Practice Mid-term Exam
Linear Systems

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Please explain all you answers.

1. Consider the nonlinear systems

$$\ddot{y} + \dot{y} + y = u^2 - 1.$$

   (a) Compute a state-space representation for the system with input u and output y

   (b) Linearize the system around the solution $y(t) = 0$, $u(t) = 1$, $\forall t \geq 0$.

2. Consider the system

$$\dot{x} = (A - bk)x + bu,$$
$$y = cx,$$

   where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

   where k_1 and k_2 are constant scalars.

   (a) Compute the system's transfer function.

   (b) Determine values for k_1 and k_2 such that the transfer function is equal to

$$\frac{s}{s^2 + s + 1}$$

   Hint: Your answer to (a) should appear as a function of the constants k_1 and k_2.

3. Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Compute the characteristic polynomial of A. Is A diagonalizable?

   (b) Compute e^{At}.

   (c) Compute a value for the initial state $x(0) := \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) \end{bmatrix}'$ such that the solution to

$$\dot{x} = Ax, \qquad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

   is equal to

$$y(t) = te^{-2t}, \qquad \forall t \geq 0.$$

Hint: To solve (c), start by writing $y(t)$ as a function of $x_1(0), x_2(0), x_3(0)$ and then determine values for these constants to get the desired output.

4. Compute a matrix $A(t)$ such that

$$\Phi(t,t_0) = \begin{bmatrix} 1 & e^{\frac{t^2-t_0^2}{2}} - 1 \\ 0 & e^{\frac{t^2-t_0^2}{2}} \end{bmatrix}$$

is the state transition matrix of the homogeneous ODE

$$\dot{x} = A(t)x$$