Hybrid Control and Switched Systems

Lecture #8
Stability and convergence of hybrid systems

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Summary

Lyapunov stability of hybrid systems

Properties of hybrid systems

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$
$\mathcal{Q}_{\text{sig}} \equiv$ set of all piecewise constant signals $q:[0,T) \to \mathcal{Q}$, $T \in (0,\infty]$

Sequence property $\equiv p : \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}} \to \{\text{false,true}\}$

E.g.,

$$p(q,x) = \begin{cases} \text{true} & q(t) \in \{1,3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of signals $(q, x) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ ***satisfies*** p if $p(q, x) = \text{true}$

A hybrid automaton H ***satisfies*** p (write $H \vDash p$) if

$p(q, x) = \text{true}$, for every solution (q, x) of H

"***ensemble properties***" $\equiv$ property of the whole family of solutions
(cannot be checked just by looking at isolated solutions)
e.g., continuity with respect to initial conditions…

Lyapunov stability of ODEs (recall)

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals taking values in $\mathbb{R}^n$
Given a signal $x \in \mathcal{X}_{\text{sig}}$, $\|x\|_{\text{sig}} := \sup_{t \geq 0} \|x(t)\|$

signal norm

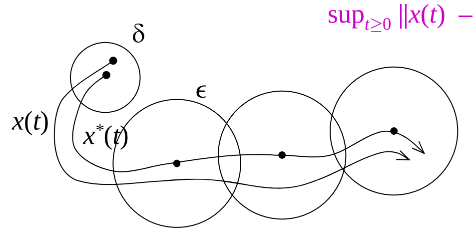
ODE can be seen as an operator

$$T : \mathbb{R}^n \to \mathcal{X}_{\text{sig}}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):
A solution x^* is (***Lyapunov) stable*** if T is continuous at $x^*_0 := x^*(0)$, i.e.,

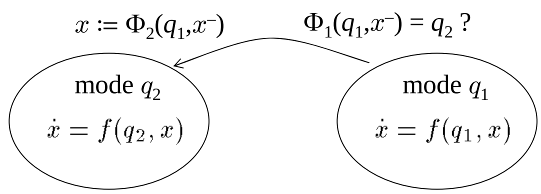
$$\forall \epsilon > 0\ \exists \delta > 0 : \|x_0 - x^*_0\| \leq \delta \ \Rightarrow\ \underbrace{\|T(x_0) - T(x^*_0)\|_{\text{sig}} \leq \epsilon}_{\sup_{t \geq 0} \|x(t)\ -\ x^*(t)\| \leq \epsilon}$$



pend.m

Lyapunov stability of hybrid systems

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



$\mathcal{X}_{\text{sig}} \equiv$ set of all piecewise continuous signals $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$
$\mathcal{Q}_{\text{sig}} \equiv$ set of all piecewise constant signals $q:[0,T) \to \mathcal{Q}$, $T \in (0,\infty]$

Hybrid automaton can be seen as an operator

$$\mathrm{T} : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Definition (continuity definition):
A solution (q^*, x^*) is (***Lyapunov) stable*** if T is continuous at $(q^*(0), x^*(0))$.

To make sense of *continuity* we need ways to measure "distances" in $\mathcal{Q} \times \mathbb{R}^n$ and $\mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$

Lyapunov stability of hybrid systems

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

A few possible "metrics"

$$d_{\text{disc}}\big((q_1,x_1),(q_2,x_2)\big) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 \\ +\infty & \text{otherwise} \end{cases}$$

one cares very much about the discrete states matching

$$d_{\text{cont}}\big((q_1,x_1),(q_2,x_2)\big) := \|x_1 - x_2\|$$

one does not care at all about the discrete states matching

Definition (continuity definition):
A solution (q^*, x^*) is (***Lyapunov) stable*** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e.,

$$\forall \epsilon > 0\ \exists \delta > 0 : d_0\big(\,(q^*(0), x^*(0))\,,\,(q(0), x(0))\,\big) \le \delta$$

