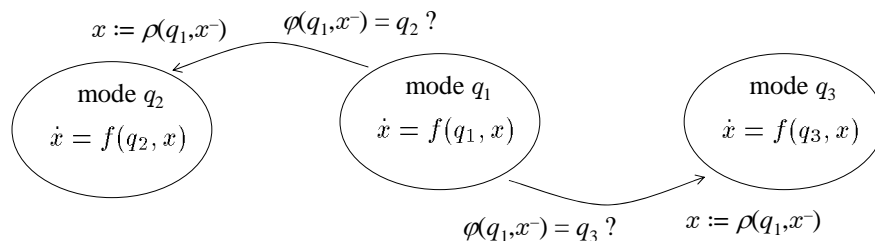


Example #2: Thermostat



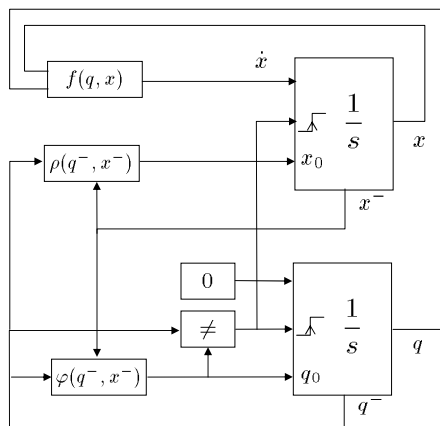
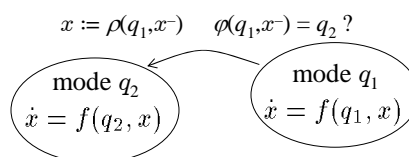
Simulation of hybrid automaton

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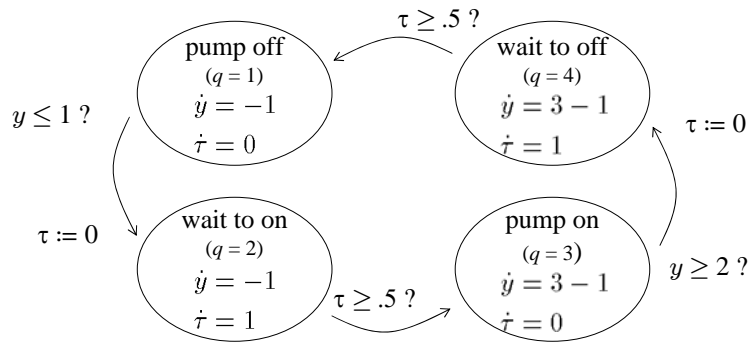
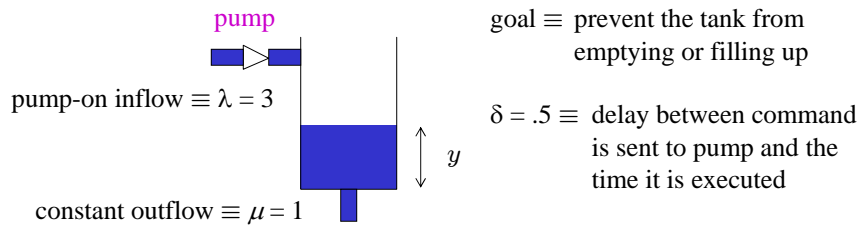


Simulation of hybrid automaton

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Example #5: Tank system

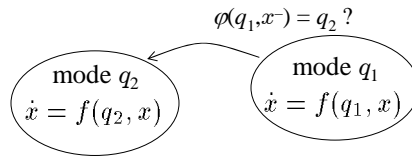


Example #5: Tank system

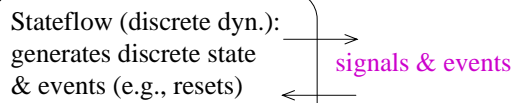
tank.mdl
 tank/f(q)
 tank/rho
 tank/phi
 Scope x
 Scope q

Stateflow

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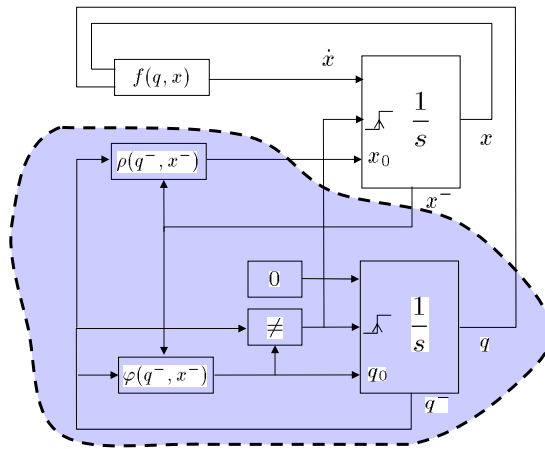
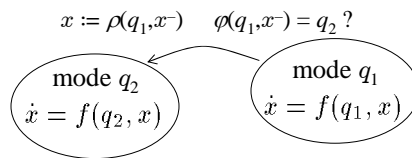


