Hybrid Control and Switched Systems

Lecture #8
Stability and convergence of hybrid systems
(topological view)

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Summary

Lyapunov stability of hybrid systems

Properties of hybrid systems

$\mathcal{X}_{sig} \equiv$ set of all piecewise continuous signals $x:[0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$
$\mathcal{Q}_{sig} \equiv$ set of all piecewise constant signals $q:[0,T) \to \mathcal{Q}$, $T \in (0,\infty]$

Sequence property $\equiv p : \mathcal{Q}_{sig} \times \mathcal{X}_{sig} \to \{\text{false},\text{true}\}$

E.g.,

$$p(q,x) = \begin{cases} \text{true} & q(t) \in \{1,3\},\ x(t) \geq x(t+3),\ \forall t \\ \text{false} & \text{otherwise} \end{cases}$$

A pair of signals $(q, x) \in \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$ ***satisfies*** p if $p(q, x) = \text{true}$

A hybrid automaton H ***satisfies*** p (write $H \vDash p$) if

$p(q, x) = \text{true}$, for every solution (q, x) of H

"***ensemble properties***" $\equiv$ property of the whole family of solutions
(cannot be checked just by looking at isolated solutions)
e.g., continuity with respect to initial conditions…

Lyapunov stability of ODEs (recall)

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$\mathcal{X}_{sig} \equiv$ set of all piecewise continuous signals taking values in $\mathbb{R}^n$
Given a signal $x \in \mathcal{X}_{sig}$, $\|x\|_{sig} := \sup_{t\geq 0} \|x(t)\|$

signal norm

ODE can be seen as an operator

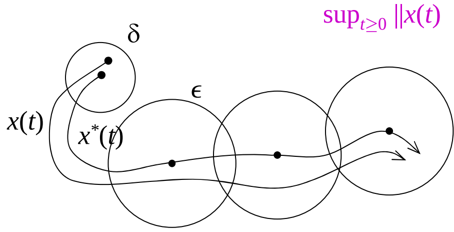
$$T : \mathbb{R}^n \to \mathcal{X}_{sig}$$

that maps $x_0 \in \mathbb{R}^n$ into the solution that starts at $x(0) = x_0$

Definition (continuity definition):
A solution x^* is (***Lyapunov) stable*** if T is continuous at $x^*_0 := x^*(0)$, i.e.,

$$\forall \epsilon > 0\ \exists \delta > 0 : \|x_0 - x^*_0\| \leq \delta \ \Rightarrow\ \|T(x_0) - T(x^*_0)\|_{sig} \leq \epsilon$$

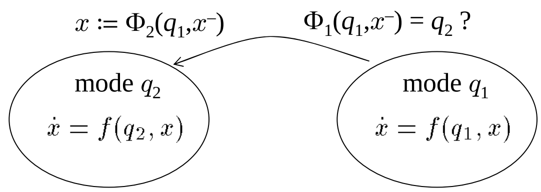
$\sup_{t\geq 0} \|x(t) - x^*(t)\| \leq \epsilon$



pend.m

Lyapunov stability of hybrid systems

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



$\mathcal{X}_{sig} \equiv$ set of all piecewise continuous signals $x:[0,T) \to \mathbb{R}^n$, $T\in(0,\infty]$
$\mathcal{Q}_{sig} \equiv$ set of all piecewise constant signals $q:[0,T)\to \mathcal{Q}$, $T\in(0,\infty]$

Hybrid automaton can be seen as an operator

$$\mathrm{T} : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Definition (continuity definition):
A solution (q^*,x^*) is (***Lyapunov) stable*** if T is continuous at $(q^*(0), x^*(0))$.

To make sense of *continuity* we need ways to measure "distances" in $\mathcal{Q} \times \mathbb{R}^n$ and $\mathcal{Q}_{sig} \times \mathcal{X}_{sig}$

Lyapunov stability of hybrid systems

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

A few possible "metrics"

$$d\big((q_1,x_1),(q_2,x_2)\big) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 \\ +\infty & \text{otherwise} \end{cases}$$

one cares very much about the discrete states matching

$$d\big((q_1,x_1),(q_2,x_2)\big) := \|x_1 - x_2\|$$

one does not care at all about the discrete states matching

Definition (continuity definition):
A solution (q^*,x^*) is (***Lyapunov) stable*** if T is continuous at $(q_0^*, x_0^*):=(q^*(0), x^*(0))$, i.e.,

$$\forall \epsilon > 0\ \exists \delta > 0 : d\big((q^*(0), x^*(0)) , (q(0), x(0)) \big) \leq \delta$$
$$\Downarrow$$
$$\sup_{t\geq 0} d\big((q^*(t), x^*(t)) , (q(t), x(t)) \big) \leq \epsilon$$

1. If the solution starts close to (q^*,x^*) it will remain close to it forever

2. ϵ can be made arbitrarily small by choosing δ sufficiently small

Note: may actually not be metrics on $\mathcal{Q} \times \mathbb{R}^n$ because one may want "zero-distance" between points. However, still define a topology on $\mathcal{Q} \times \mathbb{R}^n$, which is what is really needed to make sense of continuity…

