Hybrid Control and Switched Systems

Lecture #7
Stability and convergence of ODEs

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Summary

Lyapunov stability of ODEs
- epsilon-delta and beta-function definitions
- Lyapunov's stability theorem
- LaSalle's invariance principle
- Stability of linear systems

Properties of hybrid systems

$\mathcal{X}_{sig}$ ≡ set of all piecewise continuous signals x:$[0,T) \to \mathbb{R}^n$, $T\in(0,\infty]$
$\mathcal{Q}_{sig}$ ≡ set of all piecewise constant signals q:$[0,T)\to \mathcal{Q}$, $T\in(0,\infty]$

Sequence property $\equiv p : \mathcal{Q}_{sig} \times \mathcal{X}_{sig} \to$ {false,true}

E.g.,

$$p(q,x) = \begin{cases} \text{true} & q(t) \in \{1,3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of <u>signals</u> $(q, x) \in \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$ ***satisfies*** p if $p(q, x)$ = true

A hybrid <u>automaton</u> H ***satisfies*** p (write $H \vDash p$) if

$p(q, x)$ = true, for every solution (q, x) of H

"***ensemble properties***" ≡ property of the whole family of solutions
(cannot be checked just by looking at isolated solutions)
e.g., continuity with respect to initial conditions…

Lyapunov stability (ODEs)

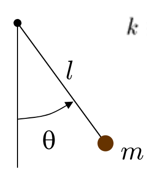
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

thus $x(t) = x_{eq}\ \forall\, t \geq 0$ is a solution to the ODE

E.g., pendulum equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



$k \equiv$ friction coefficient

$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$

two equilibrium points:
$x_1 = 0,\ x_2 = 0$ (down)
and $x_1 = \pi,\ x_2 = 0$ (up)

Lyapunov stability (ODEs)

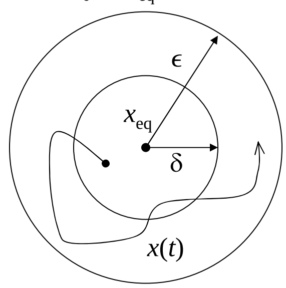
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

thus $x(t) = x_{eq}\ \forall\, t \geq 0$ is a solution to the ODE

Definition (ϵ–δ definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (***Lyapunov) stable*** if

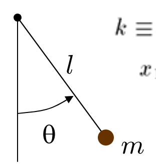
$$\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x(t_0) - x_{eq}\| \leq \delta \;\Rightarrow\; \|x(t) - x_{eq}\| \leq \epsilon\ \forall\, t \geq t_0 \geq 0$$



1. if the solution starts close to x_{eq} it will remain close to it forever
2. ϵ can be made arbitrarily small by choosing δ sufficiently small

Example #1: Pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$



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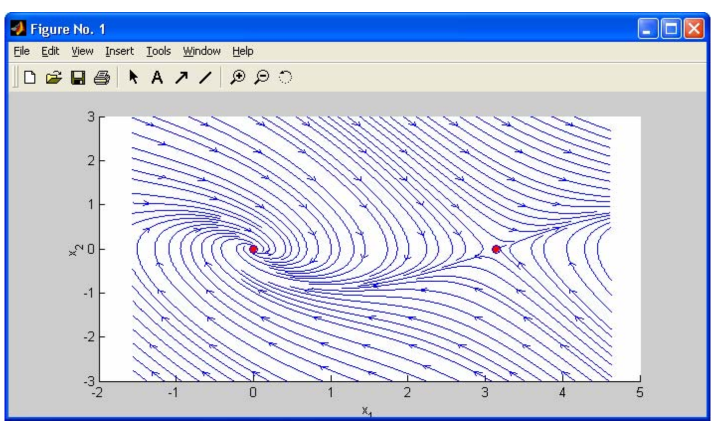


x_{eq}=(0,0)
stable

x_{eq}=(π,0)
unstable

pend.m

Lyapunov stability – continuity definition

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$\mathcal{X}_{sig} \equiv$ set of all piecewise continuous signals taking values in $\mathbb{R}^n$
Given a signal $x \in \mathcal{X}_{sig}$, $\|x\|_{sig} := \sup_{t \geq 0} \|x(t)\|$
signal norm

