Exercise 15 (A-invariance and controllability). Consider the continuous-time LTI system
\[ \dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^k. \]  
(AB-CLTI)

Prove the following two statements:

(a) The controllable subspace $C$ of the system (AB-CLTI) is $A$-invariant.

(b) The controllable subspace $C$ of the system (AB-CLTI) contains $\text{Im} B$.  

Exercise 16 (Satellite). The equations of motion of a satellite linearized around a steady-state solution, are given by
\[ \dot{x} = Ax + Bu, \quad A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \]

where the state vector $x := [x_1 \ x_2 \ x_3 \ x_4]$ includes the perturbation $x_1$ in the orbital radius, the perturbation $x_2$ in the radial velocity, the perturbation $x_3$ in the angle, and the perturbation $x_4$ in the angular velocity; and the input vector $u := [u_1 \ u_2]$ includes the radial thruster $u_1$ and a tangential thruster $u_2$.

(a) Show that the system is controllable from the input vector $u$.

(b) Can the system still be controlled if the radial thruster does not fire? What if it is the tangential thruster that fails?  

Exercise 17 (Controllable canonical form). Consider a system in controllable canonical form
\[ A = \begin{bmatrix} -\alpha_1 I_{k \times k} & -\alpha_2 I_{k \times k} & \cdots & -\alpha_{n-1} I_{k \times k} & -\alpha_n I_{k \times k} \\ I_{k \times k} & 0_{k \times k} & \cdots & 0_{k \times k} & 0_{k \times k} \\ 0_{k \times k} & I_{k \times k} & \cdots & 0_{k \times k} & 0_{k \times k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{k \times k} & 0_{k \times k} & \cdots & I_{k \times k} & 0_{k \times k} \end{bmatrix}_{nk \times nk}, \]
\[ B = \begin{bmatrix} I_{k \times k} \\ 0_{k \times k} \\ \vdots \\ 0_{k \times k} \end{bmatrix}_{nk \times k}, \quad C = [N_1 \ N_2 \ \cdots \ N_{n-1} \ N_n]_{m \times nk}. \]

Show that such a system is always controllable.  