

Hybrid Control and Switched Systems

Lecture #9 Analysis tools for hybrid systems: Impact maps

João P. Hespanha

University of California
at Santa Barbara

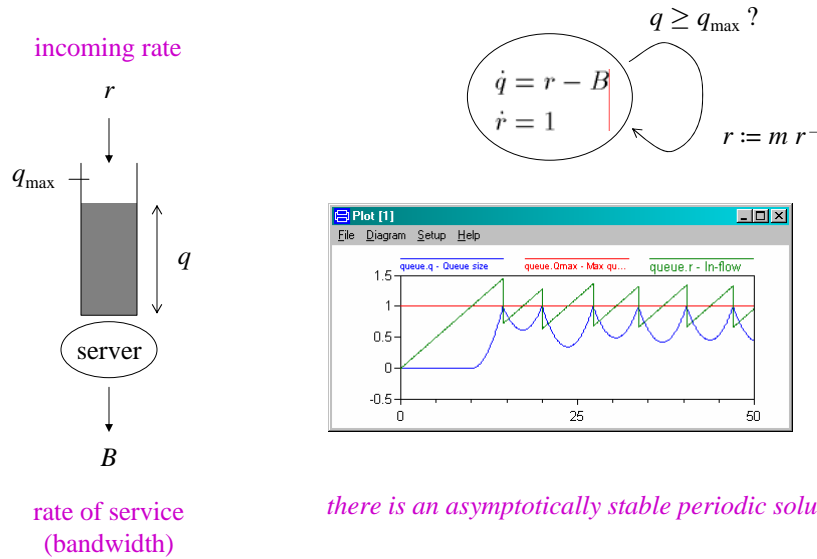


Summary

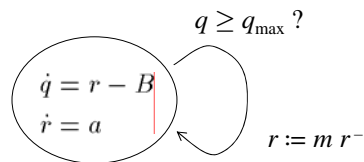
Analysis tools for hybrid systems–Impact maps

- Fixed-point theorem
- Stability of periodic solutions

Example #7: Server system with congestion control



Example #7: Server system with congestion control



For given time t_0 , and initial conditions q_0, r_0

$$r(t) = r_0 + a(t - t_0)$$

$$q(t) = q_0 + (r_0 - B)(t - t_0) + \frac{a(t - t_0)^2}{2}$$

If a transition occurred at time t_0 , when will the next one occur?

$$q(t_0) = q^-(t_1) = q_{\max} \Rightarrow \begin{cases} t_1 = \frac{2(B-r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

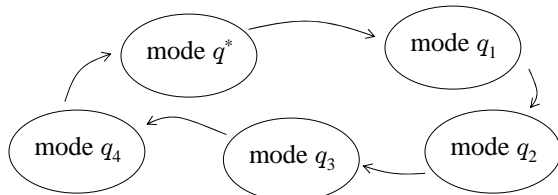
right after one jump $q = q_{\max}$
 $r = r_0$

impact, return, or Poincaré map \longrightarrow

right after next jump $q = q_{\max}$
 $r = m(2B - r_0)$

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$



Recurring mode $\equiv q^* \in \mathcal{Q}$ such that for every initialization there are infinitely many transitions into q^* , i.e., $\square \diamond \{q^* = q, (q, x) \neq \Phi(q^-, x^-)\}$

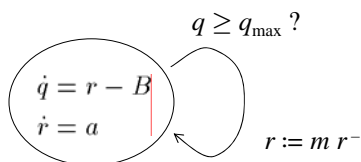
always eventually jump into mode q^*
(not necessarily from a different mode)

Definition: Impact, return, or Poincaré map (Poincaré from ODEs)

Function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that if t_k and t_{k+1} are consecutive times for which there is a transition into q^* then $x(t_{k+1}) = F(x(t_k))$

time-to-impact map \equiv function $\Pi : \mathbb{R}^n \rightarrow (0, \infty)$ such that $t_{k+1} - t_k = \Pi(x(t_k))$

Example #7: Server system with congestion control



If a transition occurred at time t_0 , when will the next one occur?

