Hybrid Control and Switched Systems

Lecture #5
Properties of hybrid systems

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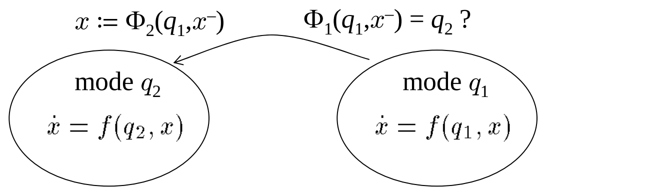
Summary

Properties of hybrid automata
- sequence properties
- safety properties
- liveness properties
- ensemble properties

Solution to a hybrid automaton

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Definition: A ***solution*** to the hybrid automaton is a pair of right-continuous signals

$$x : [0,\infty) \to \mathbb{R}^n \qquad q : [0,\infty) \to \mathcal{Q}$$

such that

1. x is piecewise differentiable & q is piecewise constant
2. on any interval (t_1, t_2) on which q is constant — continuous evolution

$$x(t) = x(t_1) + \int_{t_1}^{t} f\big(q(t_1), x(\tau)\big) d\tau \qquad \forall t \in [t_1, t_2)$$

3. $\big(q(t), x(t)\big) = \Phi\big(q^-(t), x^-(t)\big) \quad \forall t \geq 0$ — discrete transitions

Hybrid signals

Definition: A ***hybrid time trajectory*** is a (finite or infinite) sequence of closed intervals

$$\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i,\ \tau'_i = \tau_{i+1},\ i = 1,2, \dots \}$$

(if τ is finite the last interval may by open on the right)

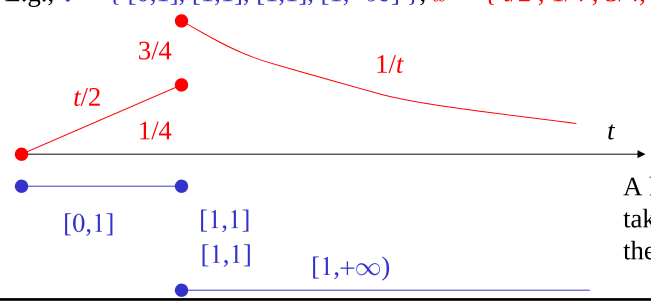
$\mathcal{T} \equiv$ set of hybrid time trajectories

Definition: For a given $\tau = \{ [\tau_i, \tau'_i] : \tau_i \leq \tau'_i,\ \tau_{i+1} = \tau'_i,\ i = 1,2, \dots \} \in \mathcal{T}$ a ***hybrid signal defined on*** $\boldsymbol{\tau}$ with values on $\mathcal{X}$ is a sequence of functions

$$x = \{ x_i : [\tau_i, \tau'_i] \to \mathcal{X} \quad i = 1,2, \dots \}$$

$x : \tau \to \mathcal{X} \equiv$ hybrid signal defined on τ with values on $\mathcal{X}$

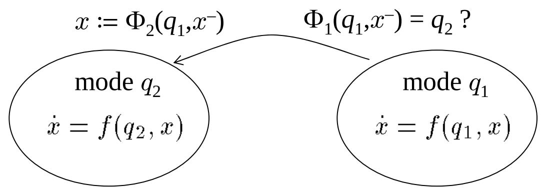
E.g., $\tau := \{ [0,1], [1,1], [1,1], [1,+\infty] \}$, $x := \{ t/2 , 1/4 , 3/4, 1/t \}$



A hybrid signal can take multiple values for the same time-instant

Execution of a hybrid automaton

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Definition: An ***execution*** of the hybrid automaton is a pair of hybrid signals

$$x : \tau \to \mathbb{R}^n \qquad q : \tau \to \mathcal{Q} \qquad \tau = \{ [\tau_i, \tau'_i] : i = 1, 2, \dots \} \in \mathcal{T}$$

such that

1. on any $[\tau_i, \tau'_i] \in \tau$, q_i is constant and — continuous evolution

$$x_i(t) = x_i(t_1) + \int_{\tau_i}^{t} f\big(q_i(\tau_i), x_i(\tau)\big) d\tau \qquad \forall t \in [\tau_i, \tau'_i]$$

