

# Hybrid Control and Switched Systems

## Lecture #9 Analysis tools for hybrid systems: Impact maps

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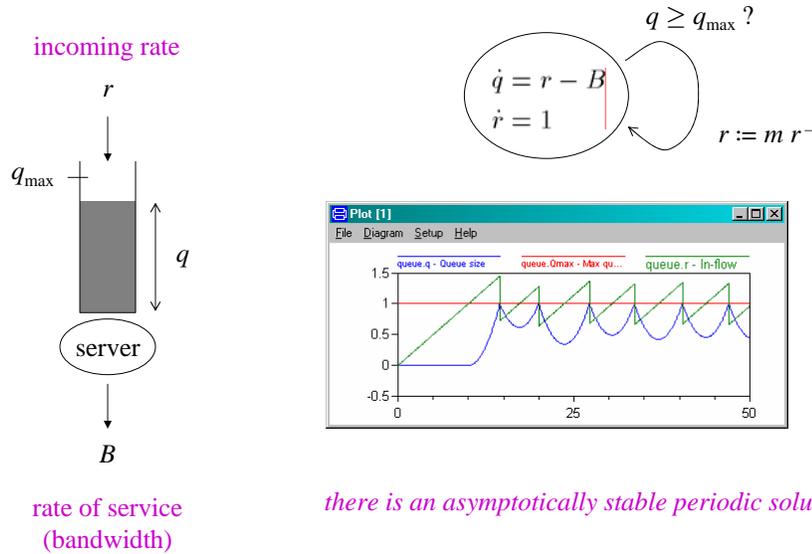


### Summary

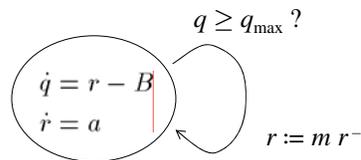
Analysis tools for hybrid systems–Impact maps

- Fixed-point theorem
- Stability of periodic solutions

## Example #7: Server system with congestion control



## Example #7: Server system with congestion control



For given time  $t_0$ , and initial conditions  $q_0, r_0$

$$r(t) = r_0 + a(t - t_0)$$

$$q(t) = q_0 + (r_0 - B)(t - t_0) + \frac{a(t - t_0)^2}{2}$$

If a transition occurred at time  $t_0$ , when will the next one occur?

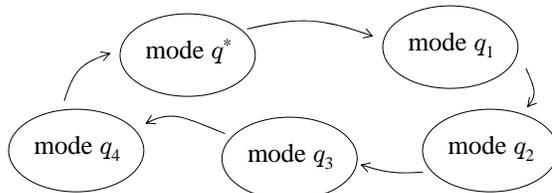
$$q(t_0) = q^-(t_1) = q_{\max} \Rightarrow \begin{cases} t_1 = \frac{2(B-r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

right after one jump      impact, return, or Poincaré map      right after next jump

$$\begin{array}{ccc} q = q_{\max} & \longrightarrow & q = q_{\max} \\ r = r_0 & & r = m(2B - r_0) \end{array}$$

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$



**Recurring mode**  $\equiv q^* \in \mathcal{Q}$  such that for every initialization there are infinitely many transitions into  $q^*$ , i.e.,  $\square \diamond \{q^* = q, (q, x) \neq \Phi(q^-, x^-)\}$

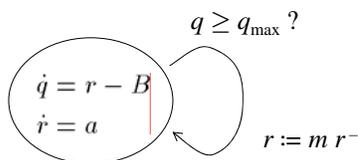
always eventually      jump into mode  $q^*$   
(not necessarily from a different mode)

**Definition: Impact, return, or Poincaré map (Poincaré from ODEs)**

Function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that if  $t_k$  and  $t_{k+1}$  are consecutive times for which there is a transition into  $q^*$  then  $x(t_{k+1}) = F(x(t_k))$

time-to-impact map  $\equiv$  function  $\Pi : \mathbb{R}^n \rightarrow (0, \infty)$  such that  $t_{k+1} - t_k = \Pi(x(t_k))$

## Example #7: Server system with congestion control



If a transition occurred at time  $t_0$ , when will the next one occur?