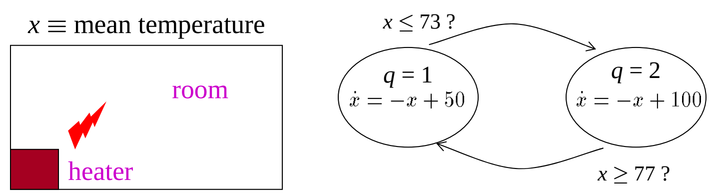
$$\Downarrow$$

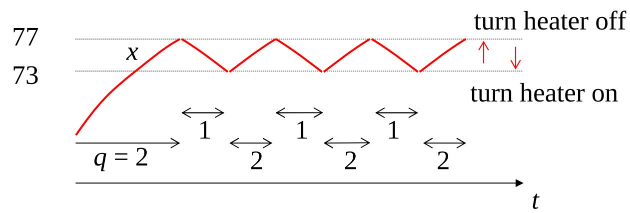
$$\sup_{t \ge 0} d_T\big(\,(q^*(t), x^*(t))\,,\,(q(t), x(t))\,\big) \le \epsilon$$

1. If the solution starts close to (q^*, x^*) (with respect to the metric d_0) it will remain close to it forever (with respect to the metric d_T)
2. ϵ can be made arbitrarily small by choosing δ sufficiently small

Note: may actually not be metrics on $\mathcal{Q} \times \mathbb{R}^n$ because one may want "zero-distance" between points. However, still define a topology on $\mathcal{Q} \times \mathbb{R}^n$, which is what is really needed to make sense of continuity. See "alternative" lecture #8 for more…

Example #2: Thermostat

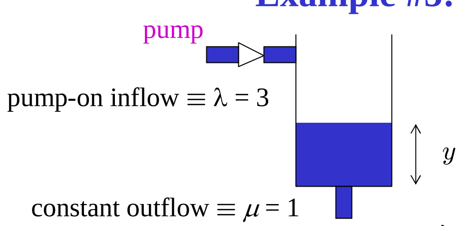




no trajectory is stable	some trajectories are stable others unstable	all trajectories are stable
for d_{disc} "metric" both on initial cond. and trajectory ($d_0 = d_T = d_{disc}$)	for d_{cont} "metric" both on initial cond. and trajectory ($d_0 = d_T = d_{cont}$)	for $d_0 = d_{disc}$ and $d_T = d_{cont}$

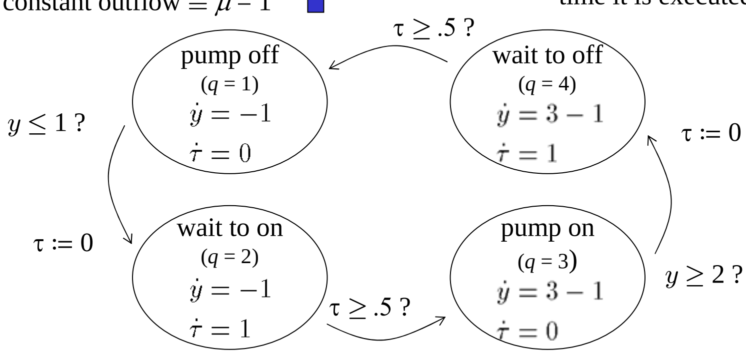
Why?

Example #5: Tank system



goal ≡ prevent the tank from emptying or filling up

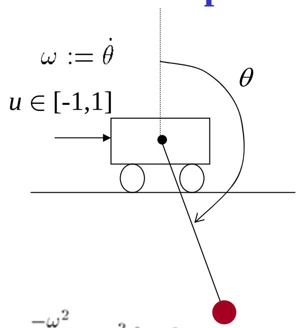
δ = .5 ≡ delay between command is sent to pump and the time it is executed



$$d_{\text{pump}}\big((q_1, x_1), (q_2, x_2)\big) := \begin{cases} \|x_1 - x_2\| & q_1, q_2 \in \{1, 2\} \text{ or } q_1, q_2 \in \{3, 4\} \\ +\infty & \text{otherwise} \end{cases}$$

this "metric" only distinguishes between modes based on the state of the pump

Example #4: Inverted pendulum swing-up



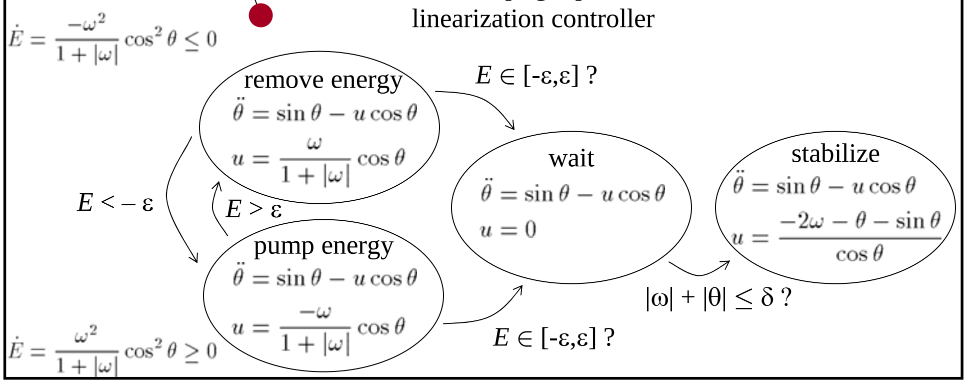
$$\ddot{\theta} = \sin\theta - u\cos\theta \qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$$

