research supported by NSF

Stochastic Hybrid Systems:
Modeling, analysis, and applications to networks and biology

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Talk outline

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1. A model for stochastic hybrid systems (SHSs)
2. Examples:
   - network traffic under TCP
   - networked control systems
3. Analysis tools for SHSs
   - Lyapunov
   - moment dynamics
4. More examples …

Collaborators:
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Students:
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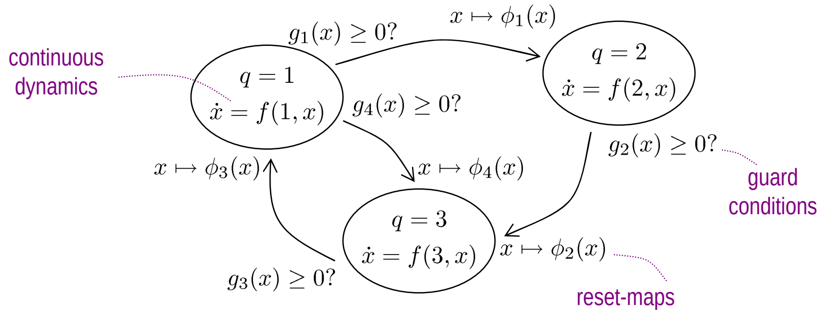
Disclaimer: This is an overview, details in papers referenced…

Deterministic Hybrid Systems

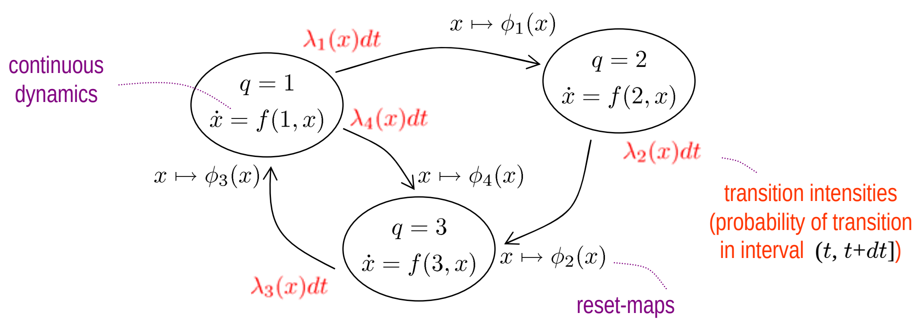


$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state
$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

} right-continuous by convention

we assume here a deterministic system so the invariant sets would be the exact complements of the guards

Stochastic Hybrid Systems



$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state
$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

Continuous dynamics:

$$\dot{x} = f(q,x,t)$$

Transition intensities:

$$\lambda_\ell(q,x,t) \qquad \ell \in \{1,\ldots,m\}$$

Reset-maps (one per transition intensity):

$$(q,x) \mapsto \phi_\ell(q,x,t) \qquad \ell \in \{1,\ldots,m\}$$

Stochastic Hybrid Systems

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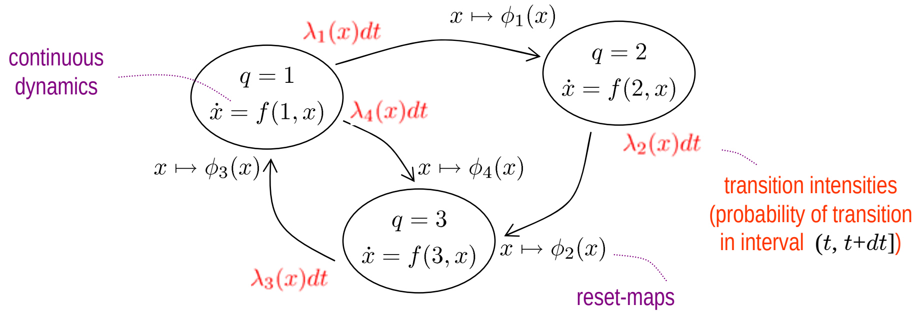
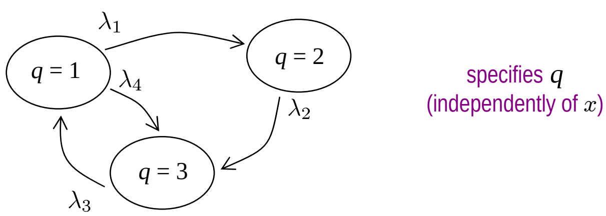


Special case: When all λ_ℓ are constant, transitions are controlled by a continuous-time Markov process



Formal model—Summary

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State space: $q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state

$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

Continuous dynamics:

$\dot{x} = f(q,x,t)$ $\qquad f : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$

Transition intensities:

$\lambda_\ell(q,x,t)$ $\qquad \lambda_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to [0,\infty) \quad \ell \in \{1,\ldots,m\}$

Reset-maps (one per transition intensity): $\qquad$ # of transitions

$(q,x) \mapsto \phi_\ell(q,x,t)$ $\qquad \phi_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathcal{Q} \times \mathbb{R}^n \quad \ell \in \{1,\ldots,m\}$

Results:

1. [*existence*] Under appropriate regularity (Lipschitz) assumptions, there exists a measure "consistent" with the desired SHS behavior
2. [*simulation*] The procedure used to construct the measure is constructive and allows for efficient generation of *Monte Carlo sample paths*
3. [*Markov*] The pair ($q(t)$, $x(t)$) $\in \mathcal{Q} \times \mathbb{R}^n$ is a (Piecewise-deterministic) Markov Process (in the sense of M. Davis, 1993)

[HSCC'04]

Stochastic Hybrid Systems with diffusion

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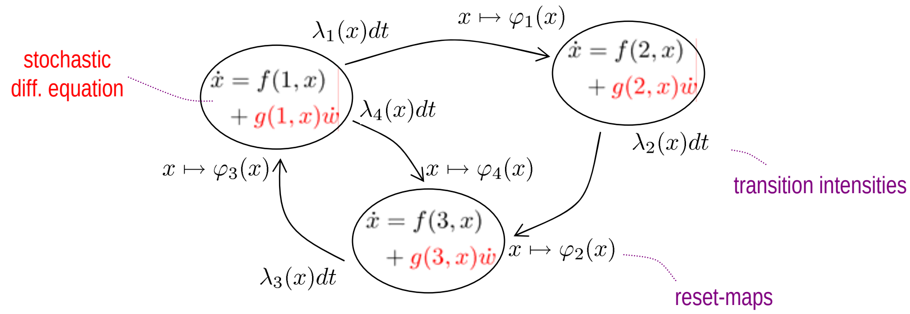

Continuous dynamics: $w \equiv$ Brownian motion process

$$\dot{x} = f(q,x,t) + g(q,x,t)\dot{w}$$

Transition intensities:

$$\lambda_\ell(q,x,t) \qquad \ell \in \{1,\ldots,m\}$$

Reset-maps (one per transition intensity):

$$(q,x) \mapsto \phi_\ell(q,x,t) \qquad \ell \in \{1,\ldots,m\}$$

Example I: Transmission Control Protocol

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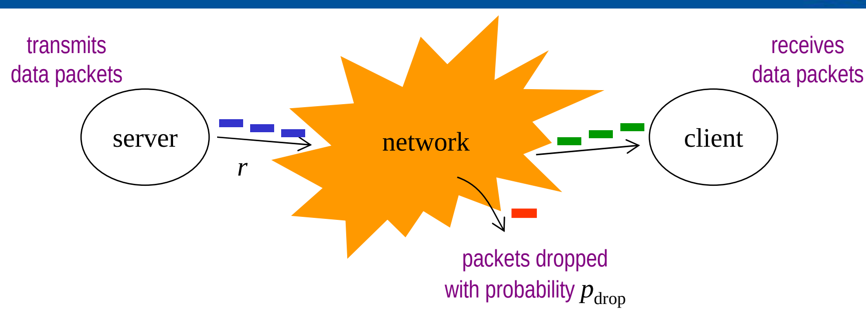
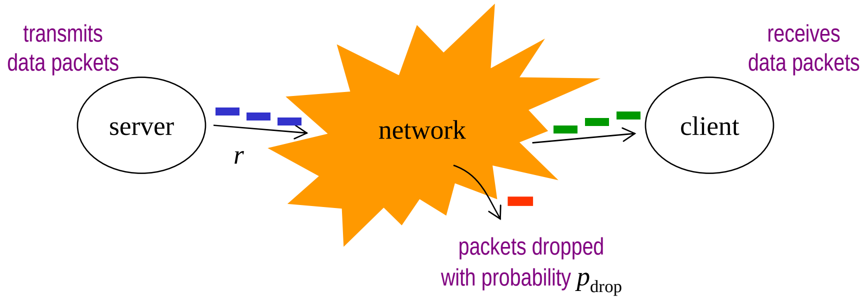


congestion control $\equiv$ selection of the rate *r* at which the server transmits packets
feedback mechanism $\equiv$ packets are dropped by the network to indicate congestion