Topological spaces

Given a set $\mathcal{X}$ and a collection $\mathcal{T}_{\mathcal{X}}$ of subsets of $\mathcal{X}$

$(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ is a ***topological space*** if

1. $\emptyset, \mathcal{X} \in \mathcal{T}_{\mathcal{X}}$
2. $\mathcal{A}, \mathcal{B} \in \mathcal{T}_{\mathcal{X}} \Rightarrow \mathcal{A} \cap \mathcal{B} \in \mathcal{T}_{\mathcal{X}}$
3. $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \in \mathcal{T}_{\mathcal{X}} \Rightarrow \cup_{i=1}^{n} \mathcal{A}_i \in \mathcal{T}_{\mathcal{X}}$ $(1 \leq n \leq \infty)$

$\mathcal{T}$ is called a topology and the sets in $\mathcal{T}$ are called ***open*** and their complements are called ***closed***

Intuitively: two elements of $\mathcal{X}$ are "arbitrarily close" if for every open set one belongs to, the other also belongs to

Examples:

$Q := \{ q_1, q_2, \ldots, q_n \}$ (finite)

$\mathcal{T}_Q := \{\emptyset, Q\}$ (***trivial topology*** – all points are close to each other)

$\mathcal{T}_Q := \{\emptyset\} \cup \{$all subsets of $Q\}$ (***discrete topology*** – no two distinct points are close to each other)

A point is only "arbitarily close" to itself (Hausdorff space)

$\mathcal{T}_Q := \{\emptyset, \{1\}, \{1,2\}\}$ of $\{1,2\}$

$\mathcal{X} := \mathbb{R}^n$

$\mathcal{T}_{\mathcal{X}} := \{$ (possibly infinite) union of all open balls $\}$ (***norm-induced topology***)

open ball $\equiv \{ x \in \mathbb{R}^n : \|x - x_0\| < \epsilon \}$

How to prove that 2. holds? (Hint: $\cup$ and $\cap$ are distributive & intersection of two open balls is in $\mathcal{T}$)

Continuity in Topological spaces

Given a set $\mathcal{X}$ and a collection $\mathcal{T}_{\mathcal{X}}$ of subsets of $\mathcal{X}$

$(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ is a ***topological space*** if

1. $\emptyset, \mathcal{X} \in \mathcal{T}_{\mathcal{X}}$
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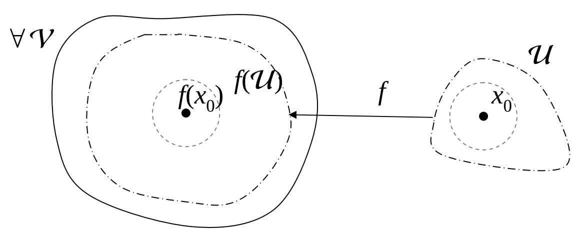
3. $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \in \mathcal{T}_{\mathcal{X}} \Rightarrow \cup_{i=1}^{n} \mathcal{A}_i \in \mathcal{T}_{\mathcal{X}}$ $(1 \leq n \leq \infty)$

$\mathcal{T}$ is called a topology and the sets in $\mathcal{T}$ are called ***open*** and their complements are called ***closed***

Given a function $f: \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y}$ topological space

f is ***continuous*** at a point x_0 in $\mathcal{X}$ if for every neighborhood (i.e., set containing an open set) $\mathcal{V}$ of $f(x_0)$ there is a neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$.



Intuitively: "arbitrarily close" points in $\mathcal{X}$ are transformed into "arbitrarily close" points in $\mathcal{Y}$

For norm-induced topologies we need only consider balls

$\mathcal{V} := \{ y : \| y - f(x_0) \| < \epsilon \}$ and $\mathcal{U} := \{ x : \| x - x_0 \| < \delta \}$

Continuity in Topological spaces

Given a function $f: \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y}$ topological space

f is ***continuous*** at a point x_0 in $\mathcal{X}$ if for every neighborhood (i.e., set containing an open set) $\mathcal{V}$ of $f(x_0)$ there is a neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$.