$$q(t_0) = q^-(t_1) = q_{\max} \Rightarrow \begin{cases} t_1 = \frac{2(B-r_0)}{a} + t_0 \\ r^-(t_1) = 2B - r_0 \\ r(t_1) = m(2B - r_0) \end{cases}$$

right after one jump

$$\begin{aligned} q &= q_{\max} \\ r &= r_0 \end{aligned}$$

right after next jump

$$\begin{aligned} q &= q_{\max} \\ r &= m(2B - r_0) \end{aligned}$$

impact map

$$F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

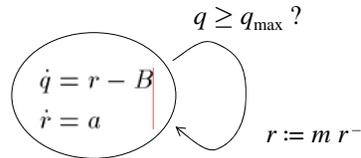
time-to-impact map

$$\Pi \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$

average rate?

fixpoints.nb

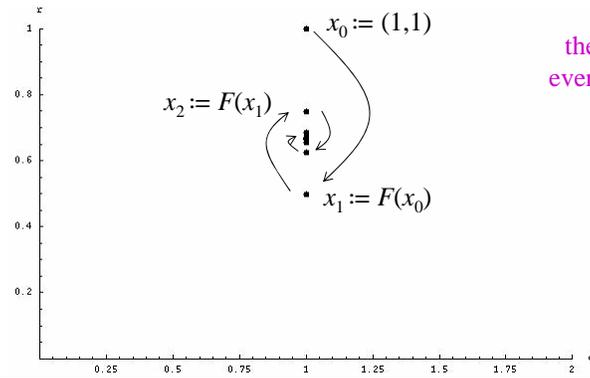
### Example #7: Server system with congestion control



impact map

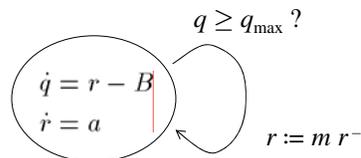
time-to-impact map

$$F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix} \quad \Pi \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



fixpoints.nb

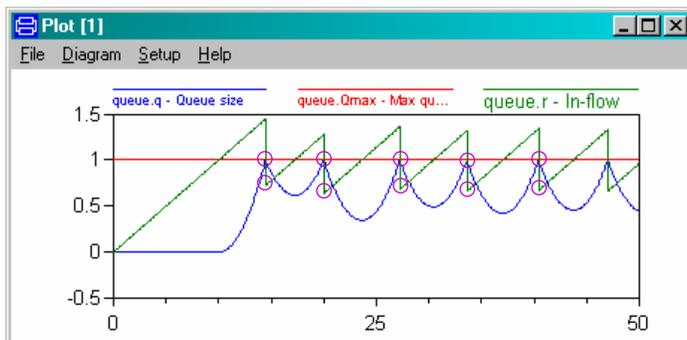
### Example #7: Server system with congestion control



impact map

time-to-impact map

$$F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix} \quad \Pi \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



the impact points eventually converge

congestion1.nb

## Contraction Mapping Theorem

**Contraction mapping**  $\equiv$  function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for which  $\exists \gamma \in [0,1)$  such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

Lipschitz coefficient

**Contraction mapping Theorem:**

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a contraction mapping then fixed-point of  $F$

1. there is one and only point  $x^* \in \mathbb{R}^n$  such that  $F(x^*) = x^*$
2. for every  $x_0 \in \mathbb{R}^n$ , the sequence  $x_{k+1} = F(x_k)$ ,  $k \geq 0$  converges to  $x^*$  as  $k \rightarrow \infty$

**Why?**

1. Consider sequence:  $x_{k+1} = F(x_k)$ ,  $k \geq 0$ . After some work (induction ...)

$$\|x_m - x_k\| \leq \frac{\gamma^m}{1 - \gamma} \|x_1 - x_2\| \quad \forall m \geq k$$

2. This means that the sequence  $x_{k+1} = F(x_k)$ ,  $k \geq 0$  is Cauchy, i.e.,

$$\forall \epsilon > 0 \quad \exists N \quad \forall m, k > N : \|x_m - x_k\| \leq \epsilon$$

and therefore  $x_k$  converges as  $k \rightarrow \infty$ .

3. Let  $x^*$  be the limit. Then

$$F(x^*) = F(\lim x_k) = \lim F(x_k) = \lim x_{k+1} = \lim x_k = x^*$$

F is continuous

## Contraction Mapping Theorem

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**Why?**

So far  $x^* := \lim F(x_k)$  exists and  $F(x^*) = x^*$  (unique??)