Example I: TCP congestion control



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feedback mechanism $\equiv$ packets are dropped by the network to indicate congestion

TCP (Reno) congestion control: packet sending rate given by

$$r(t) = \frac{w(t)}{RTT(t)}$$

congestion window (internal state of controller)
round-trip-time (from server to client and back)

- initially w is set to 1
- until first packet is dropped, w increases exponentially fast (slow-start)
- after first packet is dropped, w increases linearly (congestion-avoidance)
- each time a drop occurs, w is divided by 2 (multiplicative decrease)

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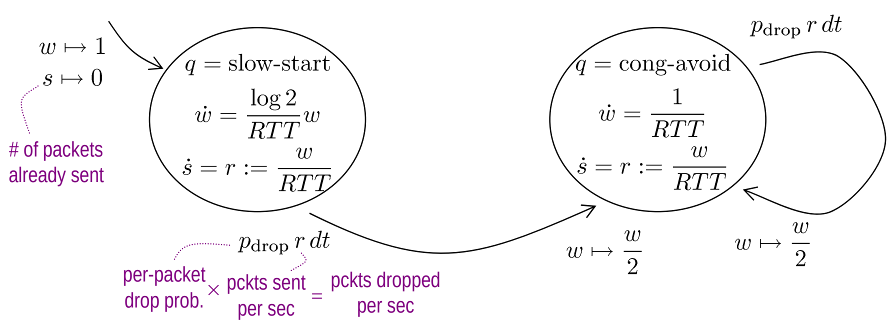

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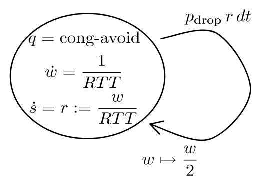
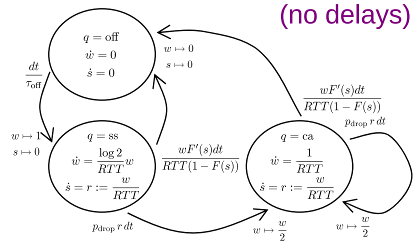
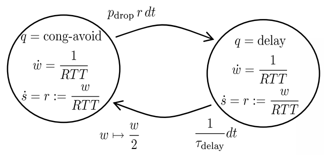
many SHS models for TCP…

long-lived TCP flows (no delays)



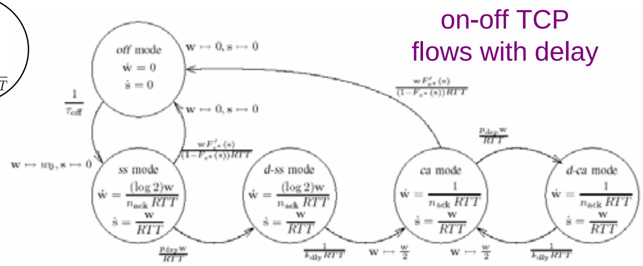
on-off TCP flows (no delays)





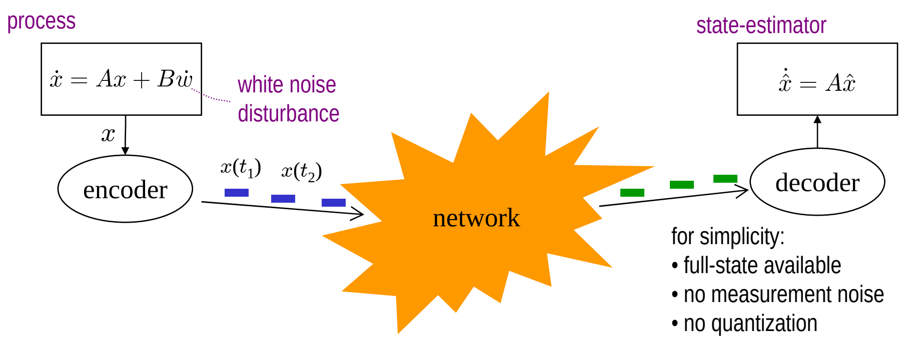
long-lived TCP flows with delay

on-off TCP flows with delay



[SIGMETRICS'03]

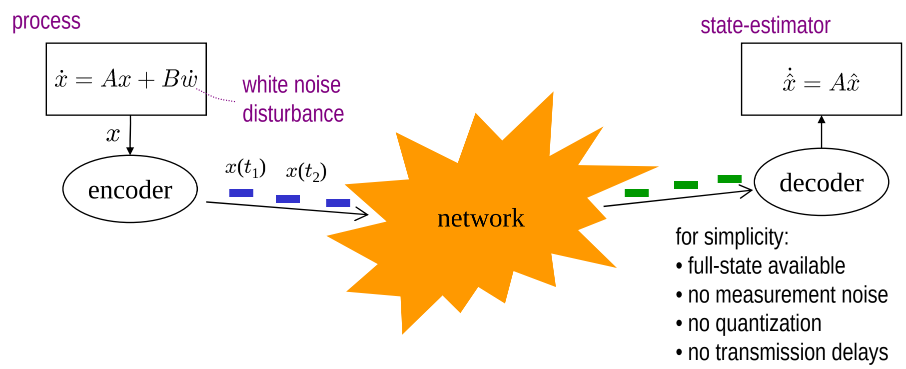
Example II: Estimation through network



for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

encoder logic ≡ determines *when* to send measurements to the network
decoder logic ≡ determines *how* to incorporate received measurements

Stochastic communication logic



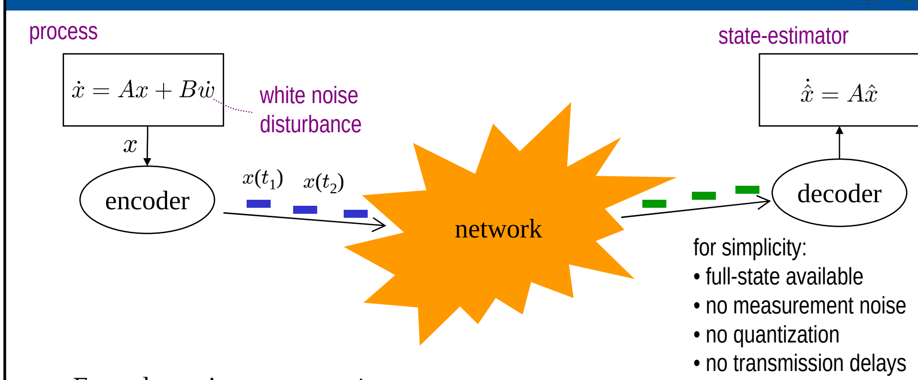
encoder logic ≡ determines *when* to send measurements to the network