2. $\big(q(\tau_{i+1}), x(\tau_{i+1})\big) = \Phi\big(q(\tau'_i), x(\tau'_i)\big)$ — discrete transitions

Sequence Properties (signals)

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals x:$[0, T) \to \mathbb{R}^n$, $T \in (0, \infty]$
$\mathcal{Q}_{\text{sig}} \equiv$ set of all piecewise constant signals q:$[0, T) \to \mathcal{Q}$, $T \in (0, \infty]$

Sequence property $\equiv p : \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}} \to \{\text{false}, \text{true}\}$

E.g.,

$$p(q, x) = \begin{cases} \text{true} & q(t) \in \{1, 3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of <u>signals</u> $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ ***satisfies*** p if $p(q, x) = \text{true}$

A hybrid <u>automaton</u> H ***satisfies*** p (write $H \vDash p$) if

$p(q, x) = \text{true}$, for every solution (q, x) of H

Sequence analysis $\equiv$ Given a hybrid automaton H and a sequence property p show that $H \vDash p$

When this is not the case, find a ***witness*** $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ such that $p(q, x) = \text{false}$

(in general for solution starting on a given set of initial states $\mathcal{H}_0 \subset \mathcal{Q} \times \mathbb{R}^n$)

Sequence Properties (hybrid signals)

$\mathcal{X}_{hsig} \equiv$ set of all hybrid signals $x = \{ x_i \}$
$\mathcal{Q}_{hsig} \equiv$ set of all hybrid signals $q = \{ q_i \}$

Sequence property $\equiv p : \mathcal{Q}_{hsig} \times \mathcal{X}_{hsig} \to \{\text{false,true}\}$

E.g.,

$$p(q,x) = \begin{cases} \text{true} & q(t) \in \{1,3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

short for:
$x_i(t) \geq x_j(t+3)\ \ \forall i : t \in [\tau_i, \tau_i']$,
$\forall j : t+3 \in [\tau_j, \tau_j']$

$$p(q,x) = \begin{cases} \text{true} & q_i \in \{1,3\},\ x_i(\tau_i') \leq x_{i+1}(\tau_{i+1}),\ \forall i \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of <u>signals</u> $(q, x) \in \mathcal{Q}_{hsig} \times \mathcal{X}_{hsig}$ ***satisfies*** p if $p(q, x) =$ true

A hybrid <u>automaton</u> H ***satisfies*** p (write $H \vDash p$) if
$p(q, x) =$ true, for every solution (q, x) of H

(in general for solution starting on a given set of initial states $\mathcal{H}_0 \subset \mathcal{Q} \times \mathbb{R}^n$)

Temporal logic formulas

Sequence properties are typically specified by temporal logic formulas

Propositional Logic (PL) primitives: $\neg\ \wedge\ \vee\ \Rightarrow\ \Leftrightarrow$

additional First-Order Logic (FOL) primitives: $\forall\ \exists$

additional Temporal Logic (TL) primitives: $\Box$ (always) $\Diamond$ (eventually)
$\circ$ (next time) μ (until)

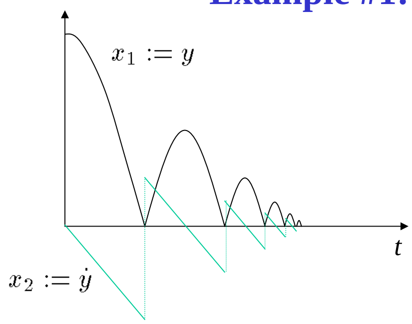
$p, q \equiv$ propositions with free time variable t

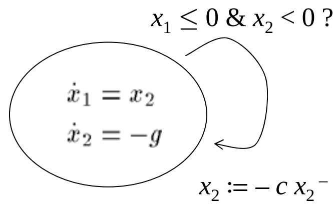
$(\Box\, p)(t_0) \Leftrightarrow \forall\, t \geq t_0,\ p(t)$
$(\Diamond\, p)(t_0) \Leftrightarrow \exists\, t \geq t_0,\ p(t)$
$(\circ\, p)(t_0) \Leftrightarrow p(t_0^+)$
$(\mu\, q,p)(t_0) \Leftrightarrow \exists\, t > t_0\ q(t) \wedge \forall\, \tau \in [t_0, t)\ p(\tau)$