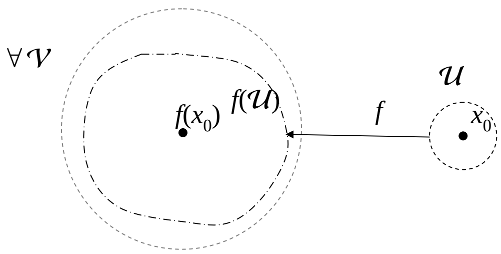
Examples: $\mathcal{X} := \mathbb{R}^n$, $\mathcal{Y} := \mathbb{R}^m$

$\mathcal{T}_\mathcal{X}, \mathcal{T}_\mathcal{Y} :=$ { (possibly infinite) union of all open balls } (norm-induced top.)

open ball $\equiv \{ x \in \mathbb{R}^n : \|x - x_0\| < \epsilon \}$

leads to the usual definition of continuity in $\mathbb{R}^n$: f continuous at x_0 if

$$\forall \epsilon > 0 \; \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \| f(x) - f(x_0)\| < \epsilon$$



Could be restated as: for every ball

$\mathcal{V} := \{ y : \| y - f(x_0) \| < \epsilon \}$

there is a ball

$\mathcal{U} := \{ x : \| x - x_0 \| < \delta \}$

such that

$x \in \mathcal{U} \Rightarrow f(x) \in \mathcal{V}$

or equivalently

$f(\mathcal{U}) \subset \mathcal{V}$

Continuity in Topological spaces

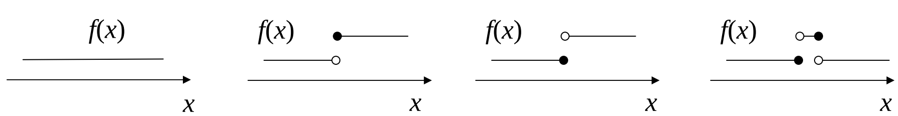
Given a function $f: \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y}$ topological space

f is ***continuous*** at a point x_0 in $\mathcal{X}$ if for every neighborhood (i.e., set containing an open set) $\mathcal{V}$ of $f(x_0)$ there is a neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$.

Examples: $\mathcal{Q} := \{ q_1, q_2, \dots, q_n \}$ (finite)

1. $\mathcal{T}_\mathcal{Q} := \{\emptyset, \mathcal{Q}\}$ (***trivial topology*** – all points are close to each other)
2. $\mathcal{T}_\mathcal{Q} := \{\emptyset\} \cup \{$all subsets of $\mathcal{Q}\}$ (***discrete topology*** – no two distinct points are close to each other)
3. $\mathcal{T}_\mathcal{Q} := \{\emptyset, \{1\}, \{1,2\}\}$ of $\{1,2\}$

Is any of these functions $f : \mathbb{R} \to \mathcal{Q}$ continuous? (usual norm-topology in $\mathbb{R}$)



Continuity in Topological spaces

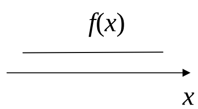
Given a function f: $\mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}$, $\mathcal{Y}$ topological space

f is ***continuous*** at a point x_0 in $\mathcal{X}$ if for every neighborhood (i.e., set containing an open set) $\mathcal{V}$ of $f(x_0)$ there is a neighborhood $\mathcal{U}$ of x_0 such that $f(\mathcal{U}) \subset \mathcal{V}$.

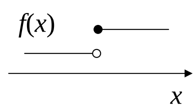
Examples: $\mathcal{Q} := \{ q_1, q_2, \ldots, q_n \}$ (finite)

1. $\mathcal{T}_\mathcal{Q} := \{\emptyset , \mathcal{Q} \}$ (***trivial topology*** – all points are close to each other)
2. $\mathcal{T}_\mathcal{Q} := \{\emptyset\} \cup \{$all subsets of $\mathcal{Q}\}$ (***discrete topology*** – no two points are close to each other)
3. $\mathcal{T}_\mathcal{Q} := \{\emptyset, \{1\}, \{1,2\} \}$ of $\{1,2\}$ (2 is ?close? to 1 but 1 is not ?close? to 2)

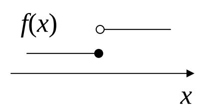
Is any of these functions $f : \mathbb{R} \to \mathcal{Q}$ *continuous? (usual norm-topology in* $\mathbb{R}$*)*



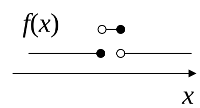
continuous for 1., 2., 3.



continuous for 1., 3.



continuous only for 1.



continuous only for 1.

(for those that don't want to leave anything to the imagination…)

Given a sets $\mathcal{Q}$, $\mathcal{X}$ with topologies $\mathcal{T}_\mathcal{Q}$ and $\mathcal{T}_\mathcal{X}$

One can construct a topology $\mathcal{T}_{\mathcal{Q}\times\mathcal{X}}$ on $\mathcal{Q}\times\mathcal{X}$:

$$\mathcal{T}_{\mathcal{Q}\times\mathcal{X}} := \{ \mathcal{A} \times \mathcal{B} : \mathcal{A} \in \mathcal{Q}, \mathcal{B} \in \mathcal{X} \}$$

Example: $\mathcal{Q} := \{1, 2\}$, $\mathcal{T}_\mathcal{Q} := \{\emptyset, \{1\}, \{1,2\} \}$

