Hybrid Control and Switched Systems

Lecture #16
Nonlinear Supervisory Control

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Summary

• Estimator-based nonlinear supervisory control
• Examples
**Motivation:** in the control of complex and highly uncertain systems, traditional methodologies based on a single controller do not provide satisfactory performance.

**Key ideas:**
1. Build a bank of alternative controllers
2. Switch among them online based on measurements

For simplicity we assume a stabilization problem, otherwise controllers should have a reference input $r$.

**Supervisor:**
- places in the feedback loop the controller that seems more promising based on the available measurements
- typically logic-based/hybrid system
Estimator-based nonlinear supervisory control

Class of admissible processes: Example #4

Process is assumed to be in the family
\[ \dot{\alpha} = a\alpha + b\beta \]
\[ \dot{\beta} = u \]
where
\[ p = (a, b) \in \mathbb{W} := [-1,1] \times \{-1,1\} \]
**Class of admissible processes**

A process is assumed to be in a family of parametric uncertainties:

\[ \mathcal{M} := \bigcup_{p \in P} \mathcal{M}_p \]

where \( \mathcal{M}_p \) is the small family of systems around a nominal process model \( N_p \).

Typically, \( \mathcal{M}_p := \{ M_p : d(M_p, N_p) \leq \epsilon_p \} \)

Most results presented here:
- independent of metric \( d \) (e.g., detectability)
- or restricted to case \( \epsilon_p = 0 \) (e.g., matching)

**Candidate controllers: Example #4**

A process is assumed to be in the family

\[ \dot{\alpha} = a\alpha^3 + b\beta \]
\[ \dot{\beta} = u \]
\[ y = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad p = (a, b) \in \mathcal{W} := [-1,1] \times \{-1,1\} \]

The state is accessible

To facilitate the controller design, one can first “back-step” the system to simplify its stabilization:

Virtual input:
\[ \dot{\alpha} = -\alpha + \gamma \quad \gamma := a + a\alpha^3 + b\beta \]
\[ \dot{\beta} = \gamma + b(u - \Psi_p(\alpha, \gamma)) \]
\[ y := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \frac{1}{b}(\gamma - a - a\alpha^3) \end{bmatrix} \]

After the coordinate transformation the new state is no longer accessible

Now the control law:
\[ u = \Psi_p(\alpha, \gamma) \]

stabilizes the system

**Candidate controllers**

\[ u = \Psi_p(\alpha, \gamma) = \Psi_p(\alpha, a + a\alpha^3 + b\beta) = -\frac{(a\alpha^3 + b\beta)(1 + 3a\alpha^2)}{b} \]
Candidate controllers

Class of admissible processes

\[ \mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p \quad T_p \equiv \text{small family of systems around a nominal process model } N_p \]

Assume given a family of candidate controllers

\[ \mathcal{C} := \bigg\{ \dot{z}_q = F_q(z_q, y), \ u = G_q(z_q, y) : q \in Q \bigg\} \]

(without loss of generality all with same dimension)

Multi-controller:

\[ \begin{align*}
\dot{x}_C &= F_\sigma(x_C, y) \\
u &= G_\sigma(x_C, y)
\end{align*} \]

Multi-estimator

\[ \begin{align*}
x_E &= A_p(x_E, y, u) \\
y_p &= C_p(x_E) \quad p \in \mathcal{P}
\end{align*} \]

How to design a multi-estimator?

we want: Matching property: there exist some \( p^* \in \mathcal{W} \) such that \( e_{p^*} \) is “small”

Typically obtained by:

process in \( \mathcal{M} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p \) \[ \implies \exists p^* \in \mathcal{W}: \text{process in } T_{p^*} \quad e_{p^*} \text{ is “small”} \]

when process “matches” \( T_{p^*} \), the corresponding error must be “small”
Candidate controllers: Example #4

Process is assumed to be
\[ \dot{\alpha} = -\alpha + \gamma \]
\[ \dot{\gamma} = -\gamma + b^*(u - \Psi_p(\alpha, \gamma)) \]
\[ y := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{b^*}(\gamma - \alpha - a^*\alpha^3) \]
\[
\begin{align*}
\gamma := \alpha + a^*\alpha^3 + b^*\beta \\
\Psi_p(\alpha, \gamma) := \frac{(\alpha - \gamma)(1 + 3a\alpha^2)}{b}
\end{align*}
\]