Some possible combinations:

1. "responsiveness" (always, eventually)
   $(\Box \Diamond\, p)(t_0) \Leftrightarrow \forall\, t_1 \geq t_0,\ \exists\, t \geq t_1\ p(t)$ → $(\Diamond\, p)(t_1)$
2. "persistence" (eventually, always)
   $(\Diamond \Box\, p)(t_0) \Leftrightarrow \exists\, t_1 \geq t_0,\ \forall\, t \geq t_1\ p(t)$

Example #1: Bouncing ball





Assuming that $x_1(0) \geq 0$, the hybrid automaton satisfies:

$\Box \{ x_1 \geq 0 \}$ (short for $(\Box \{ x_1(t) \geq 0 \})(0)$)

$\Diamond \{ x_1 = 0 \}$

$\Box \Diamond \{ x_1 = 0 \}$

$\Diamond \Box \{ x_1 < 1 \}$

$(\Box\, p)(t_0) \Leftrightarrow \forall\, t \geq t_0,\ p(t)$

$(\Diamond\, p\,)(t_0) \Leftrightarrow \exists\, t \geq t_0,\ p(t)$

$(\Box \Diamond\, p)(t_0) \Leftrightarrow \forall\, t_1 \geq t_0,\ \exists\, t \geq t_1\ p(t)$

$(\Diamond \Box\, p)(t_0) \Leftrightarrow \exists\, t_1 \geq t_0,\ \forall\, t \geq t_1\ p(t)$

Safety properties

Given a signal $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$, $x^*:[0,T^*) \to \mathbb{R}^n$ is called a ***prefix*** to x
if $T^* \leq T$ & $x^*(t) = x(t)\ \forall\, t \in [0,T^*)$

safety property $\equiv$ a sequence property p that is:

1. *nonempty*, i.e., $\exists\, (q,x)$ such that $p(q,x) =$ true
2. *prefix closed*, i.e., given signals (q,x)
   $p(q, x) \Rightarrow p(q^*, x^*)$
   for every prefix (q^*, x^*) to (q, x)
3. *limit closed*, i.e., given an infinite sequence of signals
   (q_1,x_1) , (q_2,x_2), (q_3,x_3), etc.
   each element satisfying p such that
   (q_k,x_k) is a prefix to (q_{k+1},x_{k+1}) $\forall\, k$
   then $(q,x) := \lim_{k\to\infty} (q_k,x_k)$ also satisfies p

"Something bad never happens:"

1. nontrivial

2. a prefix to a good signal is always good

3. if something bad happens, it will happen in finite time

(Technical parenthesis)

Given a signal $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$, $x^*:[0,T^*) \to \mathbb{R}^n$ is called a ***prefix*** to x
if $T^* \leq T$ & $x^*(t) = x(t)$ $\forall\, t \in [0,T^*)$

safety property ≡ …

3. *limit closed*, i.e., given an infinite sequence of signals
   (q_1,x_1), (q_2,x_2), (q_3,x_3), etc.
   each element satisfying p such that
   (q_k,x_k) is a prefix to (q_{k+1},x_{k+1}) $\forall\, k$
   then $(q,x) := \lim_{k\to\infty} (q_k,x_k)$ also satisfies p

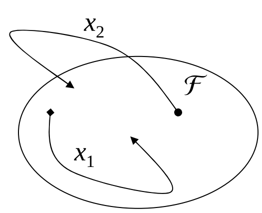
Limit in what sense?

Prefix induces a <u>relation</u> R in the set of signals $\mathcal{X}_{sig}$
$R := \{ (x^*,x) : x^* \text{ is a prefix to } x \}$
This relation is a <u>partial order</u> (for short we write $x^* \leq x$ when $(x^*,x) \in R$):
1. *reflexive*, i.e., $a \leq a$ $\forall\, a$
2. *antisymmetric*, i.e., $a \leq b$, $b \leq a \Rightarrow a = b$
3. *transitive*, i.e., $a \leq b$, $b \leq c \Rightarrow a \leq c$