$$q(t_0) = q^-(t_1) = q_{\max} \Rightarrow \begin{cases} t_1 = \frac{2(B-r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

right after one jump

$$\begin{aligned} q &= q_{\max} \\ r &= r_0 \end{aligned}$$

right after next jump

$$\begin{aligned} q &= q_{\max} \\ r &= m(2B - r_0) \end{aligned}$$

impact map

$$F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

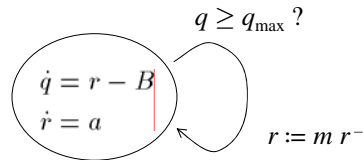
time-to-impact map

$$\Pi \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B-r)}{a}$$

average rate?

fixpoints.nb

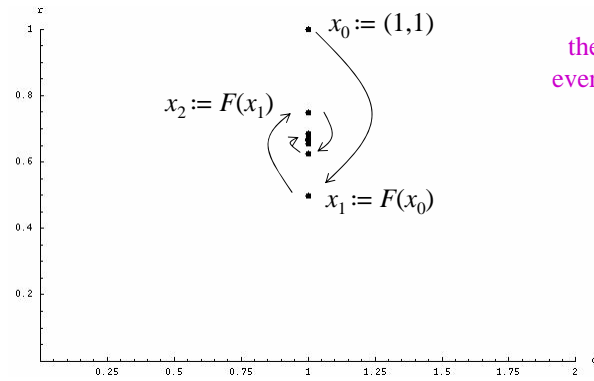
Example #7: Server system with congestion control



impact map

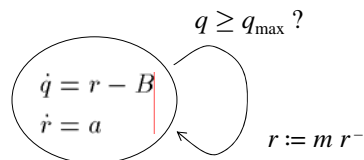
time-to-impact map

$$F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix} \quad \Pi \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



fixpoints.nb

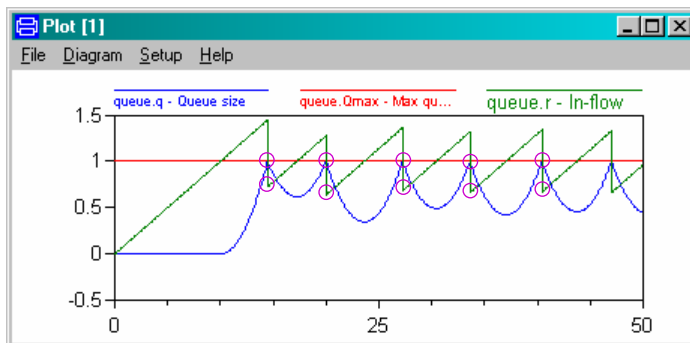
Example #7: Server system with congestion control



impact map

time-to-impact map

$$F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix} \quad \Pi \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



the impact points eventually converge

congestion1.nb

Contraction Mapping Theorem

Contraction mapping \equiv function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for which $\exists \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contraction mapping then fixed-point of F

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ converges to x^* as $k \rightarrow \infty$

Why?

1. Consider sequence: $x_{k+1} = F(x_k)$, $k \geq 0$. After some work (induction ...)

$$\|x_m - x_k\| \leq \frac{\gamma^m}{1 - \gamma} \|x_1 - x_2\| \quad \forall m \geq k$$

2. This means that the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ is Cauchy, i.e.,

$$\forall \epsilon > 0 \quad \exists N \quad \forall m, k > N : \|x_m - x_k\| \leq \epsilon$$

and therefore x_k converges as $k \rightarrow \infty$.

3. Let x^* be the limit. Then

$$F(x^*) = F(\lim x_k) = \lim F(x_k) = \lim x_{k+1} = \lim x_k = x^*$$

F is continuous

Contraction Mapping Theorem

Contraction mapping \equiv function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for which $\exists \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contraction mapping then fixed-point of F

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ converges to x^* as $k \rightarrow \infty$

Why?

So far $x^* := \lim F(x_k)$ exists and $F(x^*) = x^*$ (unique??)