$\mathcal{X} := \mathbb{R}$, $\mathcal{T}_\mathbb{R} \equiv$ norm-induced topology

some open sets: $\emptyset$, $\{ (1,x): x\in(1,2) \}$, $\{ (q,x): q=1,2, x\in(1,\infty) \}$

not open sets: $\{ (1,x): x\in(1,2] \}$, $\{ (2,x): x\in(1,2) \}$

One can construct a topology $\mathcal{T}_{\mathcal{Q}\text{sig}}$ on the set $\mathcal{Q}_{\text{sig}}$ of signals q:$[0,T)\to\mathcal{Q}$, T$\in(0,\infty]$:

$\mathcal{T}_{\mathcal{Q}\text{sig}}$:= sets of the form $\mathcal{A} := \{ q\in \mathcal{Q}_{\text{sig}} : q(t) \in \mathcal{A}(t) \; \forall \, t\}$

where the $\mathcal{A}(t)$ are a collection of open sets

Example: $\mathcal{Q} := \{1, 2\}$, $\mathcal{T}_\mathcal{Q} := \{\emptyset, \{1\}, \{1,2\} \}$

some open sets: $\{ q : q(t) = 1 \; \forall \, t\leq 1 \}$, $\{ q : q(t) = 1 \; \forall \, t \in \mathbb{Q} \}$

not open sets: $\{q : q(t) = 2 \; \forall \, t\leq 1 \}$

$\mathcal{X} := \mathbb{R}$, $\mathcal{T}_\mathbb{R} \equiv$ norm-induced topology

some open sets: $\{ x : x(t) < 0 \; \forall \, t \}$, $\{ x : |x(t)| < 1 \; \forall \, t \}$

non open sets: $\{ x : \int_0^\infty x(t) \, dt <1 \}$

Back to hybrid systems…

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Definition (continuity definition):
A solution (q^*, x^*) is ***(Lyapunov) stable*** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e., for every neighborhood $\mathcal{V}$ of $T(q_0^*, x_0^*)$ there is a neighborhood $\mathcal{U}$ of (q_0^*, x_0^*) such that $T(\mathcal{U}) \subset \mathcal{V}$

Case 1: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ trivial topology (all points close to each other)
$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm
co-domain of T: $\mathcal{T}_{\mathcal{Q}sig} \equiv$ trivial topology (all signals close to each other)
$\mathcal{T}_{\mathcal{X}sig} \equiv$ usual topology induced from sup-norm

$$\forall \epsilon > 0\ \exists \delta > 0 : \forall q(0),\ \underbrace{\| x^*(0) - x(0)\| < \delta}_{\mathcal{U}} \Rightarrow \underbrace{\forall t\ \| x^*(t) - x(t)\| < \epsilon}_{\mathcal{V}}$$

one does not care at all about the discrete states matching

$$d\big((q_1, x_1), (q_2, x_2)\big) := \|x_1 - x_2\|$$

Back to hybrid systems…

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

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Case 2: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ discrete topology (all points far from each other)
$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm
co-domain of T: $\mathcal{T}_{\mathcal{Q}sig} \equiv$ discrete topology (all signals far from each other)
$\mathcal{T}_{\mathcal{X}sig} \equiv$ usual topology induced from sup-norm

$$\forall \epsilon > 0\ \exists \delta > 0 : \underbrace{q^*(0) = q(0),\ \| x^*(0) - x(0)\| < \delta}_{\mathcal{U}} \Rightarrow \underbrace{\forall t\ \ q^*(t) = q(t),\ \| x^*(t) - x(t)\| < \epsilon}_{\mathcal{V}}$$

one cares very much about the discrete states matching

$$d\big((q_1, x_1), (q_2, x_2)\big) := \begin{cases} \|x_1 - x_2\| & q_1 = q_2 \\ +\infty & \text{otherwise} \end{cases}$$

Back to hybrid systems…

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

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Case 3: domain of T: $\mathcal{T}_{\mathcal{Q}}$ ≡ discrete topology (all points far from each other)
$\mathcal{T}_{\mathbb{R}^n}$ ≡ usual topology induced from Euclidean norm
co-domain of T: $\mathcal{T}_{\mathcal{Q}sig}$ ≡ trivial topology (all signals close to each other)
$\mathcal{T}_{\mathcal{X}sig}$ ≡ usual topology induced from sup-norm

$$\forall \epsilon > 0\ \exists \delta > 0 : \underbrace{q^*(0) = q(0),\ \|x^*(0) - x(0))\| < \delta}_{\mathcal{U}} \Rightarrow \underbrace{\forall t\ \|x^*(t) - x(t)\| < \epsilon}_{\mathcal{V}}$$

one cares very much about the discrete states matching

Back to hybrid systems…

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

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Case 4: domain of T: $\mathcal{T}_{\mathcal{Q}}$ ≡ $\{\emptyset, \{1\}, \{1,2\}\}$, $\mathcal{Q} := \{1,2\}$
$\mathcal{T}_{\mathbb{R}^n}$ ≡ usual topology induced from Euclidean norm
co-domain of T: $\mathcal{T}_{\mathcal{Q}sig}$ ≡ trivial topology (all signals close to each other)
$\mathcal{T}_{\mathcal{X}sig}$ ≡ usual topology induced from sup-norm