Simulink (continuous dynamics):
generates continuous state



Simulation of hybrid automaton

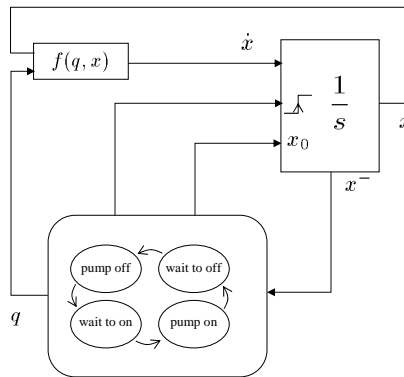
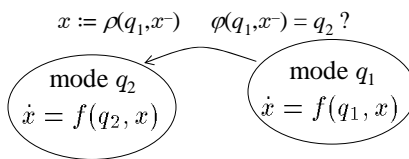
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replaced by
StateFlow
“chart”

Simulation of hybrid automaton

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 $\rho: Q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ \equiv reset map



Example #5: Tank system

tank_sf_withzerocross.mdl

Example #5: Tank system

Stateflow does not automatically do zero crossing detection, which must be done "manually"

tank_sf_nozero-cross.mdl

SHIFT

(Hybrid System Tool Interchange Format)

1. Simulation Language for Hybrid Automaton
2. Developed at UC Berkeley under the PATH (Partners for Advance Transit and Highways) project to simulate Automated Highway Systems
3. PATH provides freeware to compile SHIFT into a C-based simulator (<http://path.berkeley.edu/SHIFT/>)



Advantages:

1. Syntax matches very closely the hybrid automata formalism
2. Object-oriented and scalable (multiple copies of a automata can be created, connected, disconnected, and destroyed as the simulation runs)

Problems:

1. Currently the simulation engine is poor (limited integration algorithms, no zero-crossing detection)
2. Still in the development stages

A SHIFT program

```

type AutomataType { // declaration of automata types

  input ...          // declaration of input signals
  state ...          // declaration of internal states
  output ...         // declaration of output signals
  export ...         // declaration of events for synchronization

  discrete ...       // declaration of discrete modes
                      // & corresponding continuous dynamics
  transition ...     // definition of transition rules

  setup ...          // initializations
}

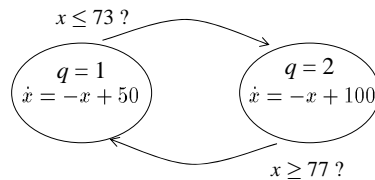
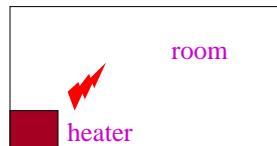
```

```
global ...          // declaration of global Automata
```

```
setup ...           // creation and initialization of global Automata
```

Example #2: Thermostat

$x \equiv$ mean temperature



thermostat.hs

```

type ThermostatType {
  state continuous number x := 50;
  discrete                    first declared is the initial mode
    heater_off { x' = -x + 50; },
    heater_on  { x' = -x + 100; };
  transition
    heater_off -> heater_on {} when { x <= 73 },
    heater_on  -> heater_off {} when { x >= 77 };
}
                                declaration of the
                                ThermostatType type

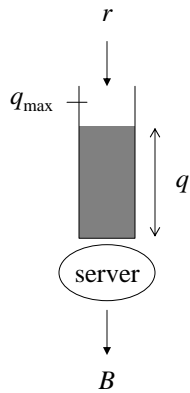
global ThermostatType thermostat;
                                declaration of a variable called
                                "thermostat" of type "ThermostatType"

setup
  define { thermostat := create(ThermostatType, x:= 60); }
                                creation (and initialization)
                                of the hybrid automaton

```

Example #7: Server system with congestion control

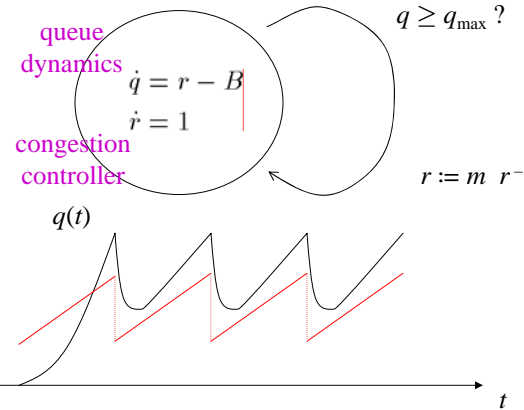
incoming rate



rate of service (bandwidth)

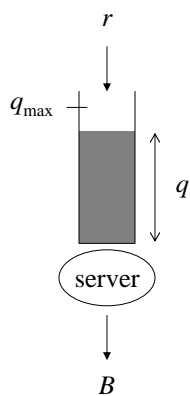
Additive increase/multiplicative decrease congestion control (AIMD):