1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from x

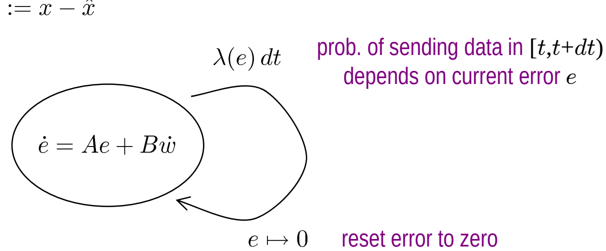
decoder logic ≡ determines *how* to incorporate received measurements

4. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

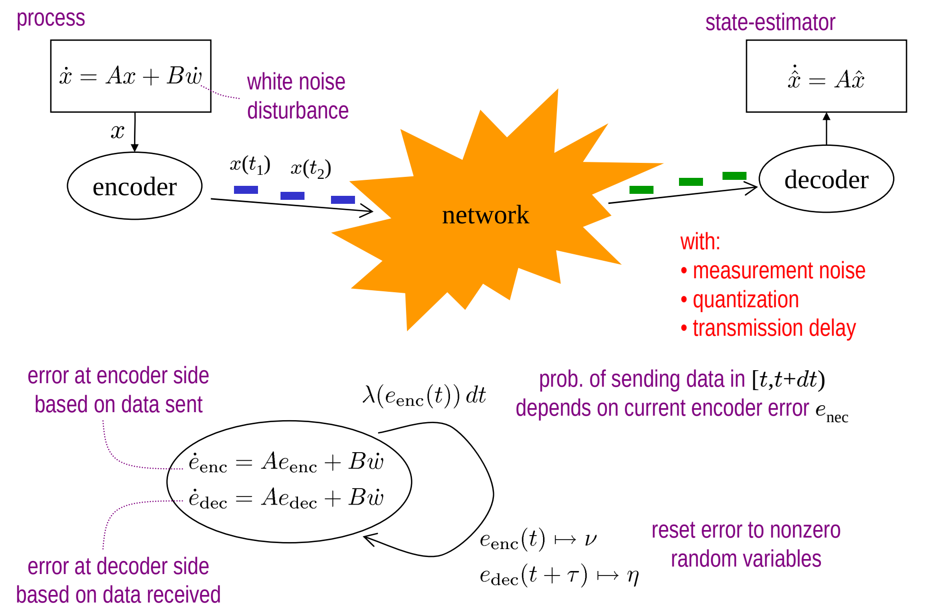
Stochastic communication logic



Error dynamics: $e := x - \hat{x}$



Stochastic communication logic

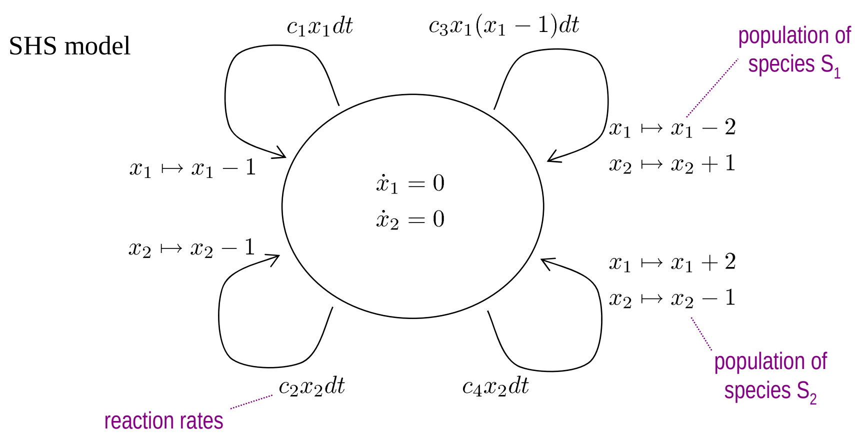


[ACTRA'04]

Example III: And now for something completely different…

Decaying-dimerizing chemical reactions (DDR):

$$S_1 \xrightarrow{c_1} 0 \qquad S_2 \xrightarrow{c_2} 0 \qquad 2\,S_1 \underset{c_4}{\overset{c_3}{\rightleftarrows}} S_2$$



Inspired by Gillespie's Stochastic Simulation Algorithm for molecular reactions [Gillespie, 76]

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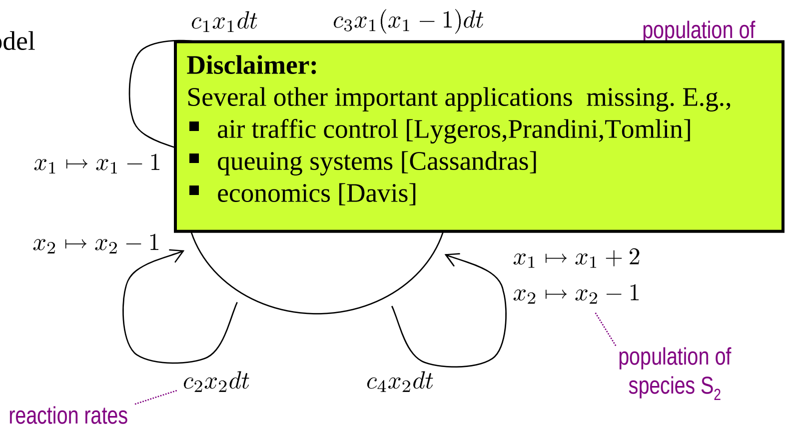
SHS model

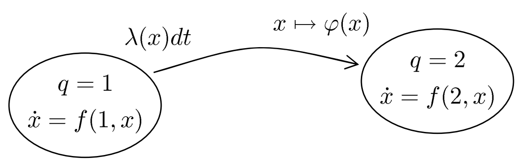


Disclaimer:
Several other important applications missing. E.g.,

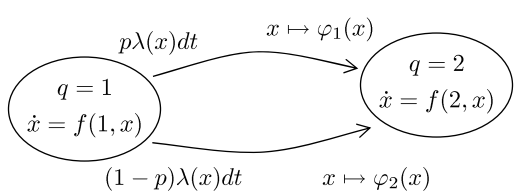
- air traffic control [Lygeros,Prandini,Tomlin]
- queuing systems [Cassandras]
- economics [Davis]

Inspired by Gillespie's Stochastic Simulation Algorithm for molecular reactions [Gillespie, 76]

Generalizations of the SHS model



1. *Stochastic resets* can be obtained by considering multiple intensities/reset-maps

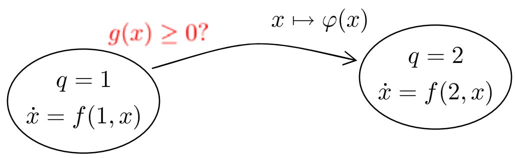


$$x \mapsto \begin{cases} \varphi_1(x) & \text{w.p. } p \\ \varphi_2(x) & \text{w.p. } 1-p \end{cases}$$

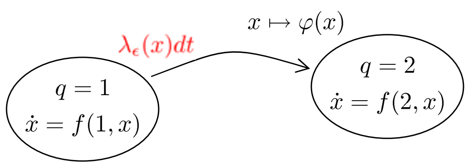
One can further generalize this to resets governed by to a continuous distribution [see paper]

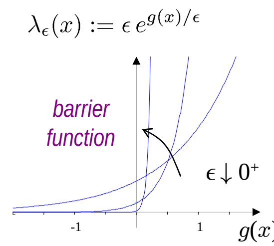
$$x \sim \mu(q, x, dx)$$

Generalizations of the SHS model



2. *Deterministic guards* can also be emulated by taking limits of SHSs

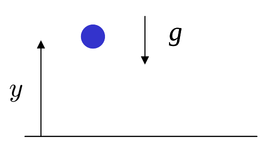


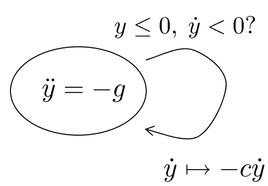


The solution to the hybrid system with a deterministic guard is obtained as $\epsilon \downarrow 0^+$