4. Suppose y^* is another fixed point:

$$\|x^* - y^*\| = \|F(x^*) - F(y^*)\| \leq \gamma \|x^* - y^*\| \quad \Rightarrow \quad (1 - \gamma) \|x^* - y^*\| \leq 0$$

x^* must be equal to y^*

Contraction Mapping Theorem

Contraction mapping \equiv function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for which $\exists \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

Contraction mapping Theorem:

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a contraction mapping then fixed-point of F

1. there is one and only point $x^* \in \mathbb{R}^n$ such that $F(x^*) = x^*$
2. for every $x_0 \in \mathbb{R}^n$, the sequence $x_{k+1} = F(x_k)$, $k \geq 0$ converges to x^* as $k \rightarrow \infty$

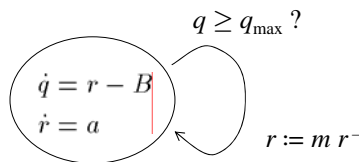
Example: $F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$

contraction as long as $m < 1$

$$\left\| F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) - F \left(\begin{bmatrix} q' \\ r' \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 0 \\ m(r' - r) \end{bmatrix} \right\| = m|r' - r| \leq m \left\| \begin{bmatrix} q' - q \\ r' - r \end{bmatrix} \right\|$$

$$F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q \\ r \end{bmatrix} \Rightarrow q = q_{\max}, r = m(2B - r) \Rightarrow q = q_{\max}, r = \frac{2mB}{1+m}$$

Example #7: Server system with congestion control

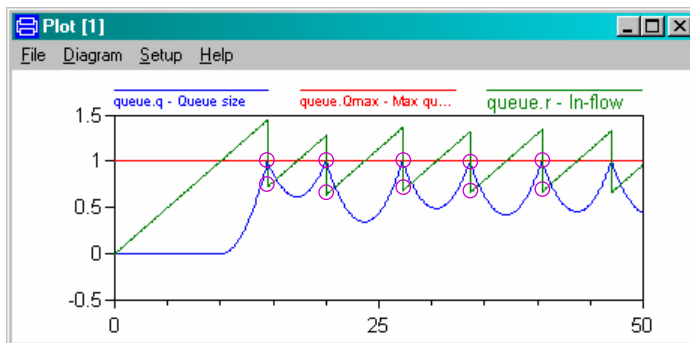


impact map

time-to-impact map

$$F \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

$$\Pi \left(\begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



the impact points eventually converge to

$$q = q_{\max}, r = \frac{2mB}{1+m}$$

congestion1.nb

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Recurring mode $\equiv q^* \in \mathcal{Q}$ such that for every initialization there are infinitely many transitions into q^* , i.e., $\square \diamond \{q^* = q, (q, x) \neq \Phi(q^-, x^-)\}$

Definition: Impact map

Function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that if t_k and t_{k+1} are consecutive times for which there is a transition into q^* then $x(t_{k+1}) = F(x(t_k))$

Theorem:

Suppose the hybrid system has a recurring mode $q^* \in \mathcal{Q}$ and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

then

- a) it has a periodic solution (may be constant)
- b) the impact points converge to the unique fixed point of F

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a recurring mode $q^* \in \mathcal{Q}$ and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point x^* of the impact map

then

- a) it has a (global) periodic solution with period $T := \Pi(x^*)$ (may be constant)
- b) the impact points converge to the unique fixed point of F

Why?

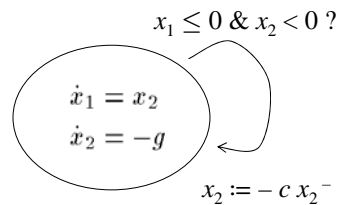
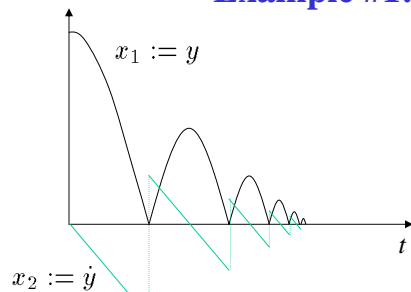
- a) Since F is a contraction it has a fixed point $x^* \in \mathbb{R}^n$