- while $q < q_{\max}$ increase r linearly
- when q reaches q_{\max} instantaneously multiply r by $m \in (0,1)$

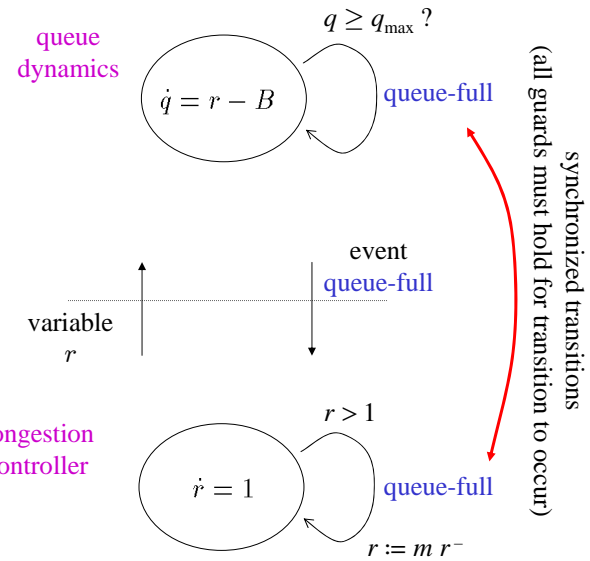


Example #7: Server system with congestion control

incoming rate

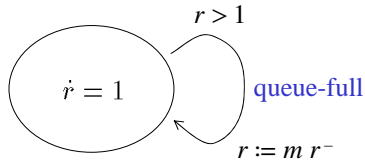


rate of service (bandwidth)



Example #7: Server system with congestion control

congestion
controller



congestion.hs

```

type CongestionControllerType {
  output continuous number r := 0;

  state number m := 0.5;           // multiplicative decrease (parameter)

  export queue_full;

  discrete
    additive_increase { r' = 1; }

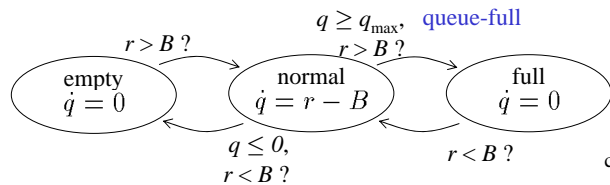
  transition
    additive_increase -> additive_increase { queue_full } when { r > 1 } do { r := m * r; }
}

```

synchronization
jump
reset
event
condition

Example #7: Server system with congestion control

queue
dynamics



congestion.hs

```

type QueueType {
  input continuous number r := 0;
  state continuous number q := 0;
  state CongestionControllerType controller; // controller to synchronize with

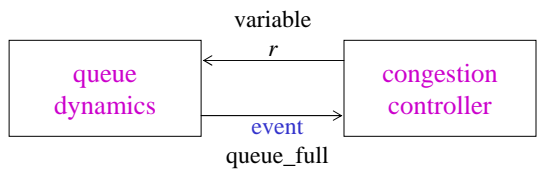
  state number B := 1; // Bandwidth (parameter)
  state number qmax := 10; // Maximum queue size (parameter)

  discrete
    empty_queue { q' = 0; }
    full_queue { q' = 0; }
    normal { q' = r - B; }

  transition
    empty_queue -> normal { } when { r > B },
    full_queue -> normal { } when { r < B };
    normal -> empty_queue { } when { q <= 0 and r <= B } do { q = 0; },
    normal -> full_queue { controller:queue_full } when { q >= Qmax and r > B } do { q = qmax; },
}