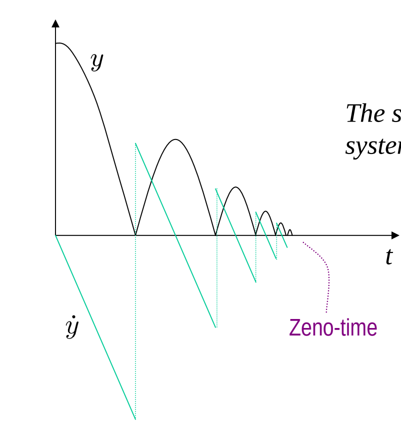
This provides a mechanism to regularize systems with chattering and/or Zeno phenomena…

Example: Bouncing-ball



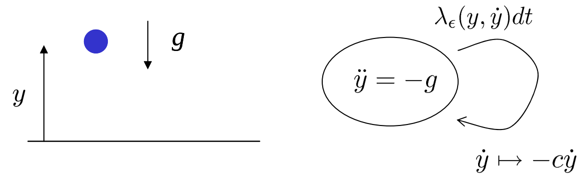


$c \in (0,1) \equiv$ energy absorbed at impact



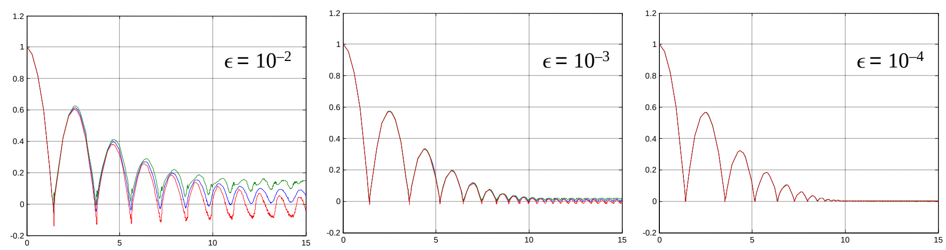
The solution of this deterministic hybrid system is only defined up to the Zeno-time

Example: Bouncing-ball



$$\lambda_\epsilon(y,\dot y) := \begin{cases} \epsilon e^{-y/\epsilon} & \dot y < 0 \\ 0 & \dot y > 0 \end{cases}$$

$c \in (0,1) \equiv$ energy absorbed at impact



mean (blue) 95% confidence intervals (red and green)

Analysis—Lie Derivative

$$\dot x = f(x,t) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $\psi : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(x,t) = \underbrace{\frac{\partial \psi}{\partial x} f(x,t) + \frac{\partial \psi}{\partial t}}_{L_f \psi}$$

derivative along solution to ODE

$L_f \psi$ Lie derivative of ψ

One can view L_f as an operator

space of scalar functions on $\mathbb{R}^n \times [0,\infty)$	$\to$	space of scalar functions on $\mathbb{R}^n \times [0,\infty)$
$\psi(x, t)$	$\mapsto$	$L_f \psi(x, t)$

L_f completely defines the system dynamics

Generator of a SHS

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$\dot{x} = f(q,x,t) + g(q,x,t)\dot{w}$ — continuous dynamics

$\lambda_\ell(q,x,t)$ — transition intensities

$(q,x) = \phi_\ell(q^-, x^-, t)$ — reset-maps

Given scalar-valued function $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}[\psi(q,x,t)] = \mathrm{E}\Big[(L\psi)(q,x,t)\Big]$$

generator for the SHS (L); Dynkin's formula (in differential form)

where

$$(L\psi)(q,x,t) := \frac{\partial \psi}{\partial x} f(q,x,t) + \frac{\partial \psi}{\partial t}$$

Lie derivative

$$+ \sum_{\ell=1}^{m} \Big(\underbrace{\psi\big(\phi_\ell(q,x,t),t\big) - \psi(q,x,t)}_{\text{instantaneous variation}}\Big)\underbrace{\lambda_\ell(q,x,t)}_{\text{intensity}}$$

reset term

$$+ \frac{1}{2}\,\mathrm{trace}\Big(g(q,x,t)'\frac{\partial^2 \psi}{\partial x^2} g(q,x,t)\Big)$$

diffusion term

L completely defines the SHS dynamics

Disclaimer: see following paper for technical assumptions [HSCC'04]

Generator of the SHS with diffusion

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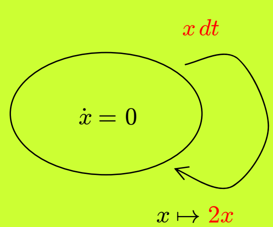
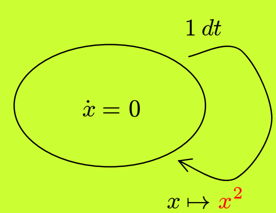
Attention:
These systems may have problems of existence of solution due to jumps!
(stochastic Zeno)

E.g.



no *global* solution

"jumping makes jumping more likely"



no *local* solution for $x(0) > 1$

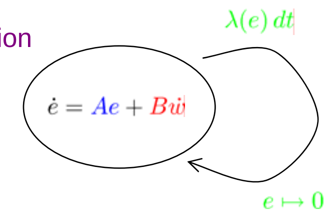
"probability of multiple jumps in short interval not sufficiently small"

L completely defines the SHS dynamics

Disclaimer: see following paper for technical assumptions [HSCC'04]

Stochastic communication logics

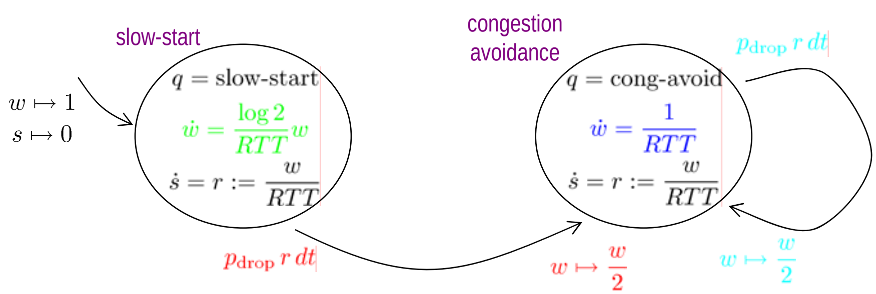
error dynamics
in remote estimation



$$(L\psi)(e,t) = \frac{\partial \psi}{\partial e} Ae + \frac{\partial \psi}{\partial t} + \Big[\psi(0,t) - \psi(e,t)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\left(B' \frac{\partial^2 \psi}{\partial e^2} B\right)$$

Long-lived TCP flows

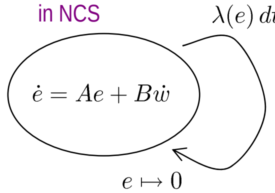
long-lived TCP flows
(with slow start)



$$(L\psi)(q,w,t) = \begin{cases} \frac{\partial \psi}{\partial w} \frac{\log 2}{RTT} w + \frac{\partial \psi}{\partial t} + \left[\psi\left(\text{ca}, \frac{w}{2}\right) - \psi(\text{ss}, w)\right] \frac{p_{\text{drop}} w}{RTT} & q = \text{ss} \\ \frac{\partial \psi}{\partial w} \frac{1}{RTT} + \frac{\partial \psi}{\partial t} + \left[\psi\left(\text{ca}, \frac{w}{2}\right) - \psi(\text{ss}, w)\right] \frac{p_{\text{drop}} w}{RTT} & q = \text{ca} \end{cases}$$

Lyapunov-based stability analysis

error dynamics in NCS



$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\psi(e)] = \mathrm{E}\Big[(L\psi)(e)\Big]$$ Dynkin's formula

$$(L\psi)(e) = \frac{\partial\psi}{\partial e}Ae + \Big[\psi(0) - \psi(e)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\Big(B'\frac{\partial^2\psi}{\partial e^2}B\Big)$$