Take any t_0 . Since $F(x^*) = x^*$,

$$q(t_0) = q^*, x(t_0) = x^* \Rightarrow q(t_0 + T) = q^*, x(t_0 + T) = F(x^*) = x^*$$

- b) Impact points are defined by the sequence $x(t_{k+1}) = F(x(t_k))$, which converges to the fixed point x^*

Example #1: Bouncing ball



impact map

$$F \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ cx_2 \end{bmatrix}$$

contraction
mapping
for $c < 1$

time-to-impact map

$$\Pi \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \frac{2x_2}{g} \quad x_2 > 0$$

time-to-impact
map is not
bounded below

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a **recurring mode** and

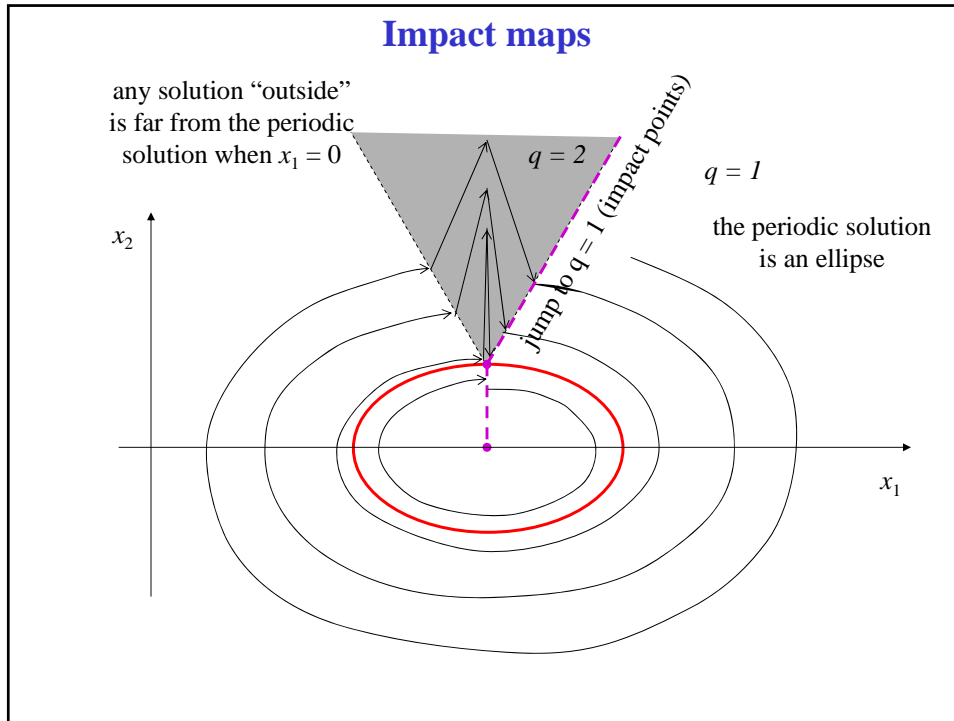
1. the corresponding **impact map is a contraction**
2. the **interval map is nonzero** on a neighborhood of the fixed point x^* of the impact map

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to the unique fixed point of F

Is the periodic solution unique?
yes/no? in what sense

Does not necessarily mean that the periodic
solution is stable (Lyapunov or Poincaré sense)



Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a **recurring mode** and

1. the corresponding **impact map is a contraction**
2. the **interval map is nonzero** on a neighborhood of the fixed point x^* of the impact map

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to the unique fixed point of F

Moreover, if

1. the **sequence of jumps** between consecutive transitions to q^* is the same for every initial condition
2. f is **locally Lipschitz** with respect to x
3. Φ_2 (continuous-state reset) is **continuous** with respect to x
4. interval map is **bounded** in a neighborhood of x^*

then

- a) any solution **converges to a periodic solution**
- b) every periodic solution is **Poincaré asymptotically stable**

