# Optimal Control of Hybrid Systems

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**Abstract**

This paper presents a method for optimal control of hybrid systems. An inequality of Bellman type is considered and every solution to this inequality gives a lower bound on the optimal value function. A discretization of this "hybrid Bellman inequality" leads to a convex optimization problem in terms of finite-dimensional linear programming. From the solution of the discretized problem, a value function that preserves the lower bound property can be constructed. An approximation of the optimal feedback control law is given and tried on some examples.

**Keywords:** hybrid systems, optimal control, linear programming, dynamic programming.

## 1. Introduction

Hybrid systems are systems that involve interaction between discrete and continuous dynamics. Such systems have been studied with growing interest and activity in recent years. One reason for the interest is that modeling and simulation of a complex system often require a combination of mathematical models from a variety of engineering disciplines. The structure of such submodels can be very different, some can be discrete and some continuous.

Very often, the same phenomenon can be described either by a discrete model or a continuous one, depending on the context and purpose of the model [1]. Consider for example an asynchronous discrete-event driven thermostat, which discretizes temperature information as {too hot, too cold, normal}.

Practical control systems typically involve switching between several different modes, depending on the range of operation. Even if the dynamics in each mode is simple and well understood, it is well known that automatic mode switching can give rise to unexpected phenomena.

Basic aspects of hybrid systems were treated in [6], [7], and [11]. For stability analysis, see [3, 8] and references therein. The reformulation of an optimal control problem in terms of linear programming has previously been used for continuous time systems in [9] and [10] and is closely connected to ideas of [12]. Related methods were discussed for discrete systems in [2] and on an abstract level for hybrid systems in [4].

This paper presents a novel computational approach to optimal control of hybrid systems, based on ideas from dynamic programming and convex optimization. Discretization of Bellman's inequality gives a lower bound on the optimal cost in terms of linear programming. A control law which is used for simulation is constructed from the lower bound. The results are demonstrated in some examples.

## 2. Problem Formulation

Define a hybrid system as

$$\begin{cases} \dot{x}(t) &= f_{q(t)}(x(t), u(t)) \\ q(t) &= \nu(x(t), q(t^-), \mu(t)) \end{cases} \tag{1}$$

where $x(t) \in X \subset \mathbf{R}^n$ is the state vector, $u(t) \in \Omega_u \subset \mathbf{R}^m$ is a continuous input signal of the system. There is also a discrete input, $\mu(t) \in \Omega_\mu$, which allows for the selection between $N$ different system modes, $q(t) \in Q = \{1, 2, \dots, N\}$. The notation $q(t^-)$ is used for the left-hand limit of $q$ at $t$. $S_{q,r}$ is a set (parameterized by $q$ and $r$) such that switching from mode $q$ to $r$ is possible when $x \in S_{q,r} \subseteq X$. The time argument, $t$, will often be omitted in the sequel for readability.

The optimal control problem is to minimize the cost function

$$J(x_0, q_0) = \int_{t_0}^{t_f} l_q(x, u)dt + \sum_{k=1}^{M} s(x(t_k), q(t_k^-), q(t_k^+)) \tag{2}$$

subject to (1) while bringing the system from an initial state $(x_0, q_0)$ at time $t_0$, to a final state $(x_f, q_f)$ at time $t_f$, where the end time, $t_f$, is free. Here, $M$ is an arbitrary finite number of switches occurring at times $t_0 < t_1 < t_2 < \ldots < t_M < t_f$ and $s(x,q,r) > 0$ is an associated cost for switching from discrete state $q$ to $r$, the continuous part being $x$ just before the switch. Note that $s(\cdot) > 0$ removes the problem of infinitely many jumps in a finite interval.

The framework developed in this paper would also allow the number of continuous states to vary with the discrete mode according to $\dot{x}_q(t) = f_{q(t)}(x_q(t), u_q(t))$, where $x_q(t) \in X_q \subset \mathbf{R}^{n(q)}$, $u_q(t) \in \Omega_{u_q} \subset \mathbf{R}^{m(q)}$. The usage of the system description (1), however, will hopefully prevent the reader from getting stuck on details.