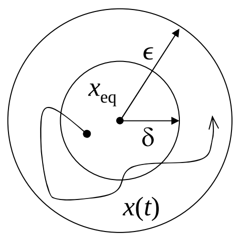
ODE can be seen as an operator

$$T : \mathbb{R}^n \to \mathcal{X}_{sig}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (***Lyapunov) stable*** if T is continuous at x_{eq}:

$$\forall \epsilon > 0 \; \exists \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \;\Rightarrow\; \underbrace{\|T(x_0) - T(x_{eq})\|_{sig} \leq \epsilon}_{\sup_{t \geq 0} \|x(t) - x_{eq}\| \leq \epsilon}$$



can be extended to nonequilibrium solutions

Stability of arbitrary solutions

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

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Given a signal $x \in \mathcal{X}_{sig}$, $\|x\|_{sig} := \sup_{t \geq 0} \|x(t)\|$
signal norm

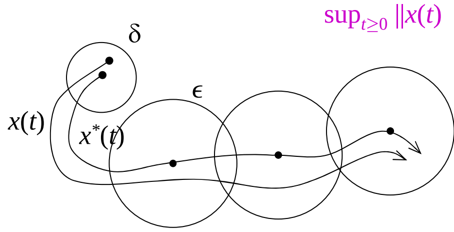
ODE can be seen as an operator

$$T : \mathbb{R}^n \to \mathcal{X}_{sig}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):
A solution $x^*:[0,T) \to \mathbb{R}^n$ is (***Lyapunov) stable*** if T is continuous at $x^*_0 := x^*(0)$, i.e.,

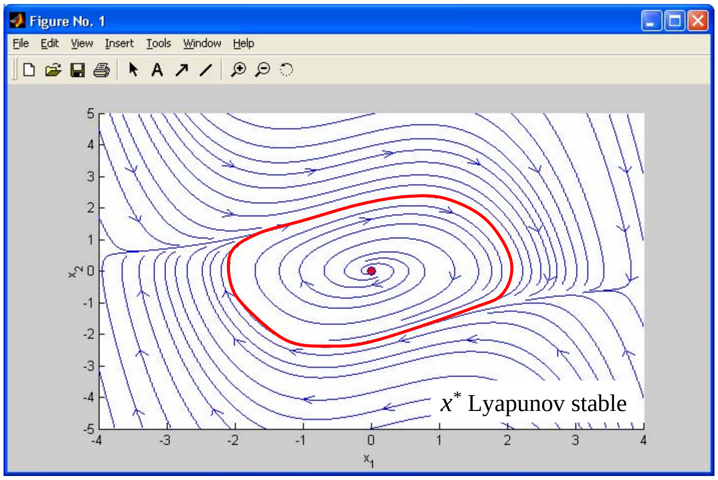
$$\forall \epsilon > 0 \; \exists \delta > 0 : \|x_0 - x^*_0\| \leq \delta \;\Rightarrow\; \underbrace{\|T(x_0) - T(x^*_0)\|_{sig} \leq \epsilon}_{\sup_{t \geq 0} \|x(t) - x^*(t)\| \leq \epsilon}$$



pend.m

Example #2: Van der Pol oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + .5(1 - x_1^2)x_2$$

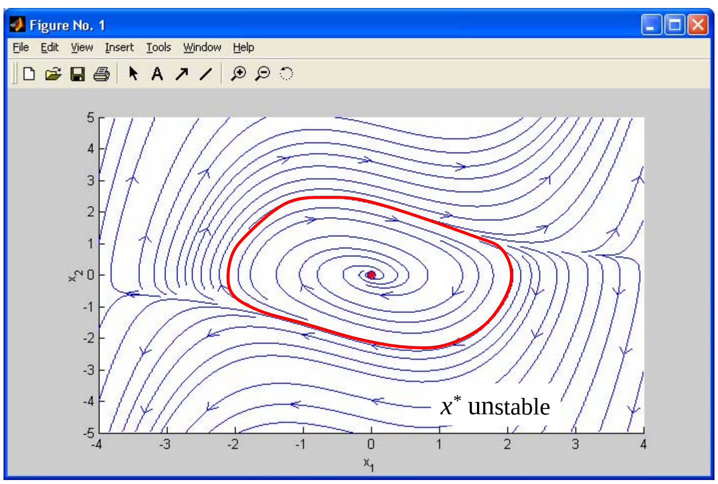


vdp.m

Stability of arbitrary solutions

E.g., Van der Pol oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - .5(1 - x_1^2)x_2$$