4. Suppose  $y^*$  is another fixed point:

$$\|x^* - y^*\| = \|F(x^*) - F(y^*)\| \leq \gamma \|x^* - y^*\| \quad \Rightarrow \quad (1 - \gamma) \|x^* - y^*\| \leq 0$$

$x^*$  must be equal to  $y^*$

## Contraction Mapping Theorem

**Contraction mapping**  $\equiv$  function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for which  $\exists \gamma \in [0,1)$  such that

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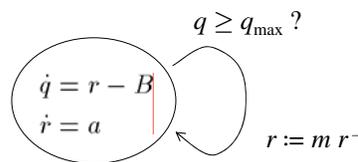
Example:  $F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$

contraction as long as  $m < 1$

$$\left\| F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) - F \left( \begin{bmatrix} q' \\ r' \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 0 \\ m(r' - r) \end{bmatrix} \right\| = m|r' - r| \leq m \left\| \begin{bmatrix} q' - q \\ r' - r \end{bmatrix} \right\|$$

$$F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q \\ r \end{bmatrix} \Rightarrow q = q_{\max}, r = m(2B - r) \Rightarrow q = q_{\max}, r = \frac{2mB}{1+m}$$

## Example #7: Server system with congestion control

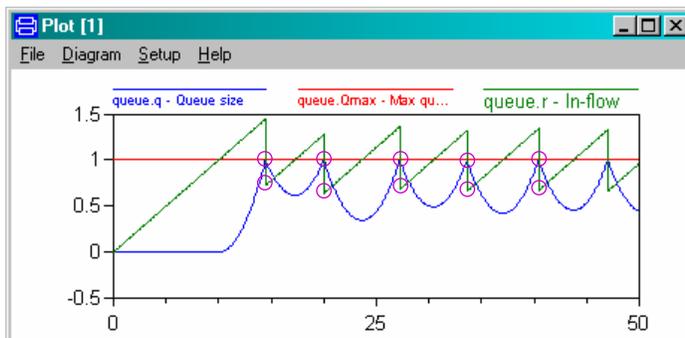


impact map

time-to-impact map

$$F \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \begin{bmatrix} q_{\max} \\ m(2B - r) \end{bmatrix}$$

$$\Pi \left( \begin{bmatrix} q \\ r \end{bmatrix} \right) = \frac{2(B - r)}{a}$$



the impact points eventually converge to

$$q = q_{\max}, r = \frac{2mB}{1+m}$$

congestion1.nb

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

**Recurring** mode  $\equiv q^* \in \mathcal{Q}$  such that for every initialization there are infinitely many transitions into  $q^*$ , i.e.,  $\square \diamond \{q^* = q, (q, x) \neq \Phi(q^-, x^-)\}$

### Definition: Impact map

Function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that if  $t_k$  and  $t_{k+1}$  are consecutive times for which there is a transition into  $q^*$  then  $x(t_{k+1}) = F(x(t_k))$

### Theorem:

Suppose the hybrid system has a recurring mode  $q^* \in \mathcal{Q}$  and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point  $x^*$  of the impact map

then

- a) it has a periodic solution (may be constant)
- b) the impact points converge to the unique fixed point of  $F$

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

### Theorem:

Suppose the hybrid system has a recurring mode  $q^* \in \mathcal{Q}$  and

1. the corresponding impact map is a contraction
2. the interval map is nonzero on a neighborhood of the fixed point  $x^*$  of the impact map

then

- a) it has a (global) periodic solution with period  $T := \Pi(x^*)$  (may be constant)
- b) the impact points converge to the unique fixed point of  $F$

### Why?

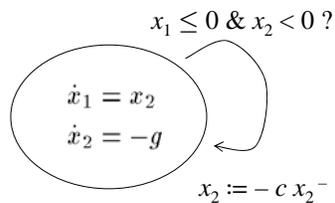
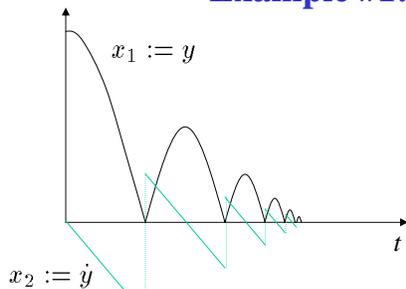
- a) Since  $F$  is a contraction it has a fixed point  $x^* \in \mathbb{R}^n$