### 3. Lower Bounds on Optimal Cost

PROPOSITION 1

Let $V_q : X \mapsto R$, $q = 1, 2, \ldots, N$ be a set of continuous, piecewise $C^1$ functions that satisfy

$$
\begin{aligned}
0 &\le \frac{\partial V_q(x)}{\partial x} f_q(x,u) + l_q(x,u) \\
&\qquad \forall x \in X,\, u \in \Omega_u,\, q \in Q && (3)\\
0 &\le V_r(x) - V_q(x) + s(x,q,r) \\
&\qquad \forall x \in S_{q,r} \;\; q, r \in Q : q \neq r && (4)\\
0 &= V_{q_f}(x_f) && (5)
\end{aligned}
$$

where $f_q(x,u)$ gives the dynamics of a hybrid system according to (1), $l_q(x,u)$ and $s(x,q,r)$ define a cost function for the system according to (2). Then, for every $(x_0, q_0)$, $V_{q_0}(x_0)$ gives a lower bound on the cost for optimally bringing the system from $(x_0, q_0)$ to $(x_f, q_f)$, $x(t) \in X \; \forall t \in [t_0, t_f]$. □

*Remark 1.* Rather than having one single value function, $V(x)$, as would be the case for a purely continuous system, the proposition gives a set of value functions, $V_q(x)$, where $q$ is the initial value of the discrete mode. Note that these functions give the cost for optimal trajectories that are allowed to switch modes — the index $q$ only implies that trajectories *starting* in mode $q$ are considered.

It is of course possible to think of $V_q(x)$ as one single function, parameterized by $x$ and $q$. For consistent notation, however, $V_q(x)$ has been chosen instead of $V(x,q)$.

*Proof.* Let $\hat{u}(\cdot)$ and $\hat{\mu}(\cdot)$ be control signals that drive the system from $(x_0, q_0)$ at time $t_0$ to $(x_f, q_f)$ at time $t_f \equiv t_{M+1}$. Let $\hat{q}(t)$ denote the mode trajectory resulting from $\hat{\mu}(t)$ and define $x_k = x(t_k)$, $x_k^- = x(t_k^-)$, and $\hat{q}_k = \hat{q}(t)$, $t_k \le t < t_{k+1}$. Then

$$
\begin{aligned}
&J(x_0, \hat{q}_0) = \\
&\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} l_{\hat{q}_k}(x, \hat{u}) dt + \sum_{k=1}^{M} s(x_k^-, \hat{q}_{k-1}, \hat{q}_k) \ge \\
&\sum_{k=0}^{M} \int_{t_k}^{t_{k+1}} -\frac{\partial V_{\hat{q}_k}(x)}{\partial x} f_{\hat{q}_k}(x, \hat{u}) dt + \\
&+ \sum_{k=1}^{M} \left\{ V_{\hat{q}_{k-1}}(x_k^-) - V_{\hat{q}_k}(x_k^-) \right\} = \\
&\sum_{k=0}^{M} \left\{ V_{\hat{q}_k}(x_k) - V_{\hat{q}_k}(x_{k+1}) \right\} + \\
&+ \sum_{k=1}^{M} \left\{ V_{\hat{q}_{k-1}}(x_k) - V_{\hat{q}_k}(x_k) \right\} = \\
&V_{\hat{q}_0}(x_0) - V_{\hat{q}_M}(x_{M+1}) = V_{\hat{q}_0}(x_0)
\end{aligned}
$$

□

Also the optimal value function, $V_q^\star(x)$ will meet the the constraints (3)-(5), under appropriate interpretation of $\partial V_q(x)/\partial x$. Hence the inequalities do not introduce any conservatism in the lower bound.

### 4. Discretization

Utilizing a computer to solve (3)-(5) for a specific control problem, a straight forward approach is to grid the state space to require the inequalities to be met at a set of evenly distributed points in $X$. This approximation will, however, not guarantee a lower bound on the optimal cost, unless the nature of $f_q$ and $V_q$ *between* the grid points is taken into consideration.