Hybrid controller:

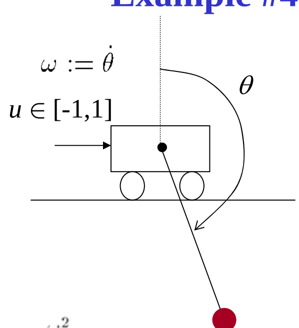
1st pump/remove energy into/from the system by applying maximum force, until E ≈ 0 (energy control) $\dot{E} = -u\omega\cos\theta$

2nd wait until pendulum is close to the upright position

3th next to upright position use feedback linearization controller



Example #4: Inverted pendulum swing-up

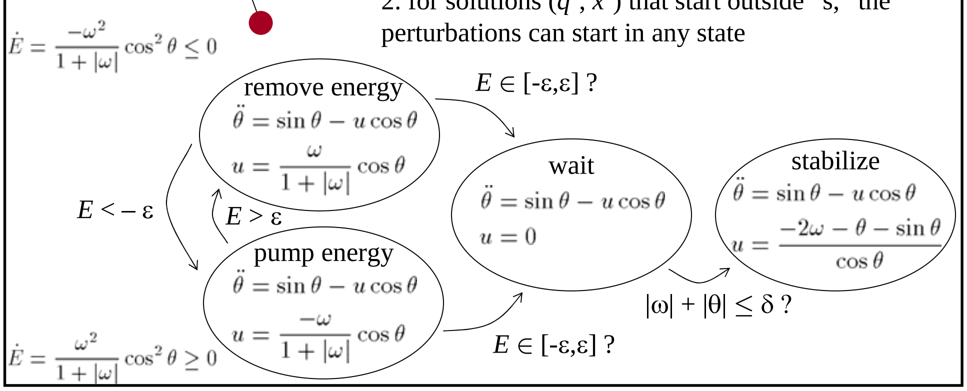


$$\ddot{\theta} = \sin\theta - u\cos\theta \qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$$

A possible "metric"

$$d_{\text{pump}}\big((q_1, x_1), (q_2, x_2)\big) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 = s \text{ or } q_1, q_2 \in \{r, p, w\} \\ +\infty & \text{otherwise} \end{cases}$$

1. for solutions (q^*, x^*) that start in "s," we only consider perturbations that also start in "s"
2. for solutions (q^*, x^*) that start outside "s," the perturbations can start in any state



Asymptotic stability for hybrid systems

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$\mathrm{T} : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

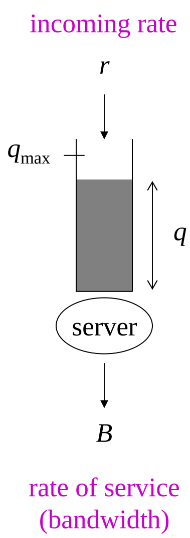
Definition (continuity definition):
A solution (q^*, x^*) is ***(Lyapunov) stable*** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e.,

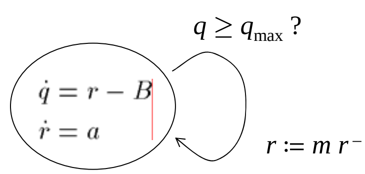
$$\forall \epsilon > 0\ \exists \delta > 0 : d_0\big(\, (q^*(0), x^*(0))\,,\, (q(0), x(0))\,\big) \leq \delta$$
$$\Downarrow$$
$$\sup_{t \geq 0} d_T\big(\, (q^*(t), x^*(t))\,,\, (q(t), x(t))\,\big) \leq \epsilon$$