vdp.m

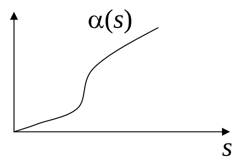
Lyapunov stability

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

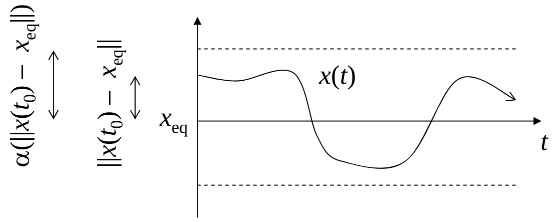
class $\mathcal{K} \equiv$ set of functions $\alpha:[0,\infty)\to[0,\infty)$ that are

1. continuous
2. strictly increasing
3. $\alpha(0)=0$



Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (***Lyapunov) stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \geq t_0 \geq 0, \; \|x(t_0) - x_{eq}\| \leq c$$



the function α can be constructed directly from the δ(ϵ) in the ϵ–δ (or continuity) definitions

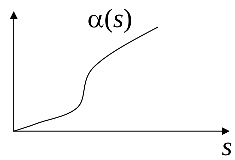
Asymptotic stability

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

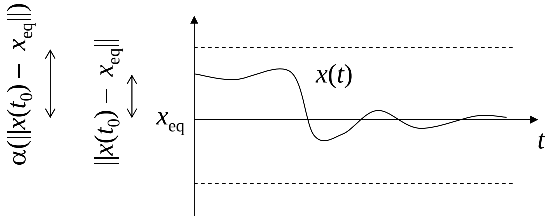
class $\mathcal{K} \equiv$ set of functions $\alpha:[0,\infty)\to[0,\infty)$ that are

1. continuous
2. strictly increasing
3. $\alpha(0)=0$



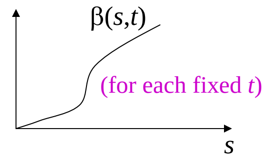
Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if it is Lyapunov stable and for every initial state the solution exists on $[0,\infty)$ and

$$x(t) \to x_{eq} \text{ as } t\to\infty.$$



Asymptotic stability

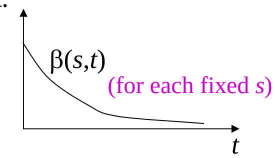
$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$



equilibrium point ≡ $x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

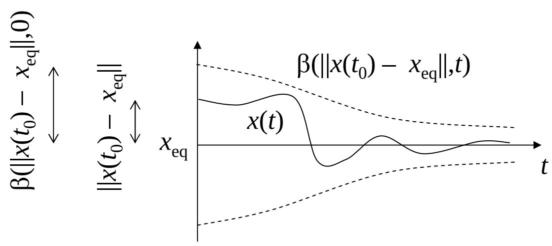
class $\mathcal{KL}$ ≡ set of functions $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ s.t.

1. for each fixed t, $\beta(\cdot,t) \in \mathcal{K}$
2. for each fixed s, $\beta(s,\cdot)$ is monotone decreasing and $\beta(s,t) \to 0$ as $t\to\infty$



Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if $\exists\ \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\ t \geq t_0 \geq 0$$