small perturbation in x (but no perturbation in q) leads to small change in x

for $q^*(0) = 1$: $\forall \epsilon > 0\ \exists \delta > 0 : q(0) = 1,\ \|x^*(0) - x(0))\| < \delta \Rightarrow \forall t\ \|x^*(t) - x(t)\| < \epsilon$

for $q^*(0) = 2$: $\forall \epsilon > 0\ \exists \delta > 0 : \forall q(0),\ \|x^*(0) - x(0))\| < \delta \Rightarrow \forall t\ \|x^*(t) - x(t)\| < \epsilon$

small perturbation in x leads to small change in x, regardless of $q(0)$

Back to hybrid systems…

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Definition (continuity definition):
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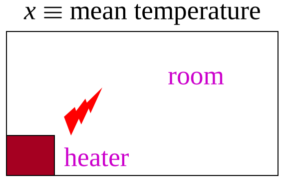
Case 4: domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv$ discrete topology (all points far from each other)
$\mathcal{T}_{\mathbb{R}^n} \equiv$ usual topology induced from Euclidean norm
co-domain of T: $\mathcal{T}_{\mathcal{Q}} \equiv \{\emptyset, \{1\}, \{1,2\}\}$, $\mathcal{Q} := \{1,2\}$ (signal version…)
$\mathcal{T}_{\mathcal{X}sig} \equiv$ usual topology induced from sup-norm

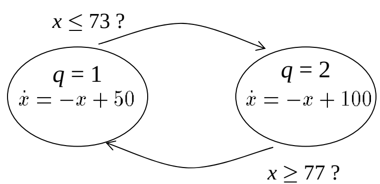
$$\forall \epsilon > 0\ \exists \delta > 0 : q^*(0) = q(0),\ \| x^*(0) - x(0) \| < \delta \Rightarrow \forall t\ q^*(t) = 2 \text{ or } q^*(t) = q(t),\ \| x^*(t) - x(t) \| < \epsilon$$

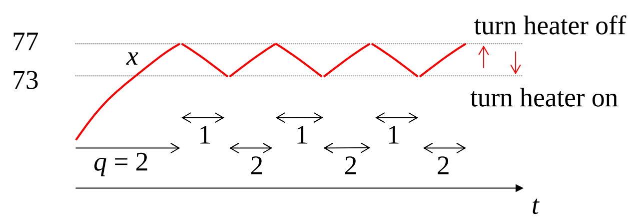
small perturbation in x (but no perturbation in q) leads to small change in x,
q^* and q may differ only when $q^* = 2$

Example #2: Thermostat

$x \equiv$ mean temperature



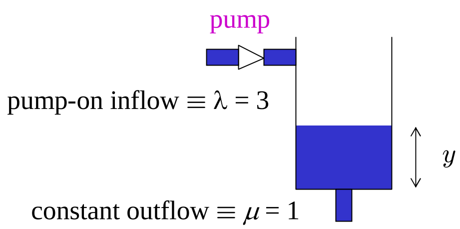




Why?

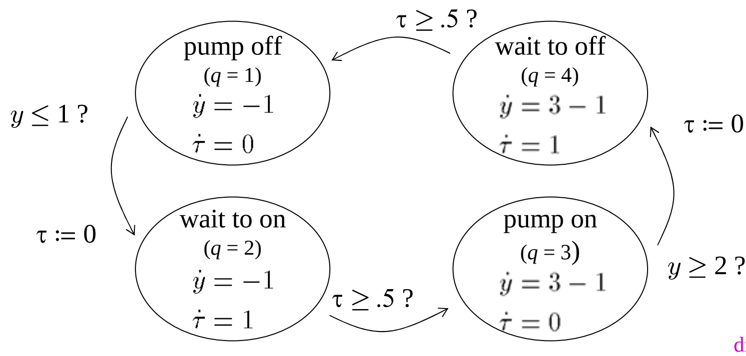
no trajectory is stable	some trajectories are stable others unstable	all trajectories are stable
for discrete topology on $\mathcal{Q}$ (all points far from each other)	for trivial topology on $\mathcal{Q}$ (all points close to each other)	for discrete topology on $\mathcal{Q}$ for the domain and the trivial topology for the codomain

Example #5: Tank system



goal ≡ prevent the tank from emptying or filling up

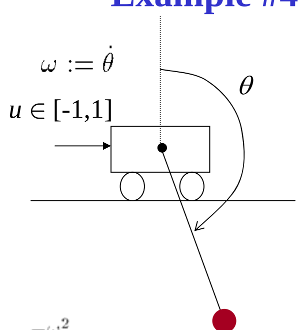
δ = .5 ≡ delay between command is sent to pump and the time it is executed