```

Example #7: Server system with congestion control



congestion.hs

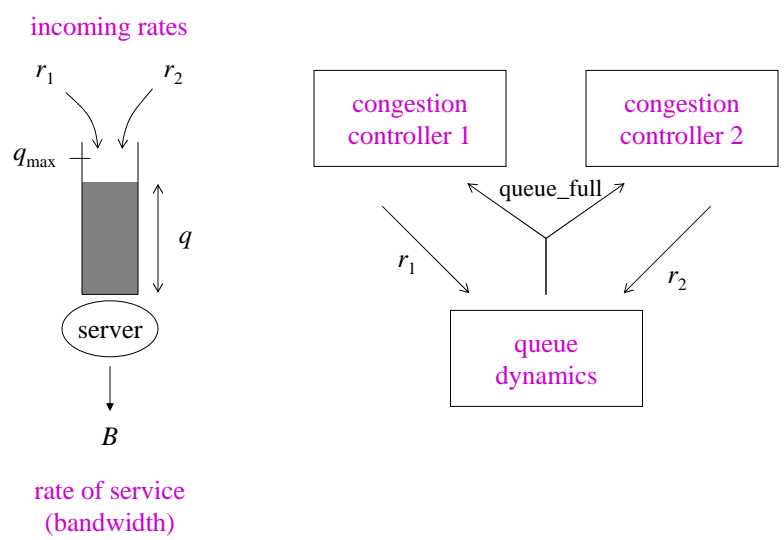
```

global QueueType queue;
global CongestionControllerType contr;
                                declaration of global variables

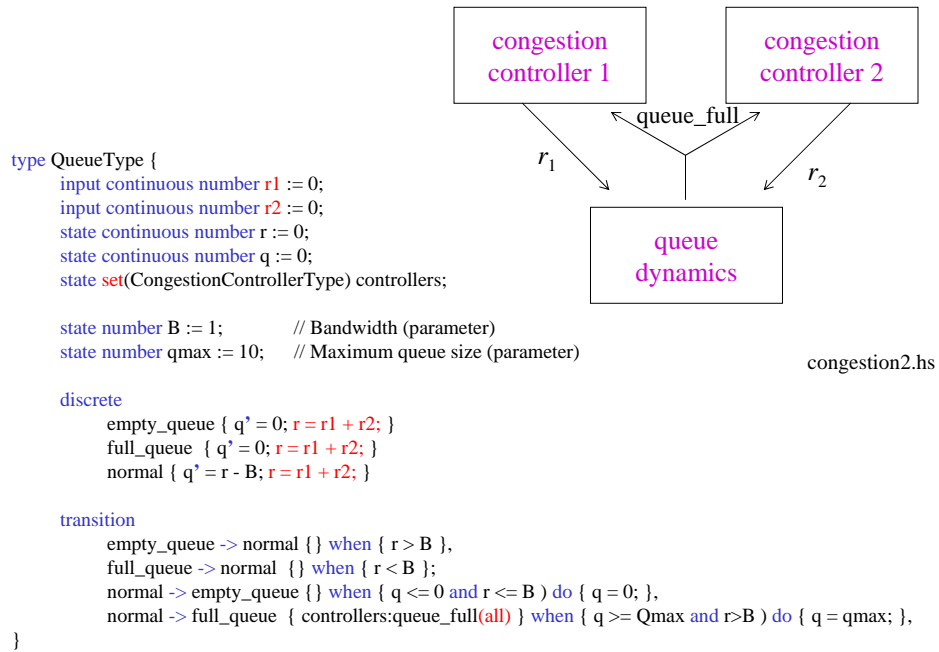
setup
  define {
    contr := create(CongestionControllerType);
    queue := create(QueueType, controller := contr);
                                creation (and initialization)
                                of the hybrid automata
  }
  connect {
    r(queue) <- r(contr);
                                inputs ← output connections
  }

```

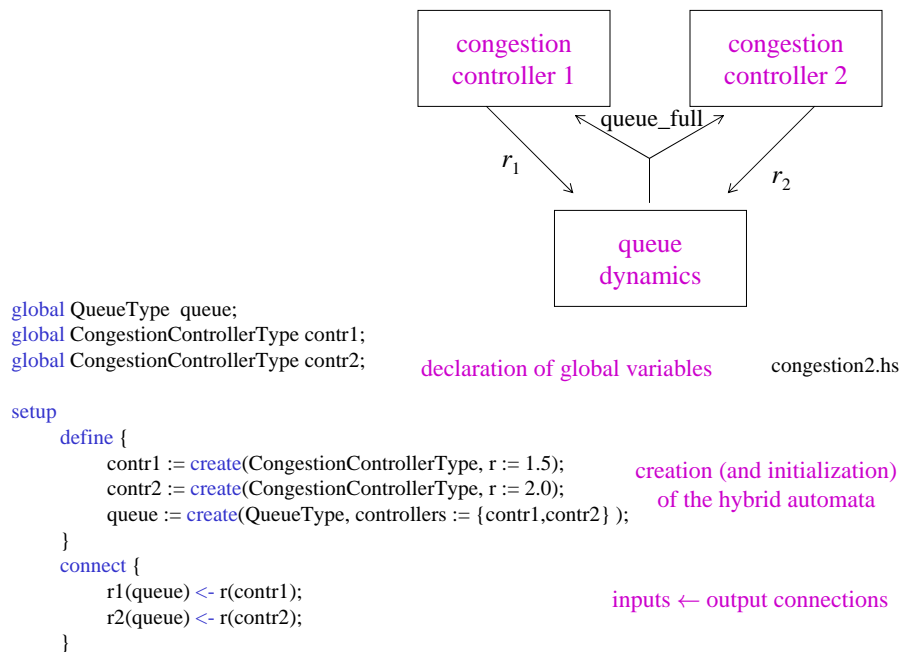
Example #10: Server with multiple congestion controllers



Example #10: Server with multiple congestion controllers

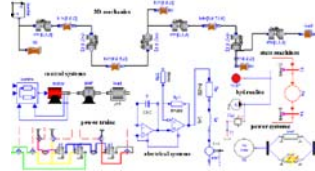


Example #10: Server with multiple congestion controllers



Modelica

1. Object-oriented language for modeling physical systems
2. Developed and promoted by the Modelica Association
(international non-profit, non-governmental organization based in Sweden)
<http://www.modelica.org/>
3. Dynasim AB sells Dymola, currently the best Modelica simulator
(interfaces with MATLAB and Simulink)



Advantages:

1. Object-oriented and scalable
2. Large number of component libraries available (electric circuits, mechanical system, thermo-hydraulics, power systems, robotics, petri-nets, etc.)
3. Numerically stable simulation engine
(allows differential algebraic equations, automatic zero-crossing detection)
4. Heavily used in industry, especially in Europe
5. UCSB has a site license!