We have ***exponential stability*** when

$$\beta(s,t) = c\, e^{-\lambda t}\, s$$

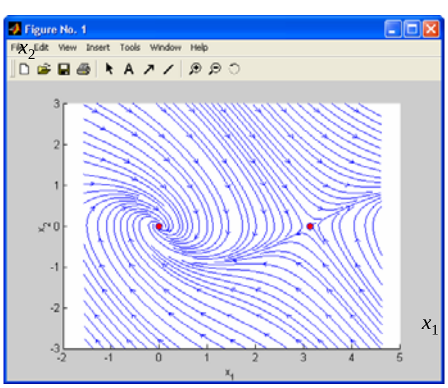
with $c,\lambda > 0$

linear in s and negative exponential in t

Example #1: Pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

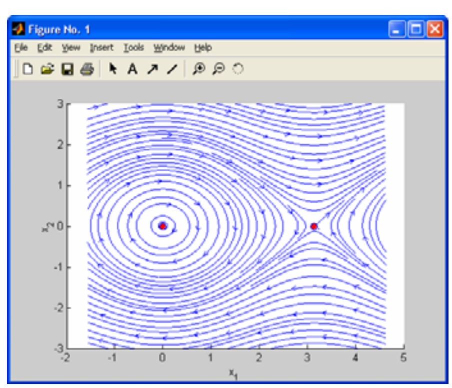
$k > 0$ (with friction)



$x_{eq}=(0,0)$ asymptotically stable

$x_{eq}=(\pi,0)$ unstable

$k = 0$ (no friction)



$x_{eq}=(0,0)$ stable but not asymptotically

$x_{eq}=(\pi,0)$ unstable

pend.m

Example #3: Butterfly

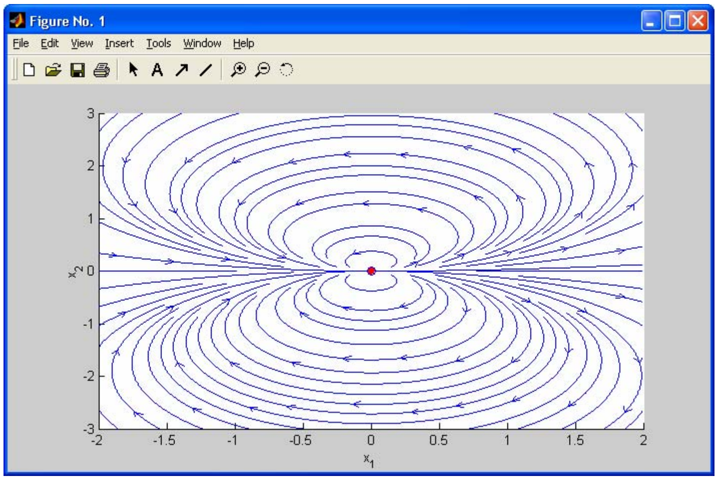
Why was Mr. Lyapunov so picky? Why shouldn't boundedness and convergence to zero suffice?

Convergence by itself does not imply stability, e.g.,

$$\dot{x}_1 = x_1^2 - x_2^2$$
$$\dot{x}_2 = 2x_1x_2$$

equilibrium point ≡ (0,0)

all solutions converge to zero but x_{eq}= (0,0) system is not stable



converge.m

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is (***Lyapunov) stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \geq t_0 \geq 0, \; \|x(t_0) - x_{eq}\| \leq c$$

Suppose we could show that $\|x(t) - x_{eq}\|$ always decreases along solutions to the ODE. Then

$$\|x(t) - x_{eq}\| \leq \|x(t_0) - x_{eq}\| \;\; \forall\, t \geq t_0 \geq 0$$

we could pick $\alpha(s) = s$ ⇒ Lyapunov stability

We can draw the same conclusion by using other measures of how far the solution is from x_{eq}:

$V: \mathbb{R}^n \to \mathbb{R}$ positive definite ≡ $V(x) \geq 0 \;\; \forall\, x \in \mathbb{R}^n$ with = 0 only for $x = 0$
$V: \mathbb{R}^n \to \mathbb{R}$ radially unbounded ≡ $x \to \infty \Rightarrow V(x) \to \infty$

$$V(x - x_{eq}) \quad \begin{cases} = 0 & x = x_{eq} \\ > 0 & x \neq x_{eq} \\ \to \infty & \|x - x_{eq}\| \to \infty \end{cases}$$

provides a measure of how far x is from x_{eq} (not necessarily a metric–may not satisfy triangular inequality)