Multi-estimator:
\[
\begin{align*}
\dot{\alpha}_p &= -\alpha_p + (\alpha + a\alpha^3 + b\beta) \\
\dot{\gamma}_p &= -\gamma_p + b(u - \Psi_p(\alpha, \alpha + a\alpha^3 + b\beta)) \\
e_p &= \begin{bmatrix} \alpha_p - \alpha \\ \gamma_p - (\alpha + a\alpha^3 + b\beta) \end{bmatrix}
\end{align*}
\]

for \( p = p^* \) …
\[
\begin{align*}
e_p &= \begin{bmatrix} \alpha_p - \alpha \\ \gamma_p - (\alpha + a\alpha^3 + b\beta) \end{bmatrix} \Rightarrow \dot{e}_{p^*} &= -e_{p^*} \Rightarrow e_{p^*} \to 0
\end{align*}
\]

\( \gamma \) for \( p = p^* \)

Designing multi-estimators - I

Suppose nominal models \( N_p, p \in W \) are of the form
\[
\begin{align*}
z &= A_p(z, u) \\
y &= z \\
p \in P
\end{align*}
\]

\( \text{no exogenous} \)
\( \text{input} \ w \)
\( \text{state} \)
\( \text{accessible} \)

Multi-estimator:
\[
\begin{align*}
\dot{z}_p &= A(z_p - y) + A_p(y, u) \\
y_p &= z_p \\
p \in P
\end{align*}
\]

\( \text{asymptotically} \)
\( \text{stable} \ A \)

When process matches the nominal model \( N_{p^*} \), exponentially
\[
\begin{align*}
e_{p^*} := y_{p^*} - y = z_{p^*} - z \Rightarrow \dot{e}_{p^*} &= A\epsilon_{p^*} \Rightarrow \epsilon_{p^*} \to 0 \text{ as } t \to \infty
\end{align*}
\]

Matching property: Assume \( T = \{ N_p : p \in W \} \)
\[
\exists p^* \in W, \epsilon_{p^*} > 0 : \quad \| e_{p^*}(t) \| \leq c_0 e^{\lambda^* t}, \quad t \geq 0
\]
Designing multi-estimators - II

Suppose nominal models $N_p, p \in \mathbb{W}$ are of the form

\[
\dot{z} = A_p z + B_p w + H_p(y, u) \quad \text{asymptotically stable } A_p \quad y = C_p z + D_p w \quad p \in \mathcal{P}
\]


\[
\begin{align*}
\text{Multi-estimator:} & \\
\dot{z}_p &= A_p z_p + H_p(y, u) \quad y_p = C_p z_p \quad p \in \mathcal{P}
\end{align*}
\]

When process matches the nominal model $N_{p^*}$

\[
\begin{align*}
\dot{z}_{p^*} := \dot{z}_p - z \quad &\Rightarrow \quad \dot{z}_{p^*} = A_p \dot{z}_{p^*} - B_p w \\
\end{align*}
\]

**Matching property:** Assume $\mathcal{T} = \{ N_p : p \in \mathbb{W} \}$

\[
\exists p^* \in \mathbb{W}, c_w, c_{\lambda^*} > 0 : \quad \| e_{p^*}(t) \| \leq c_{w} e^{\lambda^* t} + c_w \quad t \geq 0
\]

with $c_w = 0$ in case $w(t) = 0, \forall t \geq 0$

State-sharing is possible when all $A_p$ are equal and $H_p(y, u)$ is separable:

\[
H_p(y, u) = M(y, u) k(p) \quad \forall p, u, y
\]

Designing multi-estimators - III

Suppose nominal models $N_p, p \in \mathbb{W}$ are of the form

\[
\begin{align*}
\ddot{z} = \zeta_p(\dot{z}) \left( A_p \xi_p^{-1}(\dot{z}) + B_p w + H_p(y, u) \right) \quad y = C_p \xi_p^{-1}(\dot{z}) + D_p w & \quad p \in \mathcal{P}
\end{align*}
\]

\[
\begin{align*}
\text{asymptotically stable } A_p \quad &\Rightarrow \quad \ddot{z} = \zeta_p(\dot{z}) = \zeta_{p^*}(\dot{z}) = \zeta_{p^*} \circ \zeta_p^{-1}
\end{align*}
\]