In the case of a two-dimensional continuous state space, introduce the notation

$$
\begin{aligned}
x_{jk} &= x_f + jhe_1 + khe_2 \\
X^{jk} &= \{x_{jk} + \theta_1 he_1 + \theta_2 he_2 : \; 0 \le \theta_i \le 1\} \\
\hat{X}^{jk} &= \{x_{jk} + \theta_1 he_1 + \theta_2 he_2 : \; -1 \le \theta_i \le 1\} \\
(\underline{f}_q^{jk})_i &= \min_{x \in \hat{X}^{jk}, u \in \Omega_u} (f_q(x,u))_i \\
(\overline{f}_q^{jk})_i &= \max_{x \in \hat{X}^{jk}, u \in \Omega_u} (f_q(x,u))_i \\
(\underline{l}_q^{jk})_i &= \min_{x \in \hat{X}^{jk}, u \in \Omega_u} (l_q(x,u))_i \\
V_q^{jk} &= V_q(x_{jk}) \\
\Delta_i V_q^{jk} &= (V_q(x_{jk} + he_i) - V_q(x_{jk}))/h \\
\Delta_{-i} V_q^{jk} &= (V_q(x_{jk}) - V_q(x_{jk} - he_i))/h
\end{aligned}
$$

where $e_1$ and $e_2$ are unit vectors along the coordinate axes, and $h$ is the grid size.

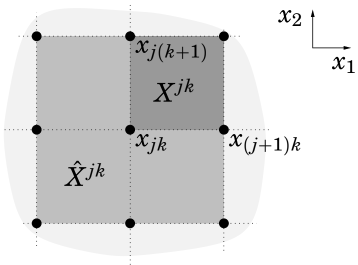


**Figure 1:** Illustration of $X^{jk}$ and $\hat{X}^{jk}$.

Introduce new vector variables, $\lambda_q^{jk} \in \mathbf{R}^n$ for $(j,k,q)$ such that $x_{jk} \in X$, $q \in Q$. The inequalities (3)-(5) can then be replaced by

$$0 \le (\lambda_q^{jk})_1 + (\lambda_q^{jk})_2 + \underline{l}_q^{jk} \tag{6}$$
$$(\lambda_q^{jk})_{|i|} \le (\underline{f}_q^{jk})_{|i|} \Delta_i V_q^{jk} \qquad i = -2,-1,1,2 \tag{7}$$
$$(\lambda_q^{jk})_{|i|} \le (\overline{f}_q^{jk})_{|i|} \Delta_i V_q^{jk} \qquad i = -2,-1,1,2 \tag{8}$$
$$0 \le V_r^{jk} - V_q^{jk} + s(x_{jk}, q, r) \qquad \forall x_{jk} \in S_{q,r} \tag{9}$$
$$0 = V_{q_f}^{00} \tag{10}$$

where (6)-(8) form a combination of backward and forward difference approximations of (3).

For $x = x_{jk} + \theta_1 h e_1 + \theta_2 h e_2 \in X^{jk}$, define the interpolating function

$$V_q(x) = (1-\theta_1)(1-\theta_2)V_q^{jk} + \theta_1(1-\theta_2)V_q^{(j+1)k} + (1-\theta_1)\theta_2 V_q^{j(k+1)} + \theta_1\theta_2 V_q^{(j+1)(k+1)} \tag{11}$$

The following result applies.

THEOREM 1—DISCRETIZATION IN $\mathbf{R}^2$
If $V_q^{jk}$ satisfy (6)-(10) for all $q \in Q$ and for all grid points $x_{jk} \in X \subset \mathbf{R}^2$ such that $X^{jk}$ intersects $X$, then the interpolating function $V_q$ defined by (11) satisfies (3)-(5) and, for every $(x_0, q_0)$, $V_{q_0}(x_0)$ is a lower bound of $J(x_0, q_0)$. □

*Remark 1.* Any function that meet the constraints, even the trivial choice $V_q(x) = 0$, is a lower bound on the true cost. Thus, to yield useful bounds, $V_q(x)$ need to be maximized subject to (6)-(10). The maximization could be carried out in either one point, $(x_0, q_0)$, or several points, $(x,q) \in X \times Q$, simultaneously.