<u>Limit</u> in the sense induced by the partial order: given $x_1 \leq x_2 \leq x_3 \leq \ldots$
$\lim_{k\to\infty} x_k = \sup \{ x_k : k \geq 1 \} =$ unique function x such that $x \geq x_k$ $\forall\, k$
& $x \leq y$ $\forall\, y$: $y \geq x_k$ $\forall\, k$ (def. of sup)

Examples

E.g., $p(q, x) = \Box\, \big(q(t),x(t)\big) \in \mathcal{F}$ where $\mathcal{F} \subset \mathcal{Q} \times \mathbb{R}^n$ is a nonempty set



x_1 satisfies p
x_2 does not

this is a safety property:
nonempty, prefix closed,
limit closed

Other safety properties:
$p(q, x) = x(t) \geq 0$ $\forall\, t$ (closed $\mathcal{F}$)
$p(q, x) = x(t) > 0$ $\forall\, t$ (open $\mathcal{F}$)

Nonsafety property:
$p(q, x) = \inf_t x(t) > 0$ (not of the form above; not limit closed, Why?)

Liveness properties

Given a signal $x:[0,T) \to \mathbb{R}^n$, $T\in(0,\infty]$, $x^*:[0,T^*) \to \mathbb{R}^n$ is called a ***prefix*** to x if $T^* \leq T$ & $x^*(t) = x(t)\ \forall\, t\in[0,T^*)$

liveness property $\equiv$ a sequence property p with the property that for every finite $(q^*, x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ there is some $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ such that:

1. (q^*, x^*) is a prefix to (q, x)
2. (q, x) satisfies p

"Something good will eventually happen:"
for any sequence there is a good continuation.

E.g., $p(q, x) = \Diamond\, (q(t),x(t))\in \mathcal{F}$ where $\mathcal{F} \subset \mathcal{Q}\times \mathbb{R}^n$ is a nonempty set

$p(q, x) = \Box\, \Diamond\, (q(t),x(t))\in \mathcal{F}$ (always, eventually: $\forall\, t_1\geq t_0,\ \exists\, t\geq t_1$)

$p(q, x) = \Diamond\, \Box\, (q(t),x(t))\in \mathcal{F}$ (eventually, always: $\exists\, t_1\geq t_0,\ \forall\, t\geq t_1$)

$p(q, x) = \exists\, L>0\ \Box\ \|x\| < L$ *what does it mean?*

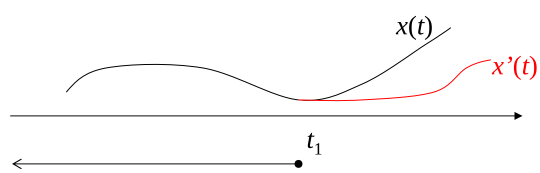
$p(q, x) = \forall\, \varepsilon>0\ \Diamond\, \Box\ \|x\| < \varepsilon$ *what does it mean?*

very rich class, more difficult to verify

Completeness of liveness/safety

Theorem 1: If p is both a liveness and a safety property then every $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ satisfies p, i.e., p is always true (trivial property)

By contradiction suppose there is a solution x that does not satisfy p



take arbitrary t_1, by liveness must have a "good" continuation x'
⇓
by safety must be "good" at least until t_1

by making $t_1 \to \infty$ we construct sequence of "good" signals that converges to x
⇓
by safety x must be "good"

Completeness of liveness/safety

Theorem 1: If p is both a liveness and a safety property then every $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ satisfies p, i.e., p is always true (trivial property)

Theorem 2: For every nonempty (not always false) sequence property p there is a safety property p_1 and a liveness property p_2 such that:
(q,x) satisfies p if and only if (q,x) satisfies both p_1 an p_2

Thus if we are able to verify safety and liveness properties we are able to verify any sequence property.

But sequence properties are not all we may be interested in…

"***ensemble properties***" $\equiv$ property of the whole family of solutions
e.g., stability (continuity with respect to initial conditions) is not a sequence property because by looking a each solution (q, x) individually we cannot decide if the system is stable. Much more on this later…

Can one find sequence properties that guarantee that the system is stable or unstable?

Next lecture…

Reachability
- transition systems
- reachability algorithm
- backward reachability algorithm
- invariance algorithm
- controller design based on backward reachability