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Moreover, if

1. the **sequence of jumps** between consecutive transitions to q^* is the same for every initial condition
2. f is **locally Lipschitz** with respect to x
3. Φ_2 (continuous-state reset) is **continuous** with respect to x
4. interval map is **bounded** in a neighborhood of x^*

then

- a) any solution **converges to a periodic solution**
- b) every periodic solution is **Poincaré asymptotically stable**

Why? (since the sequence of jumps is the same we don't need to worry about \mathcal{Q})

- a)
 - i) 1–3 guarantee continuity with respect to initial conditions (on finite interval)
 - ii) Therefore, if at an impact time t_k , x is close to the fixed point x^* it will remain close to the periodic solution until the next impact (time between impacts is bounded because of 4)
 - iii) Moreover, if $x(t_k)$ converges to x^* the whole solution converges to the periodic solution
- b) Stability from ii) convergence from iii) plus the fact that, from a Poincaré perspective, all periodic solutions are really the same (by uniqueness of fixed point)

Proving that a function is a contraction

Contraction mapping \equiv function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ for which $\exists \gamma \in [0,1)$ such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

A good tool to prove that a mapping is contraction...

Mean Value Theorem:

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

where

$$\gamma := \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial F}{\partial x}(z) \right\| \quad \frac{\partial F}{\partial x} := \left[\frac{\partial F_i}{\partial x_j} \right]_{i,j} \quad \left(\begin{array}{l} \text{constant matrix for} \\ \text{a linear or affine } F \end{array} \right)$$

A mapping may be contracting for one norm but not for another

– often most of the effort is spent in finding the “right” norm

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2} \quad \|x\|_1 := |x_1| + |x_2| \quad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

$$F \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) := \begin{bmatrix} .99x_1 \\ .49(x_1 + x_2) \end{bmatrix} \quad \begin{array}{l} \text{contraction for } \|\cdot\|_\infty \\ \text{not a contraction for } \|\cdot\|_2 \end{array}$$

Stability of difference equations

Given a discrete-time system

$$x_{k+1} = F(x_k) \quad k \geq 0, x_k \in \mathbb{R}^n$$

equilibrium point $\equiv x_{\text{eq}} \in \mathbb{R}^n$ for which $F(x_{\text{eq}}) = x_{\text{eq}}$

thus $x_k = x_{\text{eq}} \forall k \geq 0$ is a solution to the difference equation

Definition (ϵ - δ definition):

The equilibrium point $x_{\text{eq}} \in \mathbb{R}^n$ is **(Lyapunov) stable** if

$$\forall \epsilon > 0 \exists \delta > 0 : \|x_{k_0} - x_{\text{eq}}\| \leq \delta \Rightarrow \|x_k - x_{\text{eq}}\| \leq \epsilon \quad \forall k \geq k_0 \geq 0$$

The equilibrium point $x_{\text{eq}} \in \mathbb{R}^n$ is **(globally) asymptotically stable** if it is Lyapunov stable and for every initial state $x_k \rightarrow x_{\text{eq}}$ as $k \rightarrow \infty$.

(when F is a contraction we automatically have that an equilibrium point exists & global asymptotic stability with exponential convergence)

Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

Theorem:

Suppose the hybrid system has a **recurring mode** and

1. the discrete-time **impact system** $x_{k+1} = F(x_k)$ has an **asymptotically stable** equilibrium point x_{eq}
2. the **interval map is nonzero** on a neighborhood of the equilibrium point x_{eq} of the impact system

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to x_{eq}

Moreover, if

1. the **sequence of jumps** between consecutive transitions to q^* is the same for every initial condition
2. f is **locally Lipschitz** with respect to x
3. Φ_2 (continuous-state reset) is **continuous** with respect to x
4. interval map is **bounded** in a neighborhood of x_{eq}

then

- a) any solution **converges to a periodic solution**
- b) every periodic solution is **Poincaré asymptotically stable**

Next lecture...

Decoupling between continuous and discrete dynamics

- Switched systems
- Supervisors

Stability of switched systems

- Stability under arbitrary switching