The Matching property is an input/output property so the same multi-estimator can be used:

\[
\begin{align*}
\dot{z}_p &= A_p z_p + H_p(y, u) \quad y_p = C_p z_p \quad p \in \mathcal{P}
\end{align*}
\]

**Matching property:** Assume $\mathcal{T} = \{ N_p : p \in \mathbb{W} \}$

\[
\exists p^* \in \mathbb{W}, c_w, c_{\lambda^*} > 0 : \quad \| e_{p^*}(t) \| \leq c_{w} e^{\lambda^* t} + c_w \quad t \geq 0
\]

with $c_w = 0$ in case $w(t) = 0, \forall t \geq 0$
The switched system can be seen as the interconnection of the process with the “injected system” essentially the multi-controller & multi-estimator but now quite...

Constructing the injected system

1st Take a parameter estimate signal $\rho : [0, \infty) \rightarrow \mathbb{W}$.

2nd Define the signal $v := e_\rho = y_\rho - y$

3rd Replace $y$ in the equations of the multi-estimator and multi-controller by $y_\rho - v$.

\[
\begin{align*}
\dot{x}_E &= A_E(x_E, y_\rho - v, u) \\
y_p &= C_p(x_E) \quad p \in \mathcal{P} \\
\dot{x}_C &= F_\sigma(x_C, y_\rho - v) \\
u &= G_\sigma(x_C, y_\rho - v)
\end{align*}
\]
Switched system = process + injected system

Q: How to get “detectability” on the switched system?
A: “Stability” of the injected system

Stability & detectability of nonlinear systems

Stability: input $u$ “small” $\Rightarrow$ state $x$ “small”

$$\dot{x} = A(x, u) \quad A(0, 0) = ()$$

Input-to-state stable (ISS) if $\exists \beta \in \mathbb{R}_+, \gamma \in \mathbb{R}$

$$||x(t)|| \leq \beta(||x(0)||, t) + \sup_{\tau \in [0, t]} \gamma(||u(\tau)||) \quad \forall t \geq 0$$

Integral input-to-state stable (iISS) if $\exists \alpha \in \mathbb{R}_\infty, \beta \in \mathbb{R}_+, \gamma \in \mathbb{R}$

$$\alpha(||x(t)||) \leq \beta(||x(0)||, t) + \int_{\tau \in [0, t]} \gamma(||u(\tau)||) \quad \forall t \geq 0$$

Notation:
$\alpha: [0, \infty) \rightarrow [0, \infty)$ is class $\mathbb{R} \equiv$ continuous, strictly increasing, $\alpha(0) = 0$

is class $\mathbb{R}_\infty \equiv$ class $\mathbb{R}$ and unbounded

$\beta: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is class $\mathbb{R}_\mathbb{S} \equiv \beta(\cdot, t) \in \mathbb{R}$ for fixed $t$ &

$$\lim_{s \rightarrow \infty} \beta(s, t) = 0 \text{ (monotonically) for fixed } s$$
Stability & detectability of nonlinear systems

**Stability:** input $u$ “small” $\Rightarrow$ state $x$ “small”

\[
\dot{x} = A(x, u) \quad A(0, 0) = 0
\]

*Input-to-state stable (ISS)* if $\exists \beta \in \mathcal{R}, \gamma \in \mathcal{R}$

\[
\|x(t)\| \leq \beta(\|x(0)\|, t) + \sup_{\tau \in [0, t]} \gamma(\|u(\tau)\|) \quad \forall t \geq 0
\]

*Integral input-to-state stable (iISS)* if $\exists \alpha \in \mathcal{K}_\infty, \beta \in \mathcal{R}, \gamma \in \mathcal{R}$

\[
\alpha(\|x(t)\|) \leq \beta(\|x(0)\|, t) + \int_{\tau \in [0, t]} \gamma(\|u(\tau)\|) \quad \forall t \geq 0
\]

One can show:

1. for ISS systems: $u \to 0 \Rightarrow$ solution exist globally & $x \to 0$
2. for iISS systems: $\int_0^\infty \gamma(\|u\|) < \infty \Rightarrow$ solution exist globally & $x \to 0$

**Detectability:** input $u$ & output $y$ “small” $\Rightarrow$ state $x$ “small”

\[
\dot{x} = A(x, u) \quad y = C(x, u)
\]