Expected value of error:

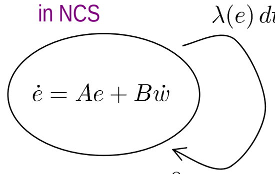
$$\psi(e) = e \quad \Rightarrow \quad (L\psi)(e) = \big(A - \lambda(e)I\big)e$$

2nd moment of the error:

$$\psi(e) = e'Pe \;\Rightarrow\; (L\psi)(e) = e'\Big[\Big(A - \frac{\lambda(e)}{2}I\Big)'P + P\Big(A - \frac{\lambda(e)}{2}I\Big)\Big]e + \mathrm{trace}\Big(B'PB\Big)$$

Lyapunov-based stability analysis

error dynamics in NCS



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For constant rate: $\lambda(e) = \gamma$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[e] = (A - \gamma I)\,\mathrm{E}[e] \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[e'Pe] \le -\mu\,\mathrm{E}[e'Pe] + c, \quad \mu, c > 0$$

assuming $(A - \gamma/2\, I)$ Hurwitz

Lyapunov-based stability analysis

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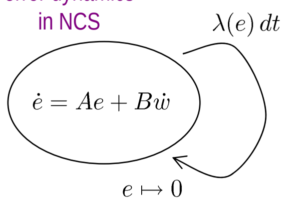
error dynamics
in NCS



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$$(L\psi)(e) = \frac{\partial \psi}{\partial e} Ae + \Big[\psi(0) - \psi(e)\Big]\lambda(e) + \frac{1}{2}\,\mathrm{trace}\left(B' \frac{\partial^2 \psi}{\partial e^2} B\right)$$

One can show…

For constant rate: $\lambda(e) = \gamma$

both always true $\forall\, \gamma \geq 0$ if A Hurwitz (no jumps needed for boundedness)

1. $\mathrm{E}[\,e\,] \to 0$ as long as $\gamma > \Re[\lambda(A)]$
2. $\mathrm{E}[\,\|e\|^m\,]$ bounded as long as $\gamma > 2\,m\,\Re[\lambda(A)]$

getting more moments bounded requires higher jump intensities

For polynomial rates: $\lambda(e) = (e'\,P\,e)^k \quad P > 0,\ k \geq 0$

1. $\mathrm{E}[\,e\,] \to 0$ (always)
2. $\mathrm{E}[\,\|e\|^m\,]$ bounded $\forall\, m$

Moreover, one can achieve the same $\mathrm{E}[\,\|e\|^2\,]$ with a smaller number of transmissions…

[ACTRA'04, CDC'04]

Analysis—Moments for SHS state

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$\dot{x} = f(q, x, t)$ — continuous dynamics

$\lambda_\ell(q, x, t)$ — transition intensities

$(q, x) = \phi_\ell(q^-, x^-, t)$ — reset-maps

$\mathbf{z}$ (scalar) random variable with mean μ and variance σ^2

$$\mathrm{P}\left(\mathbf{z} \geq \epsilon \;\middle|\; \mathbf{z} \geq 0\right) \leq \frac{\mu}{\epsilon}$$

Markov inequality
($\forall\, \epsilon > 0$)

$$\mathrm{P}\left(|\mathbf{z} - a| \geq \epsilon\right) \leq \frac{\mathrm{E}[|\mathbf{z} - a|^n]}{\epsilon^n}$$

Bienaymé inequality
($\forall\, \epsilon > 0,\ a \in \mathbb{R},\ n \in \mathbb{N}$)

$$\mathrm{P}\left(|\mathbf{z} - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2}$$

Tchebychev inequality

often a few low-order moments suffice to study a SHS…

Polynomial SHSs

$\dot{x} = f(q,x,t) + g(q,x,t)\dot{w}$ — continuous dynamics

$\lambda_\ell(q,x,t)$ — transition intensities

$(q,x) = \phi_\ell(q^-, x^-, t)$ — reset-maps

Given scalar-valued function $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\psi(q,x,t)] = \mathrm{E}\Big[(L\psi)(q,x,t)\Big]$$

generator for the SHS

Dynkin's formula (in differential form)

where

$$(L\psi)(q,x,t) := \frac{\partial \psi}{\partial x} f(q,x,t) + \frac{\partial \psi}{\partial t} + \sum_{\ell=1}^{m}\Big(\psi(\phi_\ell(q,x,t),t) - \psi(q,x,t)\Big)\lambda_\ell(q,x,t) + \frac{1}{2}\operatorname{trace}\Big(g(q,x,t)'\frac{\partial^2\psi}{\partial x^2}g(q,x,t)\Big)$$

A SHS is called a *polynomial SHS* (pSHS) if its generator maps finite-order polynomial on x into finite-order polynomials on x

Typically, when

$$x \mapsto f(q,x,t) \quad x \mapsto g(q,x,t) \quad x \mapsto \lambda_\ell(q,x,t) \quad x \mapsto \phi_\ell(q,x,t)$$

are all polynomials $\forall\, q, t$

Moment dynamics for pSHS

$x(t) \in \mathbb{R}^n$ — continuous state

$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ — discrete state

Monomial test function: Given $\bar{q} \in \mathcal{Q}$ $\quad m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$

$$\psi_{\bar{q}}^{(m)}(q,x) := \begin{cases} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} & q = \bar{q} \\ 0 & q \neq \bar{q}, \end{cases}$$

for short $x^{(m)}$

Uncentered moment:

$$\mu_{\bar{q}}^{(m)}(t) := \mathrm{E}\left[\psi_{\bar{q}}^{(m)}(q(t),x(t))\right]$$

E.g,

$$\mu_{q_1}^{(0,0,0,\ldots,0)}(t) = \mathrm{P}\big(q(t) = q_1\big) \qquad \mu_{q_1}^{(1,1,0,\ldots,0)}(t) = \mathrm{E}\left[x_1(t)x_2(t)I_{q(t)=q_1}\right]$$

$$\mu_{q_1}^{(0,1,0,\ldots,0)}(t) = \mathrm{E}\left[x_2(t)I_{q(t)=q_1}\right] \qquad \mu_{q_1}^{(2,0,0,\ldots,0)}(t) = \mathrm{E}\left[x_1(t)^2 I_{q(t)=q_1}\right]$$

Moment dynamics for pSHS

$x(t) \in \mathbb{R}^n$ — continuous state

$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ — discrete state

Monomial test function: Given $\bar{q} \in \mathcal{Q}$ $\quad m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n_{\geq 0}$

$$\psi_{\bar{q}}^{(m)}(q,x) := \begin{cases} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} & q = \bar{q} \\ 0 & q \neq \bar{q}, \end{cases}$$

for short $x^{(m)}$

Uncentered moment:

$$\mu_{\bar{q}}^{(m)}(t) := \mathrm{E}\left[\psi_{\bar{q}}^{(m)}\big(q(t), x(t)\big)\right]$$