For the original, non-discretized problem, the result of a maximization of $V_q(x)$ is always identical to the optimal cost, regardless if the maximization is done at a particular initial state, or by summing the values at several initial states.

However, for the discretized problem, different choices of maximization criteria may lead to different results. Fortunately, experience from examples shows that the difference between the results of a single-point and a multi-point maximization is often small, making it possible to compute the value function in a large subset of $X \times Q$ solving *one* LP.

*Remark 2.* The restriction $x(t) \in X$ in the optimal control problem is essential. It may happen that for some initial states $x_0$ there exist no admissible solutions inside $X$. Then the maximization of $V_{q_0}(x_0)$ can lead to arbitrarily large values.

*Remark 3.* The theorem is easily extended to $\mathbf{R}^n$. Define $\mathbf{j} = (j_1, j_2, \ldots, j_n)$ and exchange $jk$ for the new multi-index $\mathbf{j}$ in the above inequalities. The limits of all summations and enumerations should also be adjusted.

*Proof.* Assume that $x \in X^{jk}$. Noting that $\Delta_1 V_q^{jk} = \Delta_{-1} V_q^{(j+1)k}$, $\Delta_2 V_q^{jk} = \Delta_{-2} V_q^{j(k+1)}$, the inequalities (6)-(8) taken at grid points $jk$, $j(k+1)$, $(j+1)k$, and $(j+1)(k+1)$ give

$$0 \le f_{q1}(x,u)\Delta_1 V_q^{jk} + f_{q2}(x,u)\Delta_2 V_q^{jk} + l_q(x,u) \tag{12}$$
$$0 \le f_{q1}(x,u)\Delta_1 V_q^{j(k+1)} + f_{q2}(x,u)\Delta_2 V_q^{jk} + l_q(x,u) \tag{13}$$
$$0 \le f_{q1}(x,u)\Delta_1 V_q^{jk} + f_{q2}(x,u)\Delta_2 V_q^{(j+1)k} + l_q(x,u) \tag{14}$$
$$0 \le f_{q1}(x,u)\Delta_1 V_q^{j(k+1)} + f_{q2}(x,u)\Delta_2 V_q^{(j+1)k} + l_q(x,u) \tag{15}$$

The gradient of $V_q$ is given by

$$\frac{\partial V_q}{\partial x} = \begin{bmatrix} (1-\theta_2)\Delta_1 V_q^{jk} + \theta_2 \Delta_1 V_q^{j(k+1)} \\ (1-\theta_1)\Delta_2 V_q^{jk} + \theta_1 \Delta_2 V_q^{(j+1)k} \end{bmatrix}^T$$

and thus, adding (12)-(15) weighted with $(1-\theta_1)(1-\theta_2)$, $(1-\theta_1)\theta_2$, $\theta_1(1-\theta_2)$, and $\theta_1\theta_2$ respectively proves that (3) is met for $x$. The inequality (4) is met since $V_q$ is a convex combination of grid points that all meet (9), and (5) is the same condition as (10). □

Note a special case in which the computational load of the local optimizations in Theorem 1 is lightened: if $f_q(x,u) = h_q(x) + g_q(x)u$ and $l_q(x,u) = o_q(x) + m_q(x)u$ while $\Omega_u = [-1,1]$, then $u$ can be entirely eliminated from (6)-(8) by replacing $\underline{f}_q^{jk}$, $\overline{f}_q^{jk}$, and $\underline{l}_q^{jk}$ with $\underline{h}_q^{jk} \pm \underline{g}_q^{jk}$, $\overline{h}_q^{jk} \pm \overline{g}_q^{jk}$, and $\underline{o}_q^{jk} \pm \underline{m}_q^{jk}$ respectively. This will double the set of equations (6)-(8), but the functions $h_q$, $g_q$, $o_q$, and $m_q$ are optimized over $\hat{X}^{jk}$ solely.