Problems:

1. Not as widely known as MATLAB/Simulink

Modelica object

```
model ModelName
  // declarations of public variables (inputs and outputs) that
  // can be accessed using "ModelName.Variable name"
  TypeName VariableName;
  ...
protected
  // declarations of internal (hidden) variables (state)
  TypeName VariableName;
  ...
equation
  // algebraic and differential equations
  Expression = Expression;
  ...
end ModelName
```

Modelica 101

Predefined types

Real x, // x is a real number
y (start = 1.1, unit = "inches") "height";
// y is a "height," initialized with 1.1 "inches"

Integer n; // n is an integer

Boolean p; // p can be either **true** or **false**

Type modifiers

parameter Real B; // B does not change during the simulation but
// can be initialized with different values

constant Real PI; // PI is a fixed constant

discrete Real q; // q is a piecewise constant variable (discrete state)

flow Real r; // r is a "flow" variable
// (connection of flows follow conservation law,
// by convention positive means flow enters component)

Modelica 101

Predefined functions/variables for use in equation

der(x) // derivative of *Real* signal x

pre(x) // left-limit of *discrete* signal x

edge(x) // true when *discrete* variable x is discontinuous

time // simulation time

Commands for use in equation

x = y // equate two variable

x + z = y // (should be interpreted as equation and not assignment)

reinit(x, 2.1); // reset variable x to to 2.1

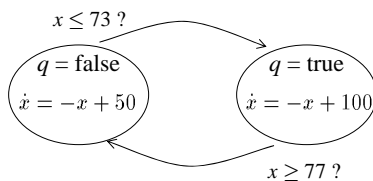
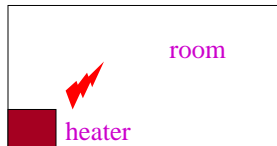
connect(x,y); // connects variables:

connect(x,z); // x = y = z (or x + y + z = 0 in case of flows)

when p then // execute command when p becomes true
command;
end when;

Example #2: Thermostat

$x \equiv$ mean temperature



model thermostat

Real x "Average temperature";
 Boolean q(start = false) "Heater state";

parameter Real xon = 73 "Turn-on temperature";
 parameter Real xoff = 77 "Turn-off temperature";

equation

```

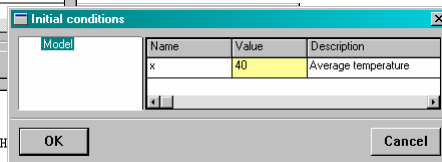
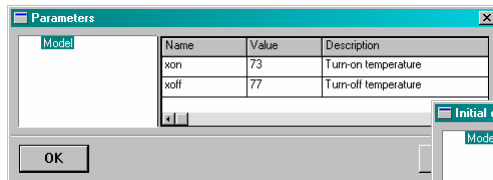
q = if not pre(q) and x <= xon then true // turn on
    else if pre(q) and x >= xoff then false // turn off
    else pre(q); // no change
der(x) = if q then 100-x
        else 50-x;
end thermostat;
  
```

thermostat.mo

Example #2: Thermostat

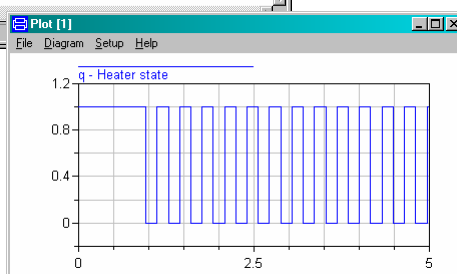
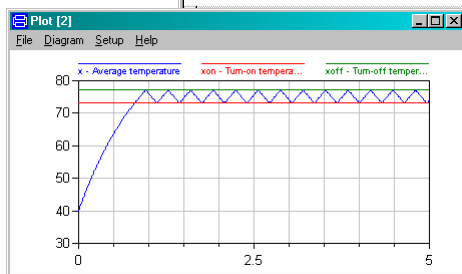
To get started:

setenv PATH /usr/local/dymola/bin:\$PATH
dymola5 thermostat.mo
 [documentation online & refs 15,16]