For polynomial SHS…

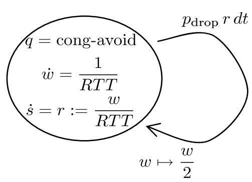
$\psi_{\bar{q}}^{(m)}(q,x)$ monomial on x $\Rightarrow$ $L\psi_{\bar{q}}^{(m)}(q,x)$ polynomial on x $\Rightarrow$ $L\psi_{\bar{q}}^{(m)}(q,x)$ linear comb. of monomial test functions

$$\dot{\mu}_{\bar{q}}^{(m)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\psi_{\bar{q}}^{(m)}\big(q,x\big)] = \mathrm{E}\left[(L\psi_{\bar{q}}^{(m)})(q,x)\right] = \sum_{i=1}^{k} \alpha_i \mu_{q_i}^{(m_i)}$$

linear moment dynamics

Moment dynamics for TCP

long-lived TCP flows

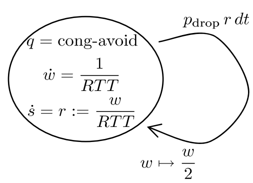


$$\mu^{(k)} := \mathrm{E}[r^k] = \mathrm{E}\left[\frac{w^k}{RTT^k}\right]$$

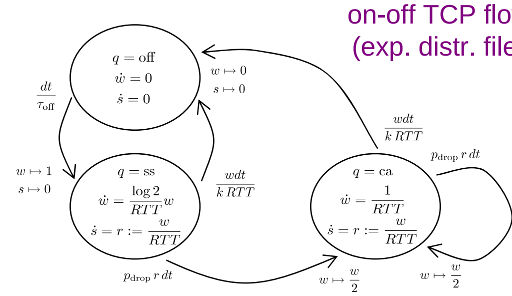
$$\dot{\mu}^{(1)} = \frac{1}{RTT^2} - \frac{p}{2}\mu^{(2)}$$
$$\dot{\mu}^{(2)} = \frac{2}{RTT^2}\mu^{(1)} - \frac{3p}{4}\mu^{(3)}$$
$$\dot{\mu}^{(3)} = \frac{3}{RTT^2}\mu^{(2)} - \frac{7p}{8}\mu^{(4)}$$
$$\vdots$$

Moment dynamics for TCP

long-lived TCP flows



on-off TCP flows (exp. distr. files)



$$\mu^{(k)} := \mathrm{E}[r^k] = \mathrm{E}\left[\frac{w^k}{RTT^k}\right]$$

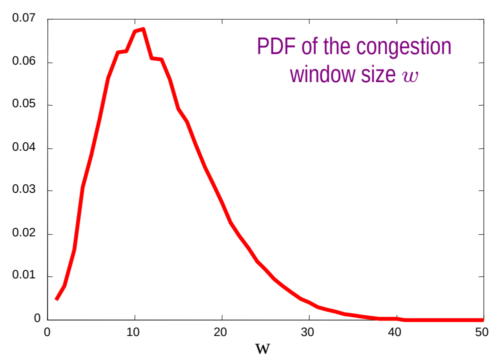
$$\dot\mu^{(1)} = \frac{1}{RTT^2} - \frac{p}{2}\mu^{(2)}$$
$$\dot\mu^{(2)} = \frac{2}{RTT^2}\mu^{(1)} - \frac{3p}{4}\mu^{(3)}$$
$$\dot\mu^{(3)} = \frac{3}{RTT^2}\mu^{(2)} - \frac{7p}{8}\mu^{(4)}$$
$$\vdots$$

$$\mu_{\text{ca}}^{(k)} := \mathrm{E}[r^k I_{q=\text{ca}}] = \mathrm{E}\left[\frac{w^k}{RTT^k} I_{q=\text{ca}}\right] \cdots$$

$$\begin{bmatrix} \dot\mu_{\text{off}}^{(0)} \\ \dot\mu_{\text{ss}}^{(0)} \\ \dot\mu_{\text{ca}}^{(0)} \\ \dot\mu_{\text{ss}}^{(1)} \\ \dot\mu_{\text{ca}}^{(1)} \\ \dot\mu_{\text{ss}}^{(2)} \\ \dot\mu_{\text{ca}}^{(2)} \\ \vdots \end{bmatrix} = \begin{bmatrix} -\tau_{\text{off}}^{-1}\mu_{\text{off}}^{(0)} + \frac{1}{k}\mu_{\text{ss}}^{(1)} + \frac{1}{k}\mu_{\text{ca}}^{(1)} \\ \tau_{\text{off}}^{-1}\mu_{\text{off}}^{(0)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(1)} \\ p\mu_{\text{ss}}^{(1)} - \frac{1}{k}\mu_{\text{ca}}^{(1)} \\ \frac{\tau_{\text{off}}^{-1} w_0}{RTT}\mu_{\text{off}}^{(0)} + \frac{\log 2}{RTT}\mu_{\text{ss}}^{(1)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(2)} \\ \frac{1}{RTT^2}\mu_{\text{ca}}^{(0)} + \frac{p}{2}\mu_{\text{ss}}^{(2)} - (\frac{1}{k}+\frac{p}{2})\mu_{\text{ca}}^{(2)} \\ \frac{\tau_{\text{off}}^{-1} w_0^2}{RTT^2}\mu_{\text{off}}^{(0)} + \frac{\log 4}{RTT}\mu_{\text{ss}}^{(2)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(3)} \\ \frac{2}{RTT^2}\mu_{\text{ca}}^{(1)} + \frac{p}{4}\mu_{\text{ss}}^{(3)} - (\frac{1}{k}+\frac{3p}{4})\mu_{\text{ca}}^{(3)} \\ \vdots \end{bmatrix}$$

Truncated moment dynamics

Experimental evidence indicates that the (steady-state) sending rate is well approximated by a Log-Normal distribution



z Log-Normal $\Rightarrow$ $\mathrm{E}[z^3] = \left(\frac{\mathrm{E}[z^2]}{\mathrm{E}[z]}\right)^3$

r Log-Normal (on each mode) $\Rightarrow$ $\mu_{\bar q,3} = \mathrm{E}[r^3 \mid q=\bar q]\mu_{\bar q,0} \approx \frac{\mathrm{E}[r^2 \mid q=\bar q]^3}{\mathrm{E}[r \mid q=\bar q]^3}\mu_{\bar q,0} = \mu_{\bar q,0}\left(\frac{\mu_{\bar q,2}}{\mu_{\bar q,1}}\right)^3$

Data from: Bohacek, A stochastic model for TCP and fair video transmission, INFOCOM'03

Moment dynamics for TCP

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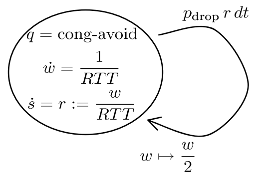
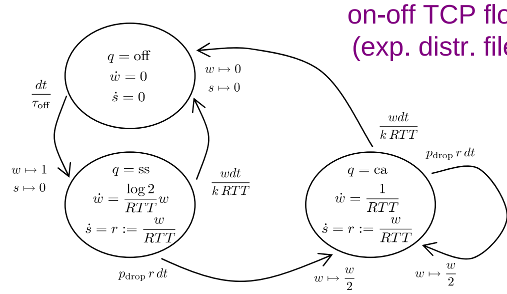
long-lived TCP flows



on-off TCP flows (exp. distr. files)



$$\mu^{(k)} := \mathrm{E}[r^k] = \mathrm{E}\left[\frac{w^k}{RTT^k}\right]$$

$$\mu_{\text{ca}}^{(k)} := \mathrm{E}[r^k I_{q=\text{ca}}] = \mathrm{E}\left[\frac{w^k}{RTT^k} I_{q=\text{ca}}\right] \cdots$$

$$\dot{\mu}^{(1)} = \frac{1}{RTT^2} - \frac{p}{2}\mu^{(1)}$$

$$\dot{\mu}^{(2)} = \frac{2}{RTT^2}\mu^{(1)} - \frac{3p}{4}\left(\frac{\mu^{(2)}}{\mu^{(1)}}\right)^3$$