## 5. Computing the Control Law

Provided that the lower bound, $V_q$, is a good enough approximation of the optimal cost, the optimal feedback control law can be calculated as

$$\begin{cases} \hat{u}(x,q) &= \underset{u \in \Omega_u}{\operatorname{argmin}} \left\{ \dfrac{\partial V_q}{\partial x} f_q(x,u) + l_q(x,u) \right\} \\ \hat{\mu}(x,q) &= \underset{\mu \in \Omega_\mu \mid x \in S_{q,\nu}}{\operatorname{argmin}} \left\{ V_\nu(x) + s(x,q,\nu) \right\} \end{cases} \tag{16}$$

where $\nu = \nu(x, q, \mu)$. Thus, the continuous input, $\hat{u}$, is computed in a standard way. The discrete input, $\hat{\mu}$, is chosen such that switching occur whenever there exist a discrete mode for which the value function has a lower value than the cost of the value function for the current mode minus the cost for switching there.

Consider the true optimal value function, $V_q^\star$. For those $(x, q, r)$ where the optimal trajectory requires mode switching, the inequality (3) will turn to equality i.e. $V_q^\star = V_r^\star + s(x, q, r)$ (this will be shown in Ex. 1). A consequence of this is that for (16) to describe correct switching between the modes, $s(x, q, q)$ has to be defined as $s(x, q, q) = \varepsilon > 0$ (rather than the real cost $s(x, q, q) = 0$). For $V_q^\star$, the proper control law is achieved as $\varepsilon$ approaches $0^+$. A small value of $\varepsilon$ suffices, however, for numerical computations.

Integration of (2) along a simulated trajectory based on (16) will provide an upper bound on the optimal cost. The better the control law, the better the estimate.

## 6. Examples

EXAMPLE 1—A CAR WITH TWO GEARS
Consider the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g_q(x_2)u, \quad q = 1,2 \quad |u| \le 1 \end{cases} \tag{17}$$

where $g_q(x)$ is plotted in Fig. 2. This could be seen as a crude model of a car, $u$ being the throttle, $g_q(x)$ the efficiency for gear number $q$.

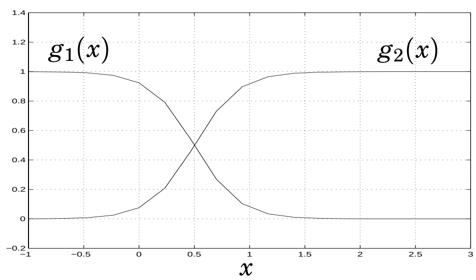


**Figure 2:** Gear efficiency at various speeds.

The problem is to bring (17) from $x_i = (-5, 0)$, $q_i = 1$ to $x_f = (0, 0)$, $q_f = 1$ in minimum time. Torque losses when using the clutch calls for an additional penalty for gear changes. Thus, the components of (2) have been chosen as $l_1(x,u) = l_2(x,u) = 1$, $s(x, 1, 2) = s(x, 2, 1) = 0.5$.

The problem is plugged into the machinery of Section 4 and $V_q(x)$ is maximized over a region $-5.5 \le x_1 \le 1.0$, $-0.5 \le x_2 \le 3.0$.

The result is shown in Figure 3 and 4 where $x_i$ and $x_f$ also have been marked. The functions look rather similar, since the cost for changing gears is only 0.5. One can see that $V_1$ has a threshold along the line $x_2 = 1$. Figure 2 reveals that the first gear is almost useless for high speeds, leading to $V_1 = V_2 + 0.5$ for $x_2 > 1$. This is the cost for using the second gear optimally after a gear switch.

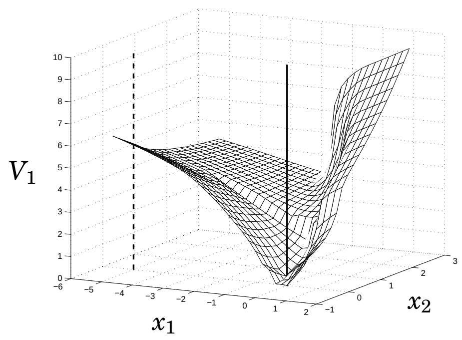


**Figure 3:** Plot of $V_1$. The initial point, $x_i$, is marked with a vertical dashed line, the final point, $x_f$, with a solid line.