A possible topology for Q: $\mathcal{T}_Q$:= {∅, {1,2}, {3,4}, {1,2,3,4} }

this topology only distinguishes between modes based on the state of the pump

Example #4: Inverted pendulum swing-up



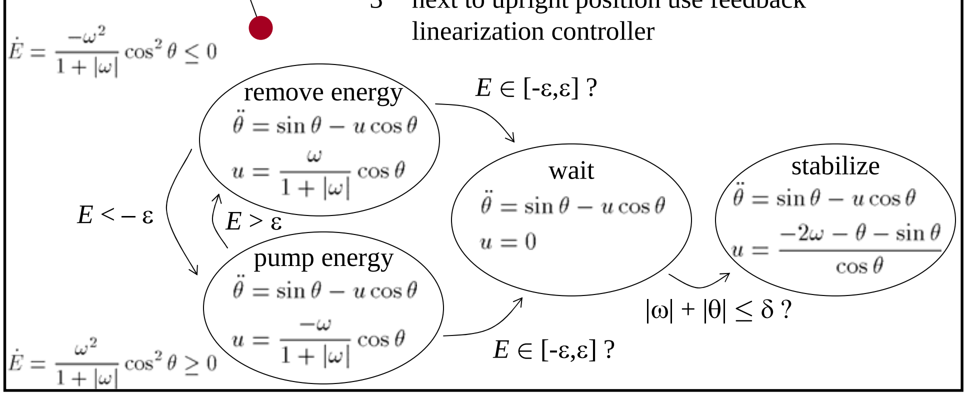
$\ddot{\theta} = \sin\theta - u\cos\theta$ $\qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$

Hybrid controller:

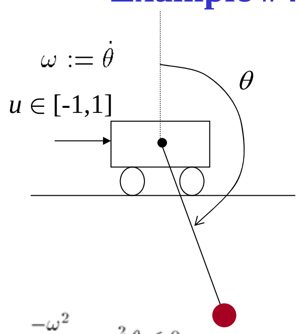
1st pump/remove energy into/from the system by applying maximum force, until E ≈ 0 (energy control) $\dot{E} = -u\omega\cos\theta$

2nd wait until pendulum is close to the upright position

3th next to upright position use feedback linearization controller



Example #4: Inverted pendulum swing-up

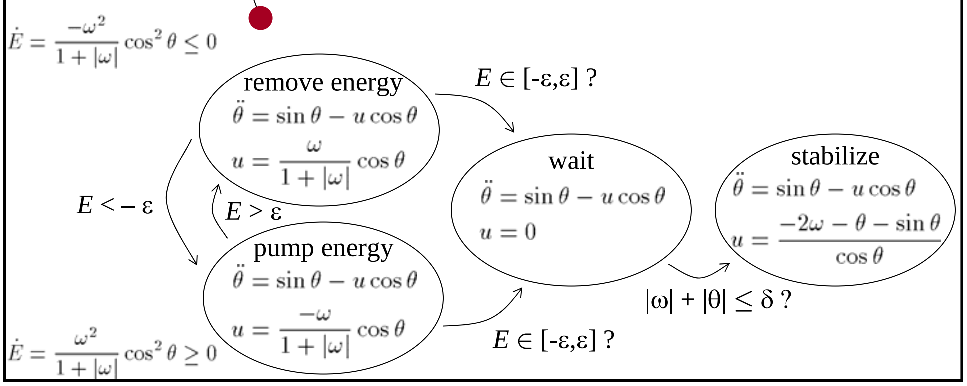


$$\ddot{\theta} = \sin\theta - u\cos\theta \qquad E := \frac{1}{2}\omega^2 + (\cos\theta - 1)$$

A possible topology for $\mathcal{Q}$:= {r,p,w,s} :

$\mathcal{T}_{\mathcal{Q}}$:= {∅, {s}, {r,p,w,s} }

1. for solutions (q^*, x^*) that start in "s," we only consider perturbations that also start in "s"
2. for solutions (q^*, x^*) that start outside "s," the perturbations can start in any state



Asymptotic stability for hybrid systems

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q,x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

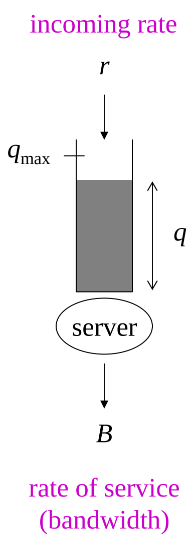
that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

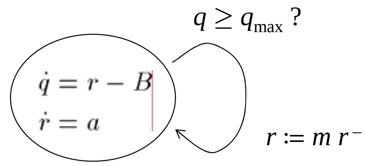
Definition (continuity definition):
A solution (q^*, x^*) is ***(Lyapunov) stable*** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$, i.e., for every neighborhood $\mathcal{V}$ of $T(q_0^*, x_0^*)$ there is a neighborhood $\mathcal{U}$ of (q_0^*, x_0^*) such that $T(\mathcal{U}) \subset \mathcal{V}$