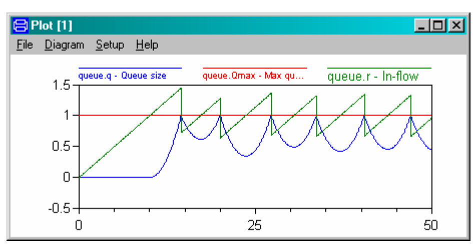
Definition:
A solution (q^*, x^*) is ***(globally) asymptotically stable*** if it is stable, every solution (q,x) exists globally, and $q \to q^*$, $x \to x^*$ as $t \to \infty$, i.e.,

$$\forall \epsilon > 0\ \exists \tau > 0 \quad \sup_{t \geq \tau} d_T\big(\, (q^*(t), x^*(t))\,,\, (q(t), x(t))\,\big) \leq \epsilon$$

Example #7: Server system with congestion control



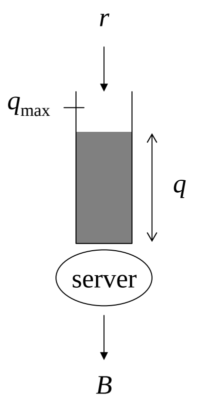




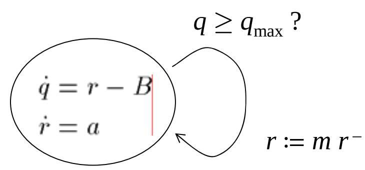
every solution exists globally and converges to a periodic solution that is stable.
Do we have asymptotic stability?

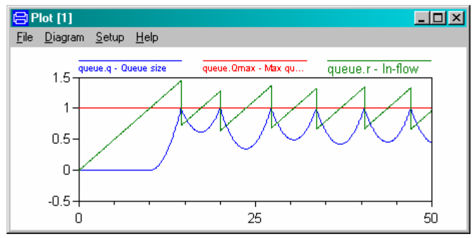
Example #7: Server system with congestion control

incoming rate



rate of service
(bandwidth)



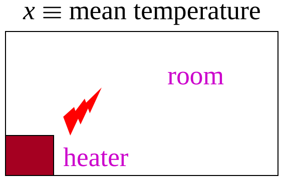


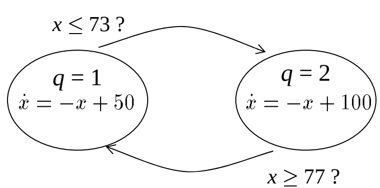
every solution exists globally and converges to a periodic solution that is stable.
Do we have asymptotic stability?

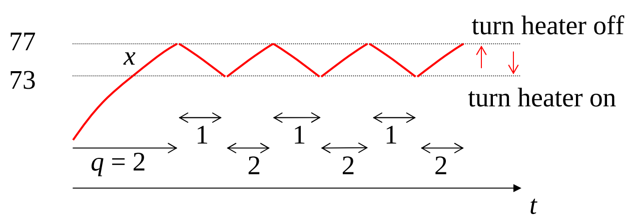
No, its difficult to have asymptotic stability for non-constant solutions due to the "*synchronization*" requirement.
(not even stability… Always?)

Example #2: Thermostat

$x \equiv$ mean temperature







no trajectory is stable

for d_{disc} "metric" both on initial cond. and trajectory
($d_0 = d_T = d_{disc}$)

all trajectories are stable but not asymptotically

for $d_0 = d_{disc}$
and $d_T = d_{cont}$

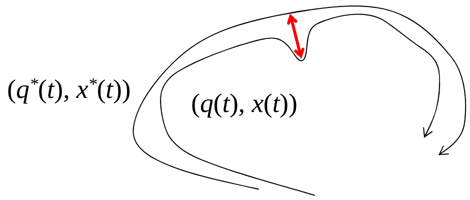
Why?

Stability of sets

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Poincaré distance between (q, x), $(q^*,x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ after t_0

$$d_{\text{P}}\big((q,x),(q^*,x^*);t_0\big) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_T\big((q(t),x(t)),(q^*(\tau),x^*(\tau))\big)$$



distance at the point t where the $(q(t), x(t))$ is the furthest apart from (q^*, x^*)

can also be viewed as the *distance from the trajectory* (q, x) *to the set* $\{(q^*(t), x^*(t)) : t \geq t_0\}$

For *constant trajectories* (q^*,x^*) its just the sup-norm:

$$d_{\text{P}}\big((q,x),(q^*,x^*);t_0\big) := \sup_{t \geq t_0} d_T\big((q(t),x(t)),(q^*,x^*)\big)$$