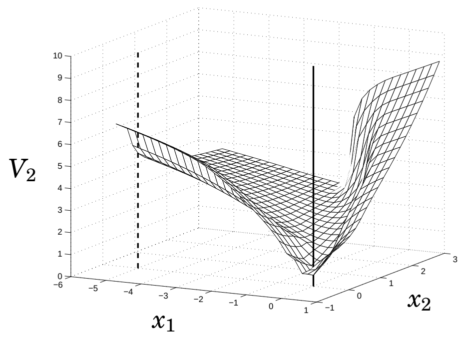


**Figure 4:** Plot of $V_2$.

Studying Fig. 5, where $V_1 - V_2$ is plotted, the strategy for changing gears is even more obvious: there is only one discrete mode allowed under optimal control when the difference hits its maximum distance. In conformity with previous reasoning, $V_1 - V_2 = 0.5$ for $x_2 > 1$, indicating the need for a change of gears when using the first gear at high speed. Analogously, the second gear should be avoided, starting with zero speed.

A simulation of the controlled system is shown in Fig. 6, where the initial point is marked with a square. The state trajectory coincides with the one of a professional rally-driver with lousy brakes. In

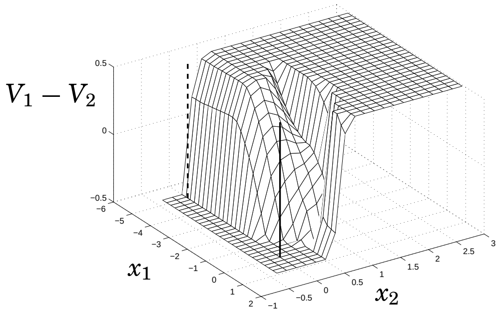


**Figure 5:** The difference between $V_1$ and $V_2$.

the beginning, maximum throttle is used on the first gear (solid line). When the speed roughly reaches the point of equal efficiency between the gears ($x_2 = 0.5$), they are switched in favor of the second gear (dashed line). At half the distance, the gas pedal is lightened to use the braking force of the engine. In the end, the first gear is used again before the origin is hit. As seen in the figure, the granularity of the discretization grid ($h = 0.18$) prevents the solution from hitting the exact origin.

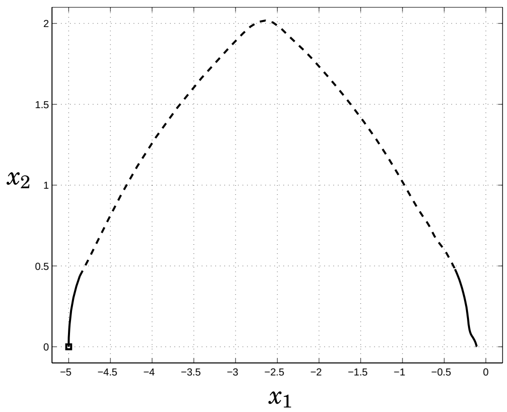


**Figure 6:** Phase portrait of a simulation. The solid line shows where gear number one has been used, the dashed line shows the second gear. The initial point is marked with a square.

□

EXAMPLE 2—ALTERNATE HEATING OF TWO FURNACES
Since the industrial power fee is determined by the highest peak of the season [5], it is desirable to spread the power consumption evenly over time. This is handled by load control, which means that the available electrical power is altered between different loads of the mill.

In this example, the temperature of two furnaces should be controlled by alternate heating. The system has two continuous states that correspond to the temperature of the furnaces and is given by $\dot{x} = f_q(x)$, where

$$f_1(x) = \begin{bmatrix} -x_1 + u_0 \\ -2x_2 \end{bmatrix} \quad f_2(x) = \begin{bmatrix} -x_1 \\ -2x_2 + u_0 \end{bmatrix}$$

$$f_3(x) = \begin{bmatrix} -x_1 \\ -2x_2 \end{bmatrix}$$

Thus, there are three discrete modes: $q = 1$ means that the first furnace is heated, $q = 2$ means that the second furnace is heated, $q = 3$ corresponds to no heating. The cost function to be minimized is

$$J(x_0, q_0) = \int_{t_0}^{\infty} \sum_{i=1}^{2} (x_i - c_i)^2 e^{-t} dt + \sum_{k=1}^{M} b e^{-t_k}$$

where the desired stationary temperature values are $c_1 = 1/4$, $c_2 = 1/8$ and the cost for switching the power is $b = 1/1000$. Since the furnaces can only be fed by a fixed amount of energy, $u_0$, it is impossible to keep them stationary at the desired temperature. Hence, the time weighting, $e^{-t}$, is necessary to get a bounded cost function.