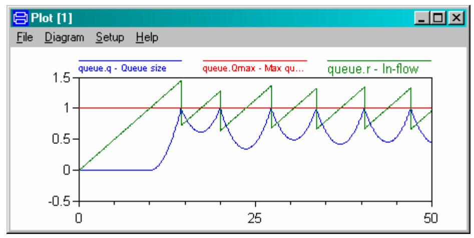
Definition:
A solution (q^*, x^*) is ***asymptotically stable*** if it is stable, every solution (q,x) exists globally, and $q \to q^*$, $x \to x^*$ as $t \to \infty$

in the sense of the topology on $\mathcal{Q}_{sig}$

Example #7: Server system with congestion control

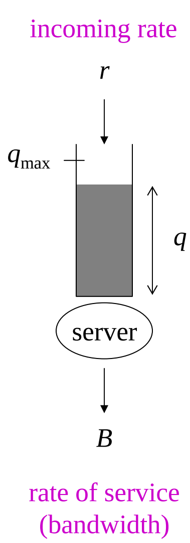


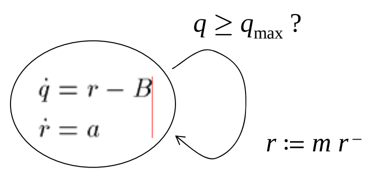


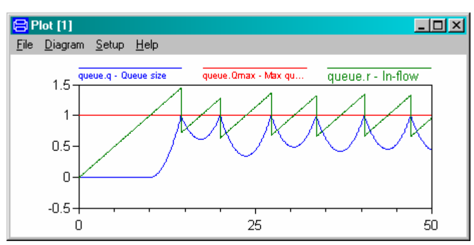


every solution exists globally and converges to a periodic solution that is stable.
Do we have asymptotic stability?

Example #7: Server system with congestion control





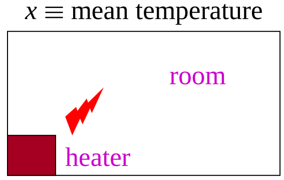


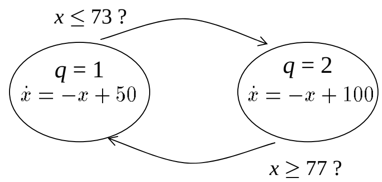
every solution exists globally and converges to a periodic solution that is stable.
Do we have asymptotic stability?

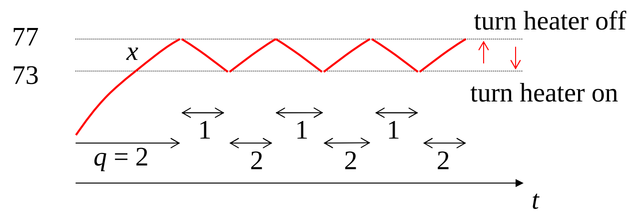
No, its difficult to have asymptotic stability for non-constant solutions due to the "*synchronization*" requirement.
(not even stability… Always?)

Example #2: Thermostat

$x \equiv$ mean temperature







no trajectory is stable

for discrete topology on $\mathcal{Q}$
(all points far from each other)

all trajectories are stable
but not asymptotically

for trivial topology on $\mathcal{Q}$
(all points close to each other)

Why?

Stability of sets

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

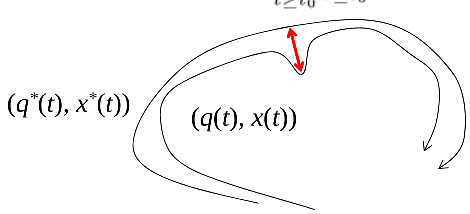
Hybrid automaton can be seen as an operator

$$\mathrm{T} : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{\mathrm{sig}} \times \mathcal{X}_{\mathrm{sig}}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Poincaré distance between $(q, x), (q^*, x^*) \in \mathcal{Q}_{\mathrm{sig}} \times \mathcal{X}_{\mathrm{sig}}$ after t_0

$$d_{\mathrm{P}}\big((q,x),(q^*,x^*);t_0\big) := \sup_{t \geq t_0} \inf_{\tau \geq t_0} d_{\mathcal{T}}\Big(\big(q(t),x(t)\big),\big(q^*(\tau),x^*(\tau)\big)\Big)$$



distance at the point t where the $(q(t), x(t))$ is the furthest apart from (q^*, x^*)

can also be viewed as the *distance from the trajectory* (q, x) *to the set* $\{(q^*(t), x^*(t)) : t \geq t_0\}$

For *constant trajectories* (q^*, x^*) its just the sup-norm:

$$d_{\mathrm{P}}\big((q,x),(q^*,x^*);t_0\big) := \sup_{t \geq t_0} d_{\mathcal{T}}\Big(\big(q(t),x(t)\big),(q^*,x^*)\Big)$$

Stability of sets

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

Poincaré distance between (q, x), $(q^*, x^*) \in \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$ after t_0