*Detectability* (or input/output-to-state stability IOSS) if $\exists \beta \in \mathcal{R}, \gamma_u, \gamma_y \in \mathcal{R}$

\[
\|x(t)\| \leq \beta(\|x(0)\|, t) + \sup_{\tau \in [0, t]} \gamma_u(\|u(\tau)\|) + \sup_{\tau \in [0, t]} \gamma_y(\|y(\tau)\|) \quad \forall t \geq 0
\]

*Integral detectable (iIOSS)* if $\exists \alpha \in \mathcal{K}_\infty, \beta \in \mathcal{R}, \gamma_u, \gamma_y \in \mathcal{R}$

\[
\alpha(\|x(t)\|) \leq \beta(\|x(0)\|, t) + \int_{\tau \in [0, t]} \gamma_u(\|u(\tau)\|) + \int_{\tau \in [0, t]} \gamma_y(\|y(\tau)\|) \quad \forall t \geq 0
\]

One can show:

1. for IOSS systems: $u, y \to 0 \Rightarrow x \to 0$
2. for iIOSS systems: $\int_0^\infty \gamma_u(\|u\|), \int_0^\infty \gamma_y(\|y\|) < \infty \Rightarrow x \to 0$
Certainty Equivalence Stabilization Theorem

**Theorem:** (Certainty Equivalence Stabilization Theorem)
Suppose the process is detectable and take fixed $\rho = p \in W$ and $\sigma = q \in X$

1. injected system ISS $\Rightarrow$ switched system detectable.
2. injected system integral ISS $\Rightarrow$ switched system integral detectable

Stability of the injected system is not the only mechanism to achieve detectability:
e.g., injected system i/o stable + process “min. phase” $\Rightarrow$ detectability of switched system

(Nonlinear Certainty Equivalence Output Stabilization Theorem)

Achieving ISS for the injected system

**Theorem:** (Certainty Equivalence Stabilization Theorem)
Suppose the process is detectable and take fixed $\rho = p \in W$ and $\sigma = q \in X$

1. injected system ISS $\Rightarrow$ switched system detectable.
2. injected system integral ISS $\Rightarrow$ switched system integral detectable

We want to design candidate controllers that make the injected system
(at least) integral ISS with respect to the “disturbance” input $v$

- “disturbance” input $v$ can be measured ($v = e\rho = y\rho - y$)
- the whole state of the injected system is measurable ($x_C, x_E$)
Designing candidate controllers: Example #4

Multi-estimator:
\[
\begin{align*}
\dot{\alpha}_p &= -\alpha_p + (\alpha + a\alpha^3 + b\beta) \\
\dot{\gamma}_p &= -\gamma_p + b(u - \Psi_p(\alpha, \alpha + a\alpha^3 + b\beta)) \\
e_p &= \begin{bmatrix} \alpha_p - \alpha \\ \gamma_p - (\alpha + a\alpha^3 + b\beta) \end{bmatrix}
\end{align*}
\]

To obtain the injected system, we use
\[
\begin{bmatrix} \alpha_p \\ \gamma_p \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha + a\alpha^3 + b\beta \end{bmatrix} - \begin{bmatrix} \dot{\alpha}_p \\ \dot{\gamma}_p \end{bmatrix}
\]

Candidate controller \( q = \chi(p) \):
\[
\begin{align*}
u &= \Psi_p(\alpha_p - \tilde{\alpha}_p, \gamma_p - \tilde{\gamma}_p) = \Psi_p(\alpha + a\alpha^3 + b\beta) \\
\dot{\alpha}_p &= -\alpha_p + \gamma_p - \tilde{\gamma}_p \\
\dot{\gamma}_p &= -\gamma_p \\
\text{Detectability property}
\end{align*}
\]

For nonlinear systems dwell-time logics do not work because of finite escape

Decision logic
Scale-independent hysteresis switching

**Theorem:** Let $\mathcal{W}$ be finite with $m$ elements. For every $p \in \mathcal{W}$

$$N_\sigma(\tau, t) \leq 1 + m + \frac{m \log (e_0^{-1} e^{\lambda t} \mu_p(t))}{\log(1 + h)} + \frac{m \lambda (t - \tau)}{\log(1 + h)} \quad \forall t > \tau \geq \zeta$$

number of switchings in $[\tau, t)$

and

$$\int_0^t e^{-\lambda(t-\tau)} \gamma_p(||e_\rho(\tau)||) d\tau \leq (1 + h) m \mu_p(t) \quad \forall t > \zeta$$