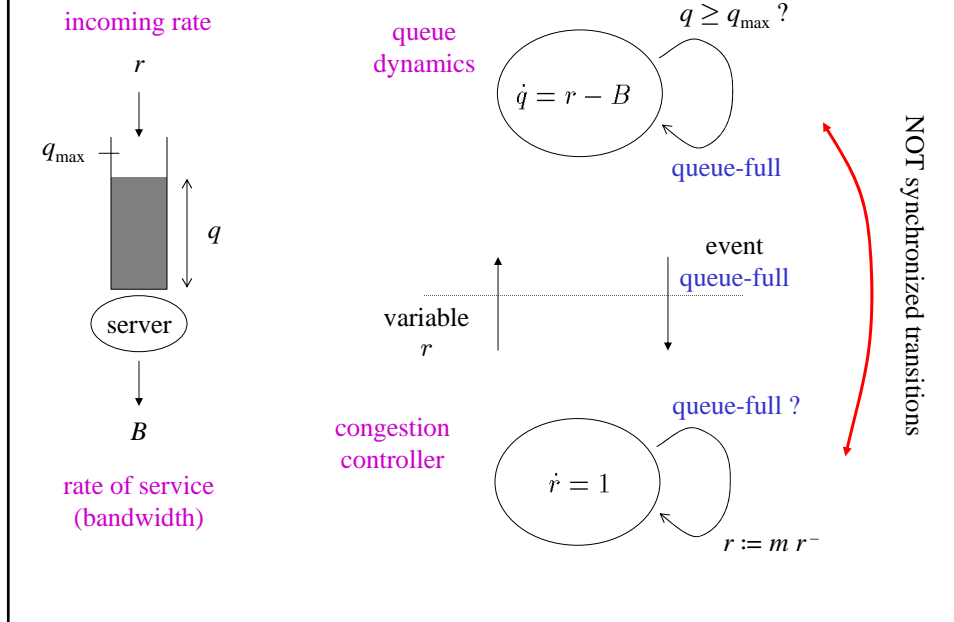


```

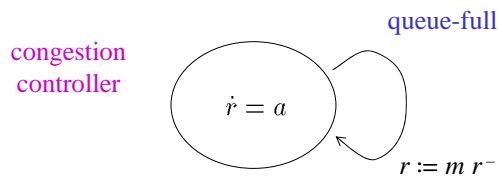
protected
Real x "Average temperature";
discrete Boolean q(start=false) "Heater state";
equation
q = if not pre(q) and x <= 73 then true else if pre(q) and x >= 77
    then false else pre(q);
der(x) = if q then -x + 100 else -x + 50;
end thermostat;
  
```



Example #7: Server system with congestion control



Example #7: Server system with congestion control



```

model Controller
  Real r(start=0)           "out-flow";
  discrete Boolean queue_full "Queue full";
  parameter Real a=.1       "Additive constant";
  parameter Real m=.5       "Multiplicative constant";

```

```

equation
  der(r) = a;
  when pre(queue_full) then
    reinit(r, r*m);
  end when;

```

```

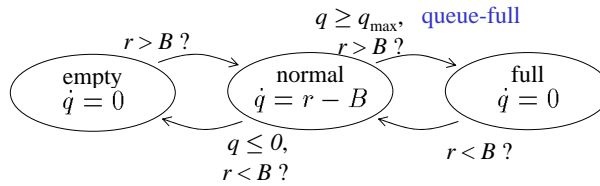
end Controller

```

congestion.mo

Example #7: Server system with congestion control

queue dynamics



model Queue

```

Real r "In-flow";
discrete Boolean queue_full(start=false) "Queue full";
parameter Real B=1 "Bandwidth";
parameter Real Qmax=1 "Max queue size";
  
```

protected

```

discrete Integer status(start=0) "Queue status"; // 0 empty, 1 normal, 2 full
Real q(start=0) "Queue size";
  
```

equation

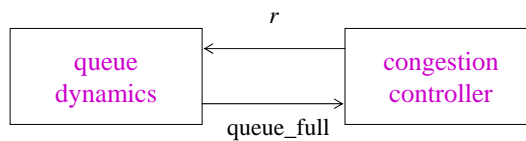
```

status = if pre(status) == 0 and r > B then 1 // no longer empty
         else if pre(status) == 2 and r < B then 1 // no longer full
         else if pre(status) == 1 and q <= 0 and r < B then 0 // became empty
         else if pre(status) == 1 and q >= Qmax and r > B then 2 // became full
         else pre(status); // no change
queue_full = pre(status) == 1 and status == 2;
der(q) = if status == 1 then r - B else 0;
  
```

end Queue

congestion.mo

Example #7: Server system with congestion control



model System

```

Queue queue;
Controller contr;
  