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

V: $\mathbb{R}^n \to \mathbb{R}$ positive definite $\equiv V(x) \geq 0 \;\; \forall\, x \in \mathbb{R}^n$ with = 0 only for $x = 0$

$$V(x - x_{\text{eq}}) \quad \begin{cases} = 0 & x = x_{\text{eq}} \\ > 0 & x \neq x_{\text{eq}} \end{cases}$$

provides a measure of how far x is from x_{eq} (not necessarily a metric–may not satisfy triangular inequality)

Q: How to check if $V(x(t) - x_{eq})$ decreases along solutions?

$$\frac{\mathrm{d}}{\mathrm{d}t} V\big(x(t) - x_{\text{eq}}\big) = \frac{\partial V}{\partial x}\big(x(t) - x_{\text{eq}}\big)\dot{x}(t)$$
$$= \frac{\partial V}{\partial x}\big(x(t) - x_{\text{eq}}\big) f\big(x(t)\big)$$

gradient of V

A: $V(x(t) - x_{eq})$ will decrease if

$$\frac{\partial V}{\partial x}\big(z - x_{\text{eq}}\big) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

can be computed without actually computing $x(t)$ (i.e., solving the ODE)

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\, \alpha \in \mathcal{K}$:

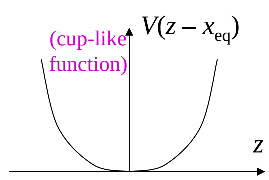
$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \;\; \forall\, t \geq t_0 \geq 0, \; \|x(t_0) - x_{eq}\| \leq c$$

Lyapunov function

Theorem (Lyapunov):
Suppose there exists a continuously differentiable, positive definite function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}\big(z - x_{\text{eq}}\big) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

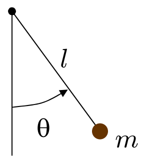
Then x_{eq} is a Lyapunov stable equilibrium.

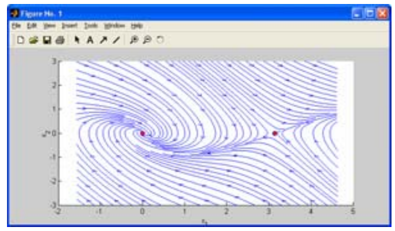


Why?
V non increasing $\Rightarrow V(x(t) - x_{eq}) \leq V(x(t_0) - x_{eq}) \;\forall\, t \geq t_0$
Thus, by making $x(t_0) - x_{eq}$ small we can make $V(x(t) - x_{eq})$ arbitrarily small $\forall\, t \geq t_0$
So, by making $x(t_0) - x_{eq}$ small we can make $x(t) - x_{eq}$ arbitrarily small $\forall\, t \geq t_0$
(we can actually compute α from V explicitly and take c = $+\infty$).

Example #1: Pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$
$$k \equiv \text{friction coefficient}$$
$$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$$





$$V(x) := \frac{g}{l}(1 - \cos x_1) + \frac{x_2^2}{2} \geq 0$$

positive definite because $V(x) = 0$ only for $x_1 = 2k\pi$ $k \in \mathbb{Z}$ & $x_2 = 0$ (all these points are really the same because x_1 is an angle)

For $x_{eq} = (0,0)$

$$\frac{\partial V}{\partial x}(x - x_{\text{eq}})f(x) = \begin{bmatrix} \frac{g}{l}\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$
$$= -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$$

Therefore x_{eq}=(0,0) is Lyapunov stable

pend.m

Example #1: Pendulum

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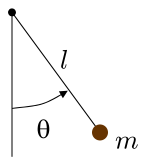

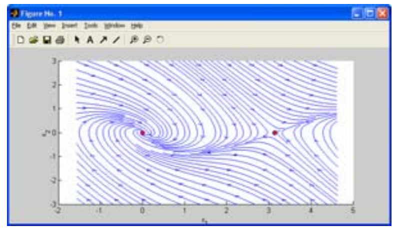

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For $x_{eq} = (\pi,0)$

$$\frac{\partial V}{\partial x}(x - x_{\text{eq}})f(x) = \begin{bmatrix} \frac{g}{l}\sin(x_1 - \pi) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$
$$= -\frac{2g}{l}x_2 \sin x_1 - \frac{k}{m}x_2^2 \gtreqless 0$$