Assume $\mathcal{W}$ is finite, the $\gamma_p$ are locally Lipschitz and

$$\exists p^* \in \mathcal{P}, c_0 > 0, \lambda^* > \lambda: \quad ||e_{p^*}(t)|| \leq c_0 e^{-\lambda^* t} \quad \forall t \in [0, T_{\text{max}}]$$

maximum interval of existence of solution

$$e^{\lambda t} \mu_p(t) = e_0 + \int_0^t e^{\lambda \tau} \gamma_p(||e_\rho(\tau)||) d\tau$$

uniformly bounded on $[0, T_{\text{max}})$

$$N_\sigma(t, \tau) \& \int_0^t e^{\lambda \tau} \gamma_p(||e_\rho(\tau)||) d\tau$$

uniformly bounded on $[0, T_{\text{max}})$

All the $\mu_p$ can be generated by a system with small dimension if $\gamma_p(||e_\rho||)$ is separable. i.e.,

$$\gamma_p(||e_\rho||) = k(p) h(y, u, x_E) \quad \forall p, u, y, x_E,$$
Scale-independent hysteresis switching

**Theorem:** Let \( \mathcal{W} \) be finite with \( m \) elements. For every \( p \in \mathcal{W} \)
\[
N(p, \tau, t) \leq 1 + m + \frac{m \log \left( \log(1 + h) \right)}{\log(1 + h)} \quad \forall t > \tau \geq \tau \text{ number of switchings in } [\tau, t)
\]
and
\[
\int_0^t e^{-\lambda(t-\tau)} \gamma_p(||e_p(\tau)||) d\tau \leq (1 + h)m\mu_p(t) \quad \forall t > \tau
\]

Assume \( \mathcal{W} \) is finite, the \( \gamma_p \) are locally Lipschitz and
\[
\exists p^* \in \mathcal{P}, c_0 > 0, \lambda^* > \lambda : \quad ||e_p^*(t)|| \leq c_0 e^{-\lambda^* t} \quad \forall t \in [0, T_{\text{max}})
\]

**Non-destabilizing property:** Switching will stop at some finite time \( T^* \in [0, T_{\text{max}}) \)
**Small error property:**
\[
\int_0^{T^*} e^{\lambda^* (t-\tau)} \gamma_p(||e_p(\tau)||) d\tau \leq C^* < \infty \quad \forall t \in [0, T_{\text{max}})
\]

Analysis

\( w = 0, \) no unmodeled dynamics

1st by the Matching property: \( \exists p^* \in \mathcal{W} \) such that \( ||e_p(t)|| \leq c_0 e^{\lambda_p t} \) \( t \geq 0 \)

2nd by the Non-destabilization property: switching stops at a finite time \( T^* \in [0, T_{\text{max}}) \) \( \Rightarrow p(t) = p^* \) & \( \sigma(t) = \gamma(p) \) \( \forall t \in [T^*, T_{\text{max}}) \)

3rd by the Small error property:
\[
\int_{T^*}^{T_{\text{max}}} e^{\lambda^* (t-\tau)} \gamma_p(||e_p(\tau)||) d\tau < \infty
\]

4th by the Detectability property:

the state \( x \) of the switched system is bounded on \([T^*, T_{\text{max}})\)
\[
\downarrow
\]
solution exists globally \( T_{\text{max}} = \infty \) & \( x \to 0 \) as \( t \to \infty \)

**Theorem:** Assume that \( \mathcal{W} \) is finite and all the \( \gamma_p \) are locally Lipschitz. The state of the process, multi-estimator, multi-controller, and all other signals converge to zero as \( t \to \infty \).
Outline

✓ Supervisory control overview
✓ Estimator-based linear supervisory control
✓ Estimator-based nonlinear supervisory control
✗ Examples

Example #4: System in strict-feedback form

Suppose nominal models $N_p, p \in \mathbb{W}$ are of the form
\[
\dot{\alpha} = a\alpha^3 + b\beta
\]
\[
\dot{\beta} = u \quad \text{state accessible}
\]
To facilitate the controller design, one can first “back-step” the system to simplify its stabilization:
\[
\dot{\alpha} = -\alpha + \gamma \quad \gamma := \alpha + a\alpha^3 + b\beta
\]
\[
\dot{\beta} = -\gamma + b(u - \Psi_p(\alpha, \gamma))
\]
\[
y := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \frac{1}{b}(\gamma - \alpha - a\alpha^3) \end{bmatrix}
\]
now the control law
\[
u = \Psi_p(\alpha, \gamma)
\]
stabilizes the system
Example #4: System in strict-feedback form