```

equation

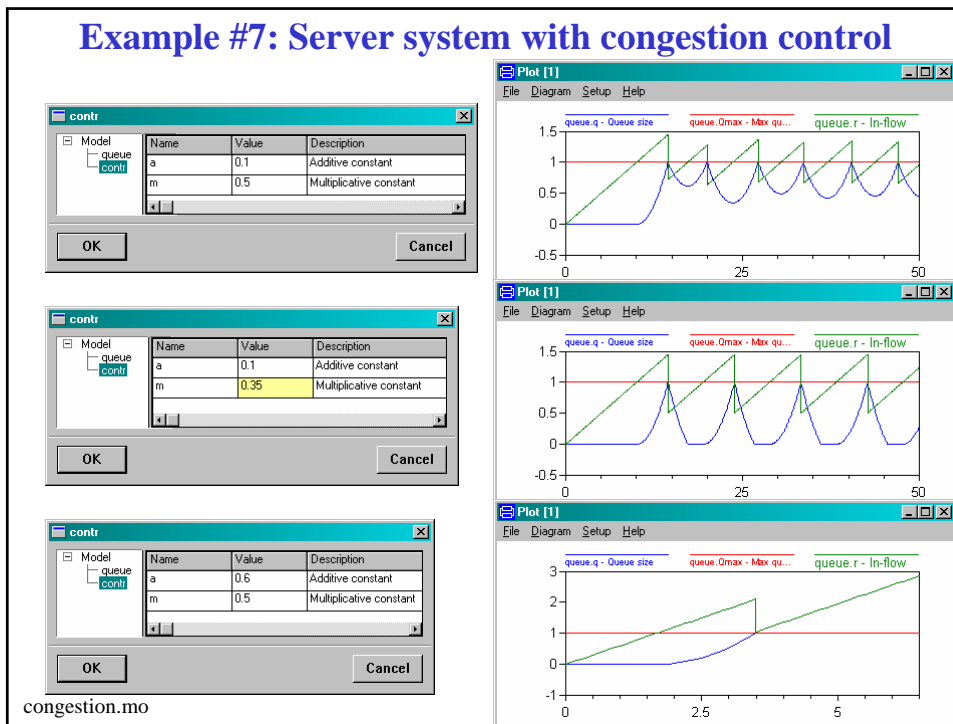
```

queue.r = contr.r;
queue.queue_full = contr.queue_full;
  
```

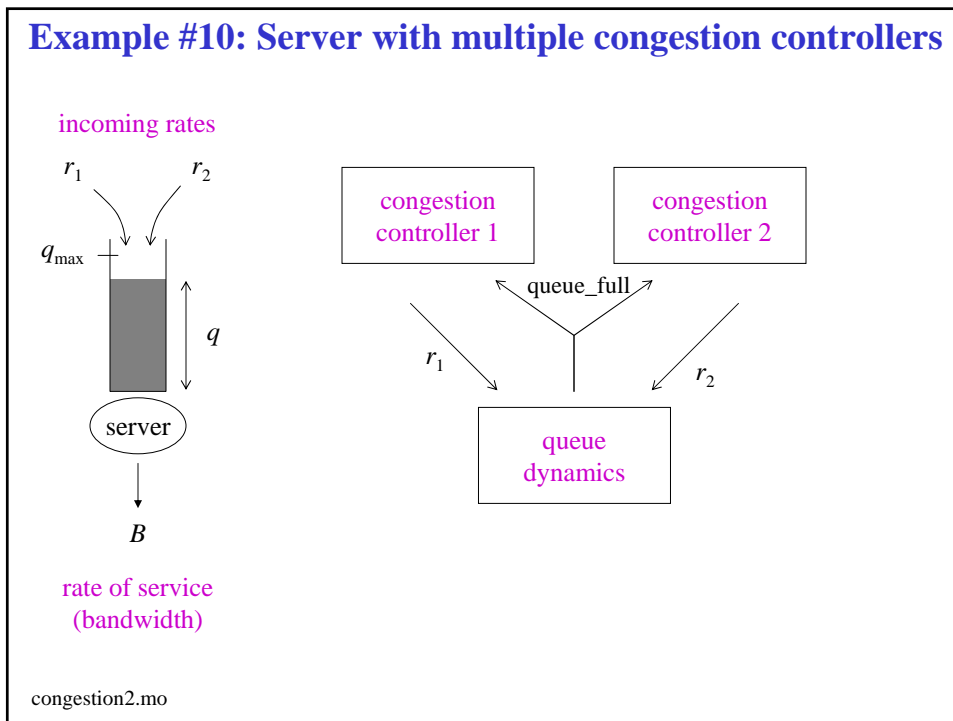
end System

congestion.mo

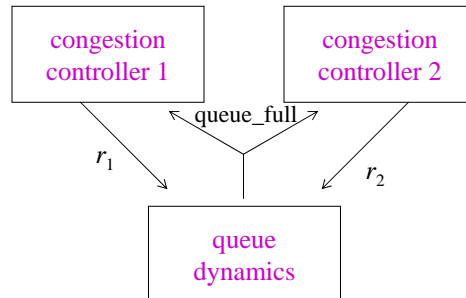
Example #7: Server system with congestion control



