PRACTICE FINAL EXAM
LINEAR SYSTEMS

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Please explain all you answers.

1. Consider the matrix

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}.$$

   Compute A^{100} and e^{At}.

   Hint: Diagonalize A.

2. (a) Show that if J is a stable Jordan block then for every $t \geq 0$ all the eigenvalues of e^{Jt} have magnitude smaller or equal to 1.

   (b) Show that if A is a stable matrix then for every $t \geq 0$ all the eigenvalues of e^{At} have magnitude smaller or equal to 1.

3. Prove that if the single-input/single-output system

$$\dot{x} = Ax, \qquad y = cx, \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}$$

   is observable then the null space of the matrix

$$\begin{bmatrix} A - \lambda I \\ c \end{bmatrix} \in \mathbb{R}^{(n+1)\times n} \tag{1}$$

   only contains the zero vector, for every $\lambda \in \mathbb{C}$.

   Hint: Prove the statement by contradiction assuming that the observability matrix is nonsingular and yet the null space of the matrix (1) *contains a nonzero vector for some* $\lambda \in \mathbb{C}$.

4. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

   (a) Is this system controllable? observable?

   (b) Compute the system's transfer matrix.

   (c) Is this system BIBO stable?

   (d) Is this system stable in the sense of Lyapunov?

5. Find a **minimal** state-space realization for the following transfer matrix:

$$\hat{G}(s) = \begin{bmatrix} \frac{2s+1}{s+1} & \frac{s+2}{s+1} \end{bmatrix}$$

6. Suppose we want find the control input u to the system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

that minimizes the quadratic cost

$$J = \int_0^\infty \|x(t)\|^2 + \|u(t)\|^2 dt.$$

As you know, the optimal control is of the form

$$u = -B'Px$$

where P is a positive definite solution to the equation

$$PA + A'P + I - PBB'P = 0. \tag{2}$$

Show that the resulting closed-loop system is asymptotically stable.

Hint: Try to find in (2) *a Lyapunov equation for the closed-loop system.*