Take any  $t_0$ . Since  $F(x^*) = x^*$ ,

$$q(t_0) = q^*, x(t_0) = x^* \Rightarrow q(t_0 + T) = q^*, x(t_0 + T) = F(x^*) = x^*$$

- b) Impact points are defined by the sequence  $x(t_{k+1}) = F(x(t_k))$ , which converges to the fixed point  $x^*$

## Example #1: Bouncing ball



impact map

$$F \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ cx_2 \end{bmatrix}$$

contraction  
mapping  
for  $c < 1$

time-to-impact map

$$\Pi \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \frac{2x_2}{g} \quad x_2 > 0$$

time-to-impact  
map is not  
bounded below

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

### Theorem:

Suppose the hybrid system has a **recurring mode** and

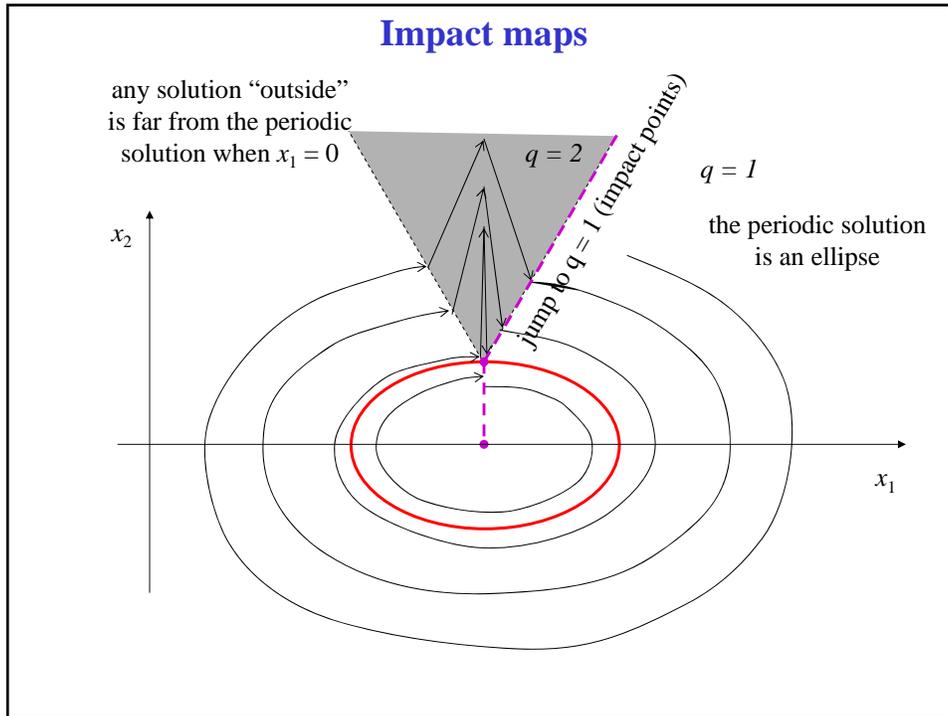
1. the corresponding **impact map is a contraction**
2. the **interval map is nonzero** on a neighborhood of the fixed point  $x^*$  of the impact map

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to the unique fixed point of  $F$

Is the periodic solution unique?  
yes/no? in what sense

Does not necessarily mean that the periodic  
solution is stable (Lyapunov or Poincaré sense)



### Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

**Theorem:**

Suppose the hybrid system has a **recurring mode** and

1. the corresponding **impact map is a contraction**
2. the **interval map is nonzero** on a neighborhood of the fixed point  $x^*$  of the impact map

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to the unique fixed point of  $F$

**Moreover, if**

1. the **sequence of jumps** between consecutive transitions to  $q^*$  is the same for every initial condition
2.  $f$  is **locally Lipschitz** with respect to  $x$
3.  $\Phi_2$  (continuous-state reset) is **continuous** with respect to  $x$
4. interval map is **bounded** in a neighborhood of  $x^*$

then

- a) any solution **converges to a periodic solution**
- b) every periodic solution is **Poincaré asymptotically stable**