Cannot conclude that x_{eq}=(π,0) is Lyapunov stable (in fact it is not!)

pend.m

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Definition (class $\mathcal{K}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(Lyapunov) stable*** if $\exists\ \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \ \ \forall\ t \geq t_0 \geq 0,\ \|x(t_0) - x_{eq}\| \leq c$$

Theorem (Lyapunov):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if = 0 only for $z = x_{eq}$ then x_{eq} is a (globally) asymptotically stable equilibrium.

Why?
V can only stop decreasing when $x(t)$ reaches x_{eq}
but V must stop decreasing because it cannot become negative
Thus, $x(t)$ must converge to x_{eq}

Lyapunov's stability theorem

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

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Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, if = 0 only for $z = x_{eq}$ then x_{eq} is a (globally) asymptotically stable equilibrium.

What if

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) = 0 \qquad \forall z \in \mathbb{R}^n$$

for other z then x_{eq} ? Can we still claim some form of convergence?

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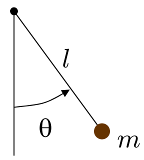
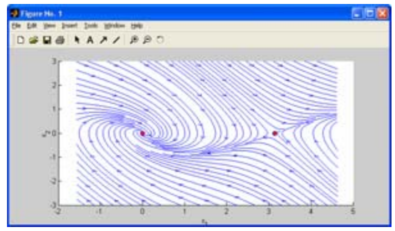
$k \equiv$ friction coefficient

$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$





$$V(x) := \frac{g}{l}(1-\cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (0,0)$

$$\frac{\partial V}{\partial x}(x - x_{\text{eq}})f(x) = \begin{bmatrix} \frac{g}{l}\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$
$$= -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$$

not strict for ($x_1 \neq 0$, $x_2=0$!)

pend.m

LaSalle's Invariance Principle

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set $\equiv x(t_0) \in M \Rightarrow x(t) \in M \; \forall\, t \geq t_0$

(in the context of hybrid systems: Reach(M) $\subset M$…)

Theorem (LaSalle Invariance Principle):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}})f(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{ z \in \mathbb{R}^n : W(z) = 0 \}$$

Note that:

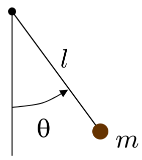
1. When $W(z) = 0$ only for $z = x_{eq}$ then $E = \{x_{eq}\}$.
   Since $M \subset E$, $M = \{x_{eq}\}$ and therefore $x(t) \to x_{eq} \Rightarrow$ asympt. stability
2. Even when E is larger then $\{x_{eq}\}$ we often have $M = \{x_{eq}\}$ and can conclude asymptotic stability.

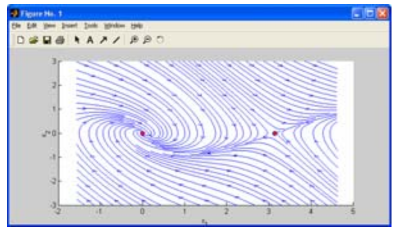
} Lyapunov theorem

Example #1: Pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$
$k \equiv$ friction coefficient

$x_1 \equiv \theta \qquad x_2 \equiv \dot{\theta}$





$$V(x) := \frac{g}{l}(1-\cos x_1) + \frac{x_2^2}{2} \geq 0$$

For $x_{eq} = (0,0)$ $\quad \frac{\partial V}{\partial x}(x - x_{eq}) f(x) = -\frac{k}{m}x_2^2 \leq 0 \qquad \forall x \in \mathbb{R}^n$

$E := \{ (x_1,x_2): x_1 \in \mathbb{R} , x_2=0\}$

Inside E, the ODE becomes

$$\dot{x}_1 = x_2 = 0$$
$$0 = \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 = -\frac{g}{l}\sin x_1$$

} define set M for which system remains inside E

Therefore x converges to $M := \{ (x_1,x_2): x_1 = k\,\pi \in \mathbb{Z} , x_2=0\}$