Stability of sets

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Poincaré distance between (q, x), $(q^*,x^*) \in \mathcal{Q}_{\text{sig}} \times \mathcal{X}_{\text{sig}}$ after t_0

$$d_{\text{P}}\big((q,x),(q^*,x^*);t_0\big) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_T\big((q(t),x(t)),(q^*(\tau),x^*(\tau))\big)$$

Definition: A solution (q^*, x^*) is ***Poincaré stable*** if

$$\forall \epsilon > 0\ \exists\, \delta > 0 : d_0\big((q(0), x(0)), (q^*(0), x^*(0))\big) \leq \delta$$
$$\Downarrow$$
$$d_{\text{P}}((q^*,x^*), (q,x); 0) = \sup_{t \geq 0} \inf_{\tau \geq 0} d_T\big((q(t), x(t)), (q^*(\tau), x^*(\tau))\big) \leq \epsilon$$

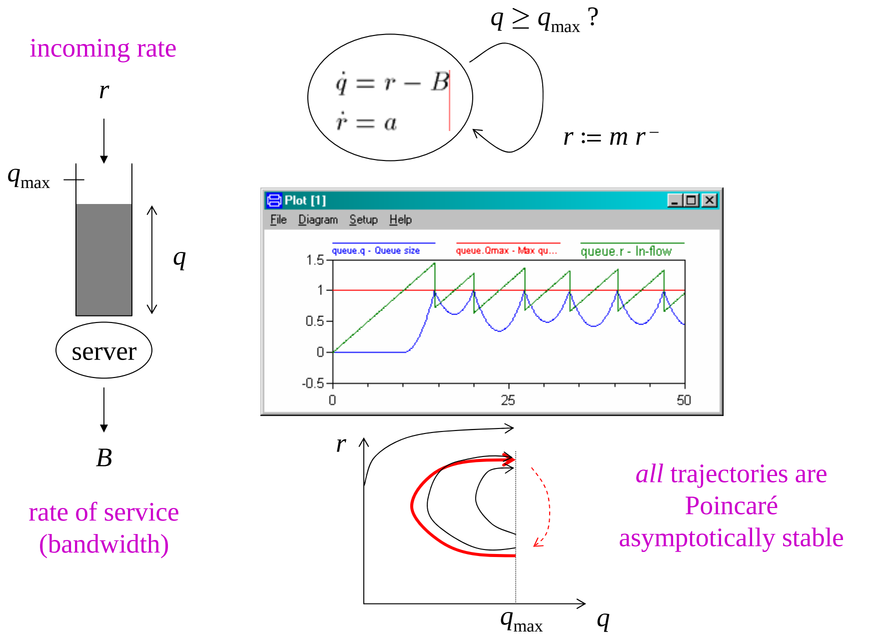
in more modern terminology one would say that the following ***set is stable***

$\{ (q^*(t), x^*(t)) : t \geq 0 \} \subset \mathcal{Q} \times \mathcal{X}$

Definition: A solution (q^*, x^*) is ***Poincaré asymptotically stable*** if it is Poincaré stable, every solution (q, x) exists globally, and $d_{\text{P}}((q,x), (q^*,x^*); t) \to 0$ as $t \to \infty$

in more modern terminology one would say that the following ***set is asymptotically stable***: $\{ (q^*(t), x^*(t)) : t \geq 0 \} \subset \mathcal{Q} \times \mathcal{X}$

Example #7: Server system with congestion control



To think about …

1. With hybrid systems there are many possible notions of stability.
   (especially due to the topology imposed on the discrete state)
   WHICH ONE IS THE BEST?
   (engineering question, not a mathematical one)

   What type of perturbations do you want to consider on the initial conditions?
   (this will define the topology on the initial conditions)
   What type of changes are you willing to accept in the solution?
   (this will define the topology on the solution)

2. Even with ODEs there are several alternatives: e.g.,

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \;\Rightarrow\; \sup_{t\geq 0} \|x(t) - x_{eq}\| \leq \epsilon$ — Lyapunov

   or

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \;\Rightarrow\; \int_0^\infty \|x(t) - x_{eq}\|\, dt \leq \epsilon$ — integral

   or

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_0^*\| \leq \delta \;\Rightarrow\; d_P(x, x^*; 0) \leq \epsilon$ — Poincaré

   (even for linear systems these definitions may differ: Why?)

Next lecture…

Analysis tools for hybrid systems

1. Impact maps
   - Fixed-point theorem
   - Stability of periodic solutions
2. Decoupling
   - Switched systems
   - Supervisors