Definition: A solution (q^*, x^*) is ***Poincaré stable*** if T is continuous at $(q_0^*, x_0^*) := (q^*(0), x^*(0))$ for the topology on $\mathcal{X}_{sig}$ induced by the Poincaré distance,

$$\forall \epsilon > 0\ \exists \delta > 0 : d_0\big((q(0), x(0)), (q^*(0), x^*(0))\big) \le \delta$$

$$\Downarrow$$

$$d_P((q^*, x^*), (q, x); 0) = \sup_{t \ge 0} \inf_{\tau \ge 0} d_T\big((q(t), x(t)), (q^*(\tau), x^*(\tau))\big) \le \epsilon$$

in more modern terminology one would say that the following ***set is stable***

$$\{ (q^*(t), x^*(t)) : t \ge 0 \} \subset \mathcal{Q} \times \mathcal{X}$$

(open sets are unions of open Poincaré balls $\{ x \in \mathcal{X}_{sig} : d_P(x - x_0) < \epsilon \}$. Show this is a topology…)

Stability of sets

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Hybrid automaton can be seen as an operator

$$T : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$$

that maps $(q_0, x_0) \in \mathcal{Q} \times \mathbb{R}^n$ into the solution that starts at $q(0) = q_0$, $x(0) = x_0$

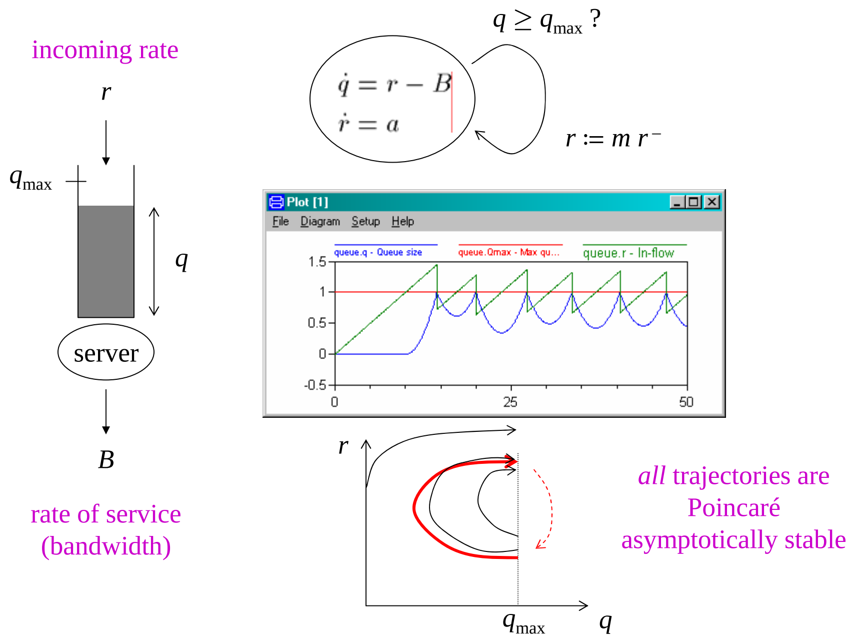
Poincaré distance between (q, x), $(q^*, x^*) \in \mathcal{Q}_{sig} \times \mathcal{X}_{sig}$ after t_0

$$d_P((q, x), (q^*, x^*); t_0) := \sup_{t \ge t_0} \inf_{\tau \ge t_0} d_T\big((q(t), x(t)), (q^*(\tau), x^*(\tau))\big)$$

Definition: A solution (q^*, x^*) is ***Poincaré asymptotically stable*** if it is Poincaré stable, every solution (q, x) exists globally, and $d_P((q, x), (q^*, x^*); t) \to 0$ as $t \to \infty$

in more modern terminology one would say that the following ***set is asymptotically stable***: $\{ (q^*(t), x^*(t)) : t \ge 0 \} \subset \mathcal{Q} \times \mathcal{X}$

Example #7: Server system with congestion control



To think about …

1. With hybrid systems there are many possible notions of stability. (especially due to the topology imposed on the discrete state)
   WHICH ONE IS THE BEST?
   (engineering question, not a mathematical one)

   What type of perturbations do you want to consider on the initial conditions?
   (this will define the topology on the initial conditions)
   What type of changes are you willing to accept in the solution?
   (this will define the topology on the signals)

2. Even with ODEs there are several alternatives: e.g.,

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \ \Rightarrow\ \sup_{t\geq 0} \|x(t) - x_{eq}\| \leq \epsilon$ — Lyapunov

   or

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_{eq}\| \leq \delta \ \Rightarrow\ \int_0^\infty \|x(t) - x_{eq}\|\, dt \leq \epsilon$ — integral

   or

   $\forall\, \epsilon > 0\ \exists\, \delta > 0 : \|x_0 - x_0^*\| \leq \delta \ \Rightarrow\ d_P(\, x, x^*; 0\,) \leq \epsilon$ — Poincaré

   (even for linear systems these definitions may differ: Why?)

Next lecture…

Analysis tools for hybrid systems

1. Impact maps
   - Fixed-point theorem
   - Stability of periodic solutions
2. Decoupling
   - Switched systems
   - Supervisors