Suppose nominal models \( N_p, p \in \mathcal{P} \) are of the form

\[
\begin{align*}
\dot{\alpha} &= -\alpha + \gamma \\
\dot{\gamma} &= -\gamma + b(u - \Psi_p(\alpha, \gamma)) \\
y &:= \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \frac{1}{b}(\gamma - \alpha - a\alpha^3) \end{bmatrix}
\end{align*}
\]

Multi-estimator:

\[
\begin{align*}
\dot{\alpha}_p &= -\alpha_p + (\alpha + a\alpha^3 + b\beta) \\
\dot{\gamma}_p &= -\gamma_p + b(u - \Psi_p(\alpha, \alpha + a\alpha^3 + b\beta)) \\
p &\in \mathcal{P}
\end{align*}
\]

When process matches the nominal model \( N_p^* \) exponentially, it is separable so we can do state-sharing. 

Candidate controller \( q = \chi(p) \):

\[
\begin{align*}
e_p &:= \begin{bmatrix} \alpha_p - \alpha \\ \gamma_p - (\alpha + a\alpha^3 + b\beta) \end{bmatrix} \\
\Rightarrow \quad e_p^* &= -e_p^* \\
\Rightarrow \quad e_p^* &\to 0
\end{align*}
\]

Detectability property

Example #4: System in strict-feedback form

\[
\begin{align*}
\dot{\alpha} &= a\alpha^3 + b\beta \\
\dot{\beta} &= u
\end{align*}
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example4_graph.png}
\end{figure}
Example #4: System in strict-feedback form

Suppose nominal models $N_p, p \in \mathcal{W}$ are of the form

$$\dot{\alpha} = a\alpha^3 + b\beta \quad \quad p := (a, b) \in \mathcal{P} \subset [-1, 1] \times [-1, 1] \setminus \{0\}$$

$$\dot{\beta} = u \quad \text{state accessible}$$

In the previous back-stepping procedure:

$$\dot{\alpha} = -\alpha + \gamma \quad \quad \gamma = \alpha + a\alpha^3 + b\beta$$

$$\dot{\gamma} = -\gamma + b(u - \Psi_p(\alpha, \gamma))$$

the controller $u = \Psi_p(\alpha, \gamma) \Rightarrow b\beta \rightarrow -\alpha - a\alpha^3$ nonlinearity is cancelled (even when $a < 0$ and it introduces damping)

One could instead make

$$b\beta \rightarrow \varphi_p(\alpha) := \begin{cases} 0 & a\alpha^2 \leq -1 \\ -\alpha - a\alpha^3 & a\alpha^2 > -1 \end{cases}$$

pointwise min-norm design

still leads to exponential decrease of $\alpha$ (without canceling nonlinearity when $a < 0$)

Example #4: System in strict-feedback form

Suppose nominal models $N_p, p \in \mathcal{W}$ are of the form

$$\dot{\alpha} = a\alpha^3 + b\beta \quad \quad p := (a, b) \in \mathcal{P} \subset [-1, 1] \times [-1, 1] \setminus \{0\}$$

$$\dot{\beta} = u \quad \text{state accessible}$$

A different recursive procedure:

$$\dot{\alpha} = a\alpha^3 + \varphi_p(\alpha) + \gamma \quad \quad \gamma := b\beta - \varphi_p(\alpha)$$

$$\dot{\gamma} = \Psi_p(\alpha, \gamma) + bu$$

In this case

$$u = \frac{1}{b} \begin{cases} 0 & \gamma\Psi_p(\alpha, \gamma) \leq -\gamma^2 \\ -\gamma + \Psi_p(\alpha, \gamma) & \gamma\Psi_p(\alpha, \gamma) > -\gamma^2 \end{cases} \Rightarrow \gamma \rightarrow 0 \quad \text{exponentially}$$

$$b\beta \rightarrow \varphi_p(\alpha) \quad \downarrow$$

$$\alpha \rightarrow 0 \quad \text{exponentially}$$

pointwise min-norm recursive design
Example #4: System in strict-feedback form

Suppose nominal models $N_p, p \in \mathcal{W}$ are of the form

$\dot{\alpha} = a\alpha^3 + \varphi_p(\alpha) + \gamma$ \quad $p := (a, b) \in \mathcal{P} \subset [-1, 1] \times [-1, 1] \setminus \{0\}$