Example #10: Server with multiple congestion controllers



Example #10: Server with multiple congestion controllers



model System

```

Queue queue;
Controller contr1;
Controller contr2;
  
```

equation

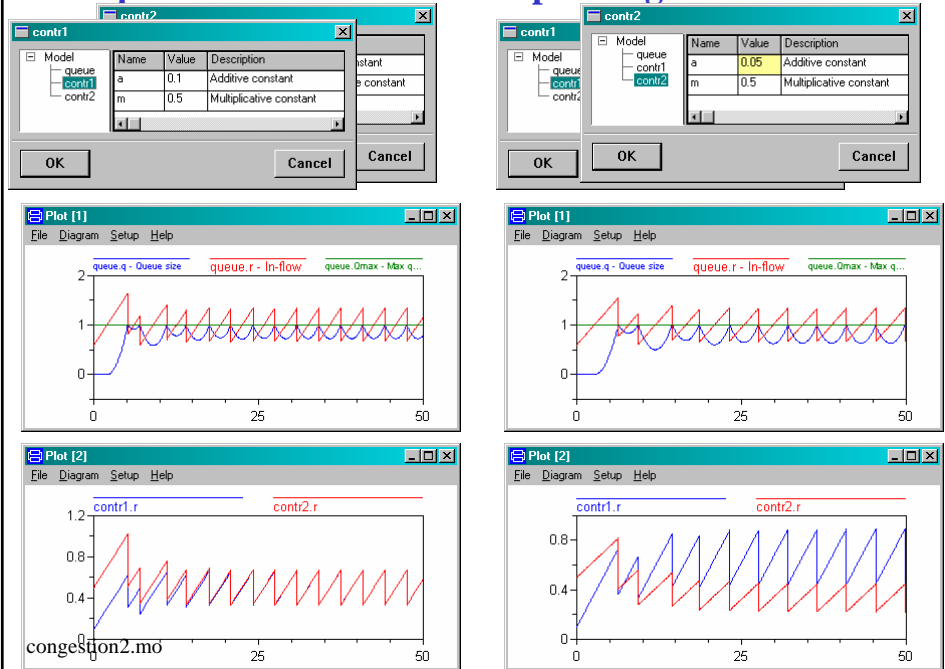
```

queue.r = contr1.r + contr2.r;
queue.queue_full = contr1.queue_full;
queue.queue_full = contr2.queue_full;
  
```

end System

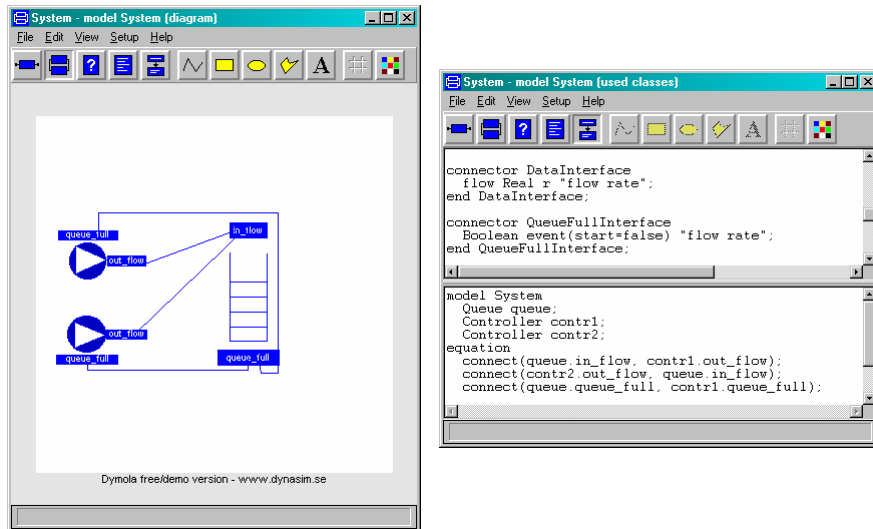
congestion2.mo

Example #10: Server with multiple congestion controllers



Example #10: Server with multiple congestion controllers

see congestion3.mo for the use of the `connect` command



congestion3.mo

Interfacing Dymola & MATLAB/Simulink

Sharing data

- Each time you simulate a Dymola model, a file "xxx.mat" is created with all the data. You can get this data into MATLAB with
`load xxx.mat`

Using Dymola models in Simulink

- Add dymola to the MATLAB path:
`path('/usr/local/dymola/mfiles',path)`
`path('/usr/local/dymola/mfiles/traj',path)`
- The Simulink block
`/usr/local/dymola/mfiles/DymolaBlockMaster.mdl`
can be used to insert a Dymola system into a Simulink diagram
Not sure if its working?

Next class...

Properties of hybrid systems

- Safety (reachability)
- Liveness
- Asymptotic properties (stability)