$$\begin{bmatrix} \dot{\mu}_{\text{off}}^{(0)} \\ \dot{\mu}_{\text{ss}}^{(0)} \\ \dot{\mu}_{\text{ca}}^{(0)} \\ \dot{\mu}_{\text{ss}}^{(1)} \\ \dot{\mu}_{\text{ca}}^{(1)} \\ \dot{\mu}_{\text{ss}}^{(2)} \\ \dot{\mu}_{\text{ca}}^{(2)} \end{bmatrix} = \begin{bmatrix} -\tau_{\text{off}}^{-1}\mu_{\text{off}}^{(0)} + \frac{1}{k}\mu_{\text{ss}}^{(1)} + \frac{1}{k}\mu_{\text{ca}}^{(1)} \\ \tau_{\text{off}}^{-1}\mu_{\text{off}}^{(0)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(1)} \\ p\mu_{\text{ss}}^{(1)} - \frac{1}{k}\mu_{\text{ca}}^{(1)} \\ \frac{\tau_{\text{off}}^{-1} w_0}{RTT}\mu_{\text{off}}^{(0)} + \frac{\log 2}{RTT}\mu_{\text{ss}}^{(1)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(2)} \\ \frac{1}{RTT^2}\mu_{\text{ca}}^{(0)} + \frac{p}{2}\mu_{\text{ss}}^{(2)} - (\frac{1}{k}+\frac{p}{2})\mu_{\text{ca}}^{(2)} \\ \frac{\tau_{\text{off}}^{-1} w_0^2}{RTT^2}\mu_{\text{off}}^{(0)} + \frac{\log 4}{RTT}\mu_{\text{ss}}^{(2)} - (\frac{1}{k}+p)\mu_{\text{ss}}^{(0)}(\frac{\mu_{\text{ss}}^{(2)}}{\mu_{\text{ss}}^{(1)}})^3 \\ \frac{2}{RTT^2}\mu_{\text{ca}}^{(1)} + \frac{p}{4}\mu_{\text{ss}}^{(0)}(\frac{\mu_{\text{ss}}^{(2)}}{\mu_{\text{ss}}^{(1)}})^3 - (\frac{1}{k}+\frac{3p}{4})\mu_{\text{ca}}^{(0)}(\frac{\mu_{\text{ca}}^{(2)}}{\mu_{\text{ca}}^{(1)}})^3 \end{bmatrix}$$

finite-dimensional nonlinear ODEs

Long-lived TCP flows

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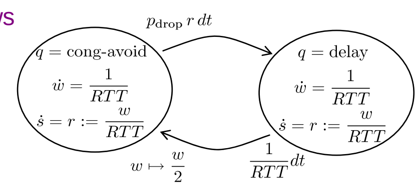
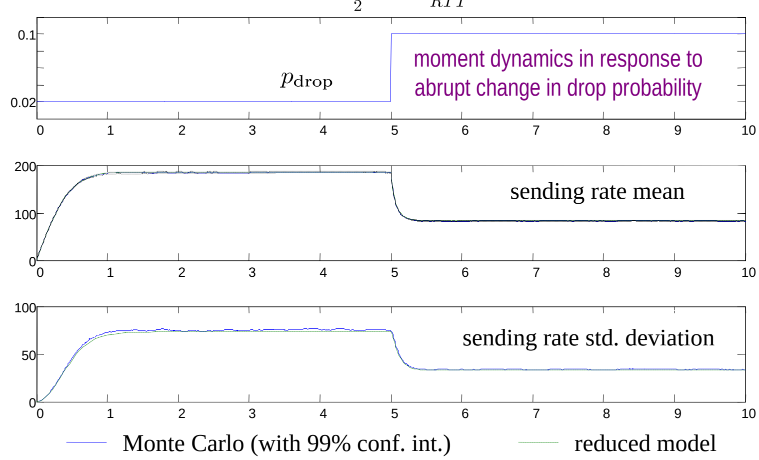
long-lived TCP flows with delay (one RTT average)





Truncated moment dynamics (revisited)

For polynomial SHS…

$$\dot{\mu}_{\bar q}^{(m)} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{E}[\psi_{\bar q}^{(m)}(q,x)] = \mathrm{E}\left[(L\psi_{\bar q}^{(m)})(q,x)\right] = \sum_{i=1}^{k} \alpha_i \mu_{q_i}^{(m_i)}$$

linear moment dynamics

Stacking all moments into an (infinite) vector μ_∞

$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

infinite-dimensional linear ODE

In TCP analysis…

$$\mu_\infty = \begin{bmatrix} \mu_{\mathrm{ss}}^{(0)} \\ \mu_{\mathrm{ca}}^{(0)} \\ \mu_{\mathrm{ss}}^{(1)} \\ \vdots \\ \mu_{\mathrm{ss}}^{(3)} \\ \mu_{\mathrm{ca}}^{(3)} \\ \vdots \end{bmatrix}$$

μ: lower order moments of interest

$\bar\mu$: moments of interest that affect μ dynamics

$$\dot\mu = A\mu + B\bar\mu$$

$\bar\mu$ approximated by nonlinear function of μ

Truncation by derivative matching

$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

infinite-dimensional linear ODE

$$\dot\mu = A\mu + B\bar\mu$$

truncated linear ODE (nonautonomous, not nec. stable)

$$\dot\nu = A\nu + B\varphi(\nu)$$

nonlinear approximate moment dynamics

Assumption: 1) μ and ν remain bounded along solutions to

$$\dot{\mu}_\infty = A_\infty \mu_\infty \quad \text{and} \quad \dot\nu = A\nu + B\varphi(\nu)$$

2) $\dot{\mu}_\infty = A_\infty \mu_\infty$ is (incrementally) asymptotically stable

Theorem: $\forall\, \delta > 0\ \exists\, N$ s.t. if $\dfrac{\mathrm{d}^k\mu}{\mathrm{d}t^k} = \dfrac{\mathrm{d}^k\nu}{\mathrm{d}t^k}, \quad \forall k \in \{1,\ldots,N\}$

then

$$\|\mu(t)-\nu(t)\| \le \beta(\|\mu(t_0)-\nu(t_0)\|, t-t_0) + \delta, \qquad \forall t \ge t_0 \ge 0$$

class $\mathcal{KL}$ function

Disclaimer: Just a lose statement. The "real" theorem is stated in [HSCC'05]

Truncation by derivative matching

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$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

infinite-dimensional linear ODE

$\dot{\mu} = A\mu + B\bar{\mu}$

truncated linear ODE (nonautonomous, not nec. stable)

$\dot{\nu} = A\nu + B\varphi(\nu)$

nonlinear approximate moment dynamics

Assumption: 1) μ and ν remain bounded along solutions to

$$\dot{\mu}_\infty = A_\infty \mu_\infty \quad \text{and} \quad \dot{\nu} = A\nu + B\varphi(\nu)$$

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Theorem: $\forall\, \delta > 0\ \exists\, N$ s.t. if $\frac{\mathrm{d}^k \mu}{\mathrm{d}t^k} = \frac{\mathrm{d}^k \nu}{\mathrm{d}t^k}, \quad \forall k \in \{1, \ldots, N\}$

then

$$\|\mu(t) - \nu(t)\| \le \beta(\|\mu(t_0) - \nu(t_0)\|, t - t_0) + \delta, \qquad \forall t \ge t_0 \ge 0$$

class $\mathcal{KL}$ function

Proof idea:
1) N derivative matches $\Rightarrow$ μ & ν match on compact interval of length T
2) stability of A_∞ $\Rightarrow$ matching can be extended to $[0,\infty)$

Truncation by derivative matching

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infinite-dimensional linear ODE

☹ Given δ, finding N is very difficult

☺ In practice, small values of N (e.g., $N=2$) already yield good results

☺ Can use

$$\frac{\mathrm{d}^k \mu}{\mathrm{d}t^k} = \frac{\mathrm{d}^k \nu}{\mathrm{d}t^k}, \qquad \forall k \in \{1, \ldots, N\}$$

to determine $\varphi(\cdot)$: $k = 1 \rightarrow$ boundary condition on φ

$k = 2 \rightarrow$ linear PDE on φ

Assu

Theorem: $\forall\, \delta > 0\ \exists\, N$ s.t. if $\frac{\mathrm{d}^k \mu}{\mathrm{d}t^k} = \frac{\mathrm{d}^k \nu}{\mathrm{d}t^k}, \quad \forall k \in \{1, \ldots, N\}$

then

$$\|\mu(t) - \nu(t)\| \le \beta(\|\mu(t_0) - \nu(t_0)\|, t - t_0) + \delta, \qquad \forall t \ge t_0 \ge 0$$

class $\mathcal{KL}$ function

Proof idea:
1) N derivative matches $\Rightarrow$ μ & ν match on compact interval of length T
2) stability of A_∞ $\Rightarrow$ matching can be extended to $[0,\infty)$