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

**Moreover, if**

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- a) any solution **converges to a periodic solution**
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**Why?** (since the sequence of jumps is the same we don't need to worry about  $\mathcal{Q}$ )

- a)
  - i) 1–3 guarantee continuity with respect to initial conditions (on finite interval)
  - ii) Therefore, if at an impact time  $t_k$ ,  $x$  is close to the fixed point  $x^*$  it will remain close to the periodic solution until the next impact (time between impacts is bounded because of 4)
  - iii) Moreover, if  $x(t_k)$  converges to  $x^*$  the whole solution converges to the periodic solution
- b) Stability from ii) convergence from iii) plus the fact that, from a Poincaré perspective, all periodic solutions are really the same (by uniqueness of fixed point)

## Proving that a function is a contraction

**Contraction mapping**  $\equiv$  function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for which  $\exists \gamma \in [0,1)$  such that

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

*A good tool to prove that a mapping is contraction...*

**Mean Value Theorem:**

$$\|F(x) - F(x')\| \leq \gamma \|x - x'\| \quad \forall x, x' \in \mathbb{R}^n$$

where

$$\gamma := \sup_{z \in \mathbb{R}^n} \left\| \frac{\partial F}{\partial x}(z) \right\| \quad \frac{\partial F}{\partial x} := \left[ \frac{\partial F_i}{\partial x_j} \right]_{i,j} \quad \left( \begin{array}{l} \text{constant matrix for} \\ \text{a linear or affine } F \end{array} \right)$$

*A mapping may be contracting for one norm but not for another*

*– often most of the effort is spent in finding the “right” norm*

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2} \quad \|x\|_1 := |x_1| + |x_2| \quad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

$$F \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) := \begin{bmatrix} .99x_1 \\ .49(x_1 + x_2) \end{bmatrix} \quad \begin{array}{l} \text{contraction for } \|\cdot\|_\infty \\ \text{not a contraction for } \|\cdot\|_2 \end{array}$$

## Stability of difference equations

Given a discrete-time system

$$x_{k+1} = F(x_k) \quad k \geq 0, x_k \in \mathbb{R}^n$$

**equilibrium point**  $\equiv x_{\text{eq}} \in \mathbb{R}^n$  for which  $F(x_{\text{eq}}) = x_{\text{eq}}$

thus  $x_k = x_{\text{eq}} \forall k \geq 0$  is a solution to the difference equation

**Definition** ( $\epsilon$ - $\delta$  definition):

The equilibrium point  $x_{\text{eq}} \in \mathbb{R}^n$  is **(Lyapunov) stable** if

$$\forall \epsilon > 0 \exists \delta > 0 : \|x_{k_0} - x_{\text{eq}}\| \leq \delta \Rightarrow \|x_k - x_{\text{eq}}\| \leq \epsilon \quad \forall k \geq k_0 \geq 0$$

The equilibrium point  $x_{\text{eq}} \in \mathbb{R}^n$  is **(globally) asymptotically stable** if it is Lyapunov stable and for every initial state  $x_k \rightarrow x_{\text{eq}}$  as  $k \rightarrow \infty$ .

(when  $F$  is a contraction we automatically have that an equilibrium point exists & global asymptotic stability with exponential convergence)

## Impact maps

$$\dot{x} = f(q, x) \quad (q, x) = \Phi(q^-, x^-) \quad q \in \mathcal{Q}, x \in \mathbb{R}^n$$

**Theorem:**

Suppose the hybrid system has a **recurring mode** and

1. the discrete-time **impact system**  $x_{k+1} = F(x_k)$  has an **asymptotically stable** equilibrium point  $x_{\text{eq}}$
2. the **interval map is nonzero** on a neighborhood of the equilibrium point  $x_{\text{eq}}$  of the impact system

then

- a) it has a **periodic solution** (may be constant)
- b) the **impact points converge** to  $x_{\text{eq}}$

**Moreover, if**

1. the **sequence of jumps** between consecutive transitions to  $q^*$  is the same for every initial condition
2.  $f$  is **locally Lipschitz** with respect to  $x$
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then

- a) any solution **converges to a periodic solution**
- b) every periodic solution is **Poincaré asymptotically stable**

## Next lecture...

Decoupling between continuous and discrete dynamics

- Switched systems
- Supervisors

Stability of switched systems

- Stability under arbitrary switching