If $V_q(x,t)$ is defined as the cost for starting in $(x,q)$ at time $t$, then the continuous part of the general time dependent Bellman inequality can be written

$$\frac{\partial V_q(x,t)}{\partial t} + \frac{\partial V_q(x,t)}{\partial x} f_q(x,u,t) + l_q(x,u,t) \geq 0 \quad (18)$$

Rewriting the functions like $V_q(x,t) = e^{-t}\tilde{V}_q(x)$ and $l_q(x,u,t) = e^{-t}\tilde{l}_q(x,u)$ for the furnace example, (18) becomes

$$-\tilde{V}_q(x) + \frac{\partial \tilde{V}_q(x)}{\partial x} f_q(x,u) + \tilde{l}_q(x,u) \geq 0 \quad (19)$$

Thus, the time dependence introduced in Bellman's inequality cancels and techniques similar to those presented above apply.

The optimal control results in a limit cycle as seen in Figure 7. The figure, that contains the phase portrait of the continuous states, shows how the temperature of one furnace always decreases as the other one is heated. By alternate heating, the temperatures first climb up to, and above the set-point and then both furnaces are turned off and the state drifts towards the origin. This procedure is then repeated over and over again, making the trajectory enclose the desired steady state (marked with a circle in the figure). The trajectory has been dashed for $t \in [0, 2.8]$ to make the limit cycle clear.

Figure 8 shows what happens when the power supply is insufficient for driving both furnaces. Mode 3 is not entered since the temperature set-points are never reached.

□

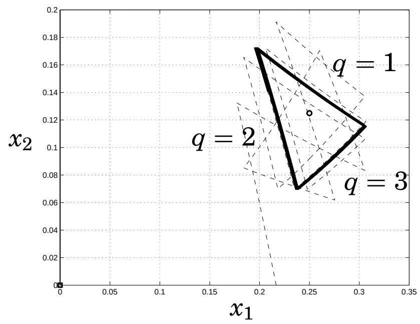


**Figure 7:** Phase portrait of the continuous states under optimal control when $u_0 = 0.8$. The mode number, $q$, has been marked for the limit cycle

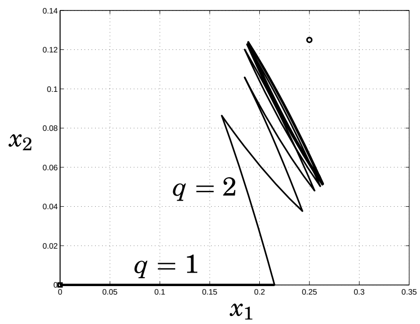


**Figure 8:** Phase portrait of the continuous states under optimal control when $u_0 = 0.4$.

## 7. Summary

An extended version of Bellman's inequality was discretized in this paper to compute a lower bound on the optimal cost function, using linear programming. Based on these computations, an approximation of the optimal control feedback law was derived.

Hybrid systems combine discrete and continuous dynamics. The analysis should therefore contain techniques that are well suited for computer science as well as control theory. The emphasis in this paper is on the continuous part, the discrete part consisting of a few system modes. At the other end of the hybrid spectrum, where purely discrete systems are found, $X$ will reduce to a single point. The first inequality of proposition 1 will then be superfluous. The set of inequalities given by (4), possibly large depending on $Q$, should be met for $S_{q,r} = \{x_f\}$. The resulting LP formulation solves the shortest-paths problem on a non-negatively weighted, directed graph — a problem that is usually attacked using Dijkstra's algorithm.

A set of MATLAB commands has been compiled by the authors to make it easy to test the above methods and implement the examples. The LP solver that is used is "PCx", developed by the Optimization Technology Center, Illinois. The MATLAB commands and a manual of usage are available free of charge upon request from the authors.

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