Moment dynamics for DDR

Decaying-dimerizing molecular reactions (DDR):

$$S_1 \xrightarrow{c_1} 0 \qquad S_2 \xrightarrow{c_2} 0 \qquad 2\,S_1 \underset{c_4}{\overset{c_3}{\rightleftarrows}} S_2$$

$$\begin{bmatrix} \dot\mu^{(1,0)} \\ \dot\mu^{(0,1)} \\ \dot\mu^{(2,0)} \\ \dot\mu^{(0,2)} \\ \dot\mu^{(1,1)} \end{bmatrix} = \begin{bmatrix} -c_1+c_3 & 2c_4 & -c_3 & 0 & 0 \\ -\frac{c_3}{2} & -c_4-c_2 & \frac{c_3}{2} & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ c_1-2c_3 & 4c_4 & -2c_1+4c_3 & 0 & 4c_4 \\ -\frac{c_3}{2} & c_4+c_2 & \frac{c_3}{2} & -2c_4-2c_2 & -c_3 \\ c_3 & -2c_4 & -\frac{3c_3}{2} & 2c_4 & -c_1+c_3-c_4-c_2 \end{bmatrix} \begin{bmatrix} \mu^{(1,0)} \\ \mu^{(0,1)} \\ \mu^{(2,0)} \\ \mu^{(0,2)} \\ \mu^{(1,1)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2c_3 & 0 \\ 0 & c_3 \\ \frac{c_3}{2} & -c_3 \end{bmatrix} \begin{bmatrix} \mu^{(3,0)} \\ \mu^{(2,1)} \end{bmatrix}$$

$$\mu^{(1,0)} := \mathrm{E}[x_1] \quad \mu^{(2,0)} := \mathrm{E}[x_1^2] \quad \mu^{(3,0)} := \mathrm{E}[x_1^3]$$

$$\mu^{(0,1)} := \mathrm{E}[x_2] \quad \mu^{(0,2)} := \mathrm{E}[x_2^2] \quad \mu^{(2,1)} := \mathrm{E}[x_1^2 x_2]$$

$$\mu^{(1,1)} := \mathrm{E}[x_1 x_2]$$

Truncated DDR model

Decaying-dimerizing molecular reactions (DDR):

$$S_1 \xrightarrow{c_1} 0 \qquad S_2 \xrightarrow{c_2} 0 \qquad 2\,S_1 \underset{c_4}{\overset{c_3}{\rightleftarrows}} S_2$$

$$\begin{bmatrix} \dot\mu^{(1,0)} \\ \dot\mu^{(0,1)} \\ \dot\mu^{(2,0)} \\ \dot\mu^{(0,2)} \\ \dot\mu^{(1,1)} \end{bmatrix} \approx \begin{bmatrix} -c_1+c_3 & 2c_4 & -c_3 & 0 & 0 \\ -\frac{c_3}{2} & -c_4-c_2 & \frac{c_3}{2} & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ c_1-2c_3 & 4c_4 & -2c_1+4c_3 & 0 & 4c_4 \\ -\frac{c_3}{2} & c_4+c_2 & \frac{c_3}{2} & -2c_4-2c_2 & -c_3 \\ c_3 & -2c_4 & -\frac{3c_3}{2} & 2c_4 & -c_1+c_3-c_4-c_2 \end{bmatrix} \begin{bmatrix} \mu^{(1,0)} \\ \mu^{(0,1)} \\ \mu^{(2,0)} \\ \mu^{(0,2)} \\ \mu^{(1,1)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2c_3 & 0 \\ 0 & c_3 \\ \frac{c_3}{2} & -c_3 \end{bmatrix} \begin{bmatrix} \left(\frac{\mu^{(2,0)}}{\mu^{(1,0)}}\right)^3 \\ \frac{\mu^{(2,0)}}{\mu^{(0,1)}}\left(\frac{\mu^{(1,1)}}{\mu^{(1,0)}}\right)^2 \end{bmatrix}$$

by matching

$$\frac{d^k\mu}{dt^k} = \frac{d^k\nu}{dt^k}, \qquad \forall k \in \{1,2\}$$

for deterministic distributions

[IJRC, 2005]

Monte Carlo vs. truncated model

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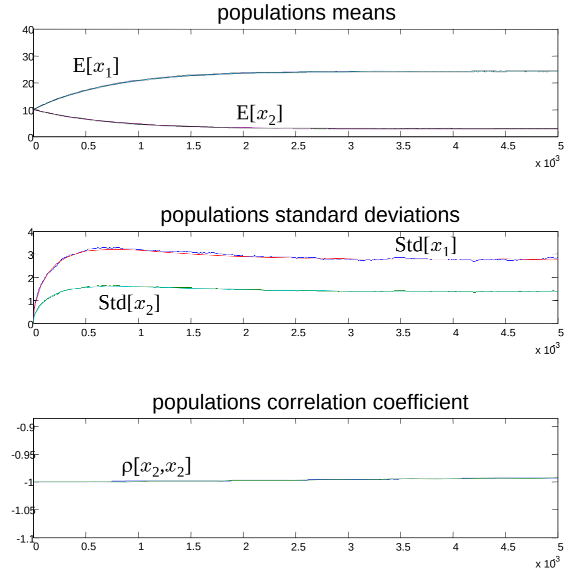


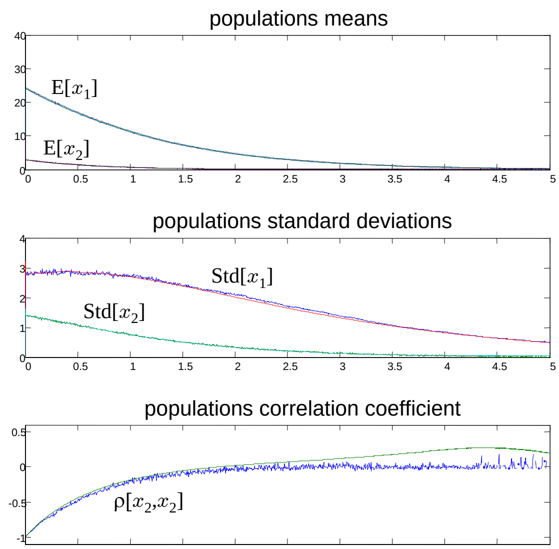
Fast time-scale
(transient)

(lines essentially
undistinguishable
at this scale)

Parameters from: Rathinam, Petzold, Cao, Gillespie, Stiffness in stochastic chemically reacting systems: The implicit tau-leaping method. J. of Chemical Physics, 2003

Monte Carlo vs. truncated model

UCSB



Slow time-scale
evolution

error only noticeable when
populations become very small
(a couple of molecules,
still adequate to study cellular
reactions)

Parameters from: Rathinam, Petzold, Cao, Gillespie, Stiffness in stochastic chemically reacting systems: The implicit tau-leaping method. J. of Chemical Physics, 2003

Conclusions

1. A simple SHS model (inspired by piecewise deterministic Markov Processes) can go a long way in modeling network traffic
2. The analysis of SHSs is generally difficult but there are tools available (generator, Lyapunov methods, moment dynamics, truncations)
3. This type of SHSs (and tools) finds use in several areas (traffic modeling, networked control systems, molecular biology, population dynamics in ecosystems)

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All papers (and some ppt presentations) available at
http://www.ece.ucsb.edu/~hespanha