However, the equilibrium point $x_{eq}=(0,0)$ is not (globally) asymptotically stable because if the system starts, e.g., at $(\pi,0)$ it remains there forever.

pend.m

Linear systems

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$$

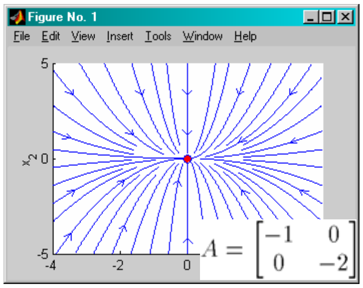
Solution to a linear ODE:

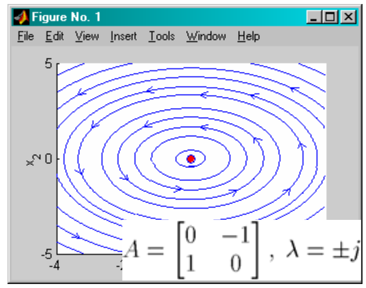
$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!}A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is

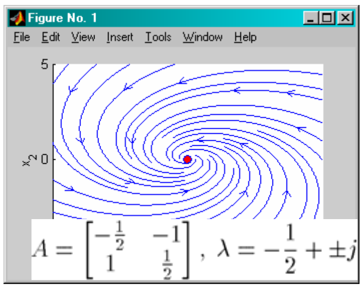
1. Lyapunov stable if and only if all eigenvalues of A have negative or zero real parts and for each eigenvalue with zero real part there is an independent eigenvector.
2. Asymptotically stable if and only if all eigenvalues of A have negative real parts. In this case the origin is actually exponentially stable

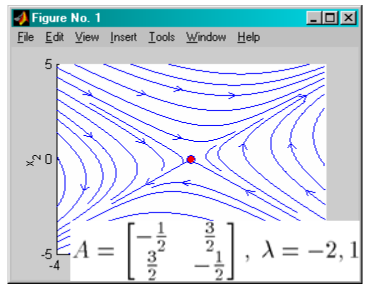
Linear systems





linear.m





Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)} x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A^k$$

Theorem: The origin $x_{eq} = 0$ is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix Q the equation

$$A' P + P A = -Q$$

Lyapunov equation

has a unique solutions P that is symmetric and positive definite

Recall: given a symmetric matrix P

P is positive definite $\equiv$ all eigenvalues are positive

P positive definite $\Rightarrow x' \mathrm{P}\, x > 0 \ \forall\, x \neq 0$

P is positive semi-definite $\equiv$ all eigenvalues are positive or zero

P positive semi-definite $\Rightarrow x' \mathrm{P}\, x \geq 0 \ \forall\, x$

Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!}A^k$$

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Lyapunov equation

has a unique solutions P that is symmetric and positive definite

Why?

1. asympt. stable ⇒ P exists and is unique (constructive proof)

$$P := \lim_{T\to\infty} \int_0^T e^{A'\tau} Q e^{A\tau} d\tau$$

$$= \lim_{T\to\infty} \int_0^T e^{A'(T-s)} Q e^{A(T-s)} ds$$

A is asympt. stable ⇒ e^{At} decreases to zero exponentiall fast ⇒ P is well defined (limit exists and is finite)

change of integration variable $\tau = T - s$

Lyapunov equation

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$$

Solution to a linear ODE:

$$x(t) = e^{A(t-t_0)}x(t_0) \qquad t \geq t_0 \qquad\qquad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!}A^k$$

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Lyapunov equation

has a unique solutions P that is symmetric and positive definite

Why?

2. P exists ⇒ asymp. stable

Consider the quadratic Lyapunov equation: $V(x) = x' P x$

V is positive definite & radially unbounded because P is positive definite

V is continuously differentiable: $\frac{\partial V}{\partial x}(x) = 2x'P$

$$\frac{\partial V}{\partial x}(x)Ax = x'(A'P + PA)x = -x'Qx < 0 \quad \forall x \neq 0$$

thus system is asymptotically stable by Lyapunov Theorem

Next lecture…

Lyapunov stability of hybrid systems