LINEAR SYSTEMS THEORY

2ND EDITION

João P. Hespanha

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Disclaimer: This is a draft and probably contains a few typos.

Comments and information about typos are welcome.
Please contact the author at hespanha@ece.ucsb.edu.

To my wife Stacy
and
to our son Rui

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Preamble

Linear systems theory is the cornerstone of control theory and a prerequisite for essentially all graduate courses in this area. It is a well-established discipline that focuses on linear differential equations from the perspective of control and estimation.

Content

The first set of lectures (1–17) covers the key topics in linear systems theory: system representation, stability, controllability and state feedback, observability and state estimation, and realization theory. The main goal of these chapters is to provide the background needed for advanced control design techniques. Feedback linearization and the LQR problem are also briefly introduced to increase the design component of this set of lectures. The preview of optimal LQR control facilitates the introduction of notions such as controllability and observability, but is pursued in much greater detail in the second set of lectures.

Three advanced foundational topics are covered in a second set of lectures (18–26): poles and zeros for MIMO systems, LQG/LQR control, and control design based on the Q parameterization of stabilizing controllers (Q design). The main goal of these chapters is to introduce advanced supporting material for modern control design techniques. Although LQG/LQR is covered in some other linear systems books, it is generally not covered at the same level of detail (in particular the frequency domain properties of LQG/LQR, loop shaping, and loop transfer recovery). In fact, there are few textbooks in print that cover the same material, in spite of the fact that these are classical results and LQG/LQR is the most widely used form of state-space control. By covering the ARE in detail, I set the stage for H-2 and H-infinity.

In writing this book, it is assumed that the reader is familiar with linear algebra and ordinary differential equations at an undergraduate level. To profit most from this textbook, the reader would also have taken an undergraduate course in classical control, but these notes are basically self-contained regarding control concepts.

Organization and Style

This book was purposely designed as a textbook, and because it is not an adaptation of a reference text, the main emphasis is on presenting material in a fashion that makes it easy for students to understand. The material is organized in lectures, and it is divided so that on average each lecture can be covered in 2 hours of class time. The sequence in which the material appears was selected to emphasize continuity and motivate the need for new concepts as they are introduced.

In writing this manuscript there was a conscious effort to reduce verbosity. This is not to say that I did not attempt to motivate the concepts or discuss their significance (on the contrary), but the amount of text was kept to a minimum. Typically, discussion, remarks, and side comments are relegated to marginal notes so that the reader can easily follow the material presented without distraction and yet enjoy the benefit of comments on the notation and terminology, or be made aware that a there is a related MATLAB® command.

Attention! When a marginal note finishes with "► p. XXX," more information about that topic can be found on page XXX.

I have also not included a chapter or appendix that summarizes background material (for example, a section on linear algebra or nonlinear differential equations). Linear algebra is a key prerequisite to this course, and it is my experience that referring a student who has a limited background in linear algebra to a brief chapter on the subject is useless (and sometime even counter-productive). I do review advanced concepts (for example, singular values, matrix norms, and the Jordan normal form), but this is done at the points in the text where these concepts are needed. I also take this approach when referring the reader to MATLAB®, by introducing the commands only where the relevant concepts appear in the text.

Learning and Teaching Using This Textbook

Lectures 1–17 can be the basis for a one-quarter graduate course on linear systems theory. At the University of California at Santa Barbara I teach essentially all the material in these lectures in one quarter with about 40 hours of class time. In the interest of time, the material in the Additional Notes sections and some of the discrete-time proofs can be skipped. For a semester-long course, one could also include a selection of the advanced topics covered in the second part of the book (Lectures 18–26).

I have tailored the organization of the textbook to simplify the teaching and learning of the material. In particular, the sequence of the chapters emphasizes continuity, with each chapter appearing motivated and in logical sequence with the preceding ones. I always avoid introducing a concept in one chapter and using it again only many chapters later. It has been my experience that even if this may be economical in terms of space, it is pedagogically counterproductive. The chapters are balanced in length so that on average each can be covered in roughly 2 hours of lecture time. Not only does this aid the instructor's planning, but it makes it easier for the students to review the materials taught in class.

As I have taught this material, I have noticed that some students start graduate school without proper training in formal reasoning. In particular, many students come with limited understanding of the basic logical arguments behind mathematical proofs. A course in linear systems provides a superb opportunity to overcome this difficulty. To this effect, I have annotated several proofs with marginal notes that explain general techniques for constructing proofs: contradiction, contraposition, the difference between necessity and sufficiency, etc. (see, for example, Note 14 on page 91). Throughout the manuscript, I have also structured the proofs to make them as intuitive as possible, rather than simply as short as possible. All mathematical derivations emphasize the aspects that give insight into the material presented and do not dwell on technical aspects of small consequence that merely bore the students. Often these technical details are relegated to marginal notes or exercises.

The book includes exercises that should be solved as the reader progresses through the material. Some of these exercises clarify issues raised in the body of the text and the reader is generally pointed to such exercises in marginal notes; for example, Exercise 8.5, which is referenced in a marginal note in page 76. Other exercises are aimed at consolidating the knowledge acquired, by asking the reader to apply algorithms or approaches previously discussed; for example, Exercise 2.10 on page 24. The book includes detailed solutions for all the exercises that appear in the sections titled "Practice Exercises," but it does not include solutions to those in the sections titled "Additional Exercises," which may be used for assignments or quizzes.

MATLAB®

Computational tools such as the MATLAB® software environment offer a significant step forward in teaching linear systems because they allow students to solve numerical problems without being exposed to a detailed treatment of numerical computations. By systematically annotating the theoretical developments with marginal notes that discuss the relevant commands available in MATLAB®, this textbook helps students learn to use these tools. An example of this can be found in MATLAB® Hint 9 on page 12, which is further expanded on page 57.

The commands discussed in the "MATLAB® Hints" assume that the reader has version R2015b of MATLAB® with Simulink®, the Symbolic Math Toolbox, and the Control System Toolbox. However, essentially all these commands have been fairly stable for several versions so they are likely to work with previous and subsequent versions for several years to come. Lecture 26 assumes that the reader has installed CVX version 1.2, which is a MATLAB® package for Disciplined Convex Programming, distributed under the GNU General Public License 2.0 [7].

Web

The reader is referred to the author's website at `www.ece.ucsb.edu/~hespanha` for corrections, updates on MATLAB® and CVX, and other supplemental material.

2nd Edition

The main thrust for publishing a second edition of this book was to facilitate the process of learning this material, both in the classroom or in a self-teaching environment. This second edition covers the same basic theoretical concepts, but we have added a large number of exercises, including solved exercises that appear in the sections titled "Practice Exercises." These exercises are based on prototypical problems and guide the student toward answers that are correct, precisely stated, and succinct. We also took this opportunity to reorganize a few lectures and correct many typos. Most typos were annoying but inconsequential, but a few could be misleading. A special thanks to all the students that got back to me with typos, requests for clarification, and other suggestions for improvement.

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