$\dot{\gamma} = \Psi_p(\alpha, \gamma) + bu$

$y := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \frac{1}{b}(\gamma + \tilde{\varphi}_p(\alpha)) \end{bmatrix}$

$\gamma = b\beta - \varphi_p(\alpha)$

Multi-estimator (option III):

$\dot{\alpha}_p = a\alpha^3 + \varphi_p(\alpha) + \gamma - (\alpha_p - \alpha)$ \quad $p \in \mathcal{P}$

$\dot{\gamma}_p = \Psi_p(\alpha, \gamma) + bu - (\gamma_p - \gamma)$

When process matches the nominal model $N_{p^*}$

$e_p := \begin{bmatrix} \dot{\alpha}_p \\ \dot{\gamma}_p \end{bmatrix} = \begin{bmatrix} \alpha_p - \alpha \\ \gamma_p - b\beta + \varphi_p(\alpha) \end{bmatrix} \Rightarrow \dot{\alpha}_{p^*} = -\tilde{e}_{p^*}$

$\dot{\gamma}_{p^*} = -\tilde{\gamma}_{p^*}$ \quad $\Rightarrow$ \quad $e_{p^*} \rightarrow 0$

Candidate controller $q = \chi(p)$:

$u = \frac{1}{b} \begin{cases} 0 & \gamma \Psi_p(\alpha, \gamma) \leq -\gamma^2 \\ -\gamma + \Psi_p(\alpha, \gamma) & \gamma \Psi_p(\alpha, \gamma) > -\gamma^2 \end{cases}$ \quad $\Rightarrow$ \quad Detectability property
Example #4: System in strict-feedback form

\[ \dot{\alpha} = a\alpha^3 + b\beta \quad \dot{\beta} = u \]

pointwise min-norm design  feedback linearization design

Example #5: Unstable-zero dynamics

\[ \dot{x} = -py^2 + u \quad \dot{y} = x + py^2 - u \]

unknown parameter  estimate  output \( y \)

(stabilization)
Example #5: Unstable-zero dynamics

\[ \dot{x} = -py^2 + u \quad \dot{y} = x + py^2 - u \]

(stabilization with noise)

Example #5: Unstable-zero dynamics

\[ \dot{x} = -py^2 + u \quad \dot{y} = x + py^2 - u \]

(set-point with noise)
Example #6: Kinematic unicycle robot

\[ \begin{align*}
\dot{x}_1 &= p_1 u_1 \cos \theta \\
\dot{x}_2 &= p_1 u_1 \sin \theta \\
\dot{\theta} &= p_2 u_2
\end{align*} \]

\(u_1\) ≡ forward velocity
\(u_2\) ≡ angular velocity

\(p_1, p_2\) ≡ unknown parameters determined by the radius of the driving wheel and the distance between them

This system cannot be stabilized by a continuous time-invariant controller. The candidate controllers were themselves hybrid.
Example #7: Induction motor in current-fed mode

\[
\begin{align*}
\dot{\lambda} &= -R\lambda + Ru \\
\dot{\omega} &= \tau - \tau_L \\
\tau &= u^T J \lambda
\end{align*}
\]

\(\lambda \in \mathbb{R}^2 \equiv \text{rotor flux}\)

\(u \in \mathbb{R}^2 \equiv \text{stator currents}\)

\(\omega \equiv \text{rotor angular velocity}\)

\(\tau \equiv \text{torque generated}\)

\(\omega\) is the only measurable output

\[
J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Unknown parameters:

\(\tau_L \in [\tau_{\text{min}}, \tau_{\text{max}}] \equiv \text{load torque}\)

\(R \in [R_{\text{min}}, R_{\text{max}}] \equiv \text{rotor resistance}\)

“Off-the-shelf" field-oriented candidate controllers:

\[
\begin{align*}
\dot{\beta} &= \frac{R}{\beta_d} \tau_d \\
\tau_d &= -\tau_d \left( K_p + K_i \int - \right) (\omega - \omega_d) \\
u &= e^{\phi J} \begin{bmatrix} \beta_d \\ \frac{\beta_d}{\beta_d} \end{bmatrix}
\end{align*}
\]