Computation of Root-Mean-Square Gains of Switched Linear Systems*

João P. Hespanha

hespanha@ece.ucsb.edu

University of California, Santa Barbara, CA 93106-9560, USA

Abstract. In this paper we compute the root-mean-square (RMS) gain of a switched linear system when the interval between consecutive switchings is large. The algorithm proposed is based on the fact that a given constant γ provides an upper bound on the RMS gain whenever there is a separation between the stabilizing and the antistabilizing solutions to a set of γ-dependent algebraic Riccati equations. The motivation for this problem is the application of robust stability tools to the analysis of hybrid systems.

1 Introduction

A switched linear system is defined by a linear differential equation for which the matrices that appear in the right-hand-side are piecewise constant. The times at which these matrices are discontinuous are called "switching times." Formally, to define a switched linear system one considers a family of n-dimensional, m-input, k-output minimal realizations

$$\{(A_p, B_p, C_p, D_p) : p \in \mathcal{P}\}, \tag{1}$$

parameterized by an index set $\mathcal{P}$, together with a family of piecewise constant signals $\mathcal{S} := \{\sigma : [0, \infty) \to \mathcal{P}\}$. For a given signal $\sigma \in \mathcal{S}$, one can then define the following linear time-varying system

$$\dot{x} = A_\sigma x + B_\sigma u, \qquad y = C_\sigma x + D_\sigma u. \tag{2}$$

The signal σ is called a *switching signal* and the times at which it is discontinuous are called *switching times*. Between switching times the dynamics of (2) are time-invariant. Typically switching signals are generated by a supervisory logic and different values of σ correspond to distinct modes of operation [1].

For different switching signals σ, the system (2) has different properties. However, it is useful to study properties of this system that remain invariant for every σ in a particular set of switching signals $\mathcal{S}$. A property that has been

* This material is based upon work supported by the National Science Foundation under Grant No. ECS-0093762.

extensively investigated for several sets of switching signals is stability. We say that (2) is *uniformly exponentially stable over* $\mathcal{S}$ when the state transition matrix $\Phi_\sigma(t,\tau)$ of the homogeneous switched system $\dot{z} = A_\sigma z$ can be bounded by[1]

$$\|\Phi_\sigma(t,\tau)\| \le c e^{-\lambda(t-\tau)}, \qquad \forall t,\tau \ge 0, \quad \forall \sigma \in \mathcal{S},$$

where the constants $c, \lambda > 0$ are chosen independently of σ in $\mathcal{S}$. In [2–10] conditions are presented that guarantee stability of (2) for every set of switching signals $\mathcal{S}$. It is also well known (cf., e.g., [11]) that (2) is uniformly exponentially stable as long as $\mathcal{S}$ only contains signals with consecutive switching times separated by a sufficiently long interval. In particular, defining $\mathcal{S}[\tau_D]$, $\tau_D > 0$ to be the set of all switching signals with interval between consecutive discontinuities no smaller than τ_D, there exists a $\tau_D^* > 0$ so that (2) is uniformly exponentially stable over any $\mathcal{S}[\tau_D]$, $\tau_D \ge \tau_D^*$. The minimum interval τ_D between consecutive switching times is called the *dwell-time*. In [12] it is shown that uniform exponential stability is preserved when the set of switching signals is enlarged to contain signals that occasionally have dwell times smaller than τ_D, provided that this does not happen on-the-average (*slow-switching on-the-average*). The reader is referred to [13, 1] for a discussion of these and related results.

Surprisingly, there has been relatively little work on the study of input-output properties that remain invariant over sets of switching signals. For example, little is known under what conditions a switched system like (2) satisfies an *Integral Quadratic Constrain* (IQC) [14]:

$$\int_0^\infty \begin{bmatrix} y(t)' & u(t)' \end{bmatrix} M \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt \ge 0, \qquad \forall u \in \mathcal{L}_2, \tag{3}$$

for every $\sigma \in \mathcal{S}$. In (3), y is computed along solutions to (2) with $x(0) = 0$. This type of constrain provides a general framework to address performance and robustness of interconnected systems and has therefore been receiving significant attention in the context of unswitched system. It is therefore timely to study conditions under which IQCs remain invariant over specific classes of switching signals. In the recent paper [15] the authors study passivity-like IQCs with

$$M = M_{\delta,\rho} := \begin{bmatrix} -\delta^2 I & I \\ I & -\rho^2 I \end{bmatrix}, \qquad \delta, \rho \ge 0,$$

for switched hybrid systems. When

$$M = M_\gamma := \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix}, \qquad \gamma > 0$$

we obtain another IQC whose importance has long been recognized as it can be used to characterize the RMS gain of a systems. In this paper we study this

[1] Given a vector a and a matrix A we denote by $\|a\|$ and $\|A\|$ the Euclidean norm of a and the largest singular value of A, respectively. Given a vector or matrix-valued measurable signal z defined on $[0,\infty)$, we denote by $\|z\|_2$ the $\mathcal{L}_2$-norm $\left(\int_0^\infty \|z(t)\| dt\right)^{\frac{1}{2}}$ of z. The set of all signals z for which $\|z\|_2$ is finite is denoted by $\mathcal{L}_2$.

IQC for switched systems. RMS gains of switched system are typically needed to evaluate the performance of closed-loop systems with hybrid controllers or processes. This analysis also finds application in the use of robust control techniques in the stability analysis of hybrid systems.

Given a set of switching signals $\mathcal{S}$, we define the *RMS gain of* (2) *over* $\mathcal{S}$ to be the smallest scalar $\mathfrak{g}_{\mathcal{S}} \geq 0$ for which (3) holds with $M = M_{\mathfrak{g}_{\mathcal{S}}}$, for every $\sigma \in \mathcal{S}$. To be more precise,

$$\mathfrak{g}_{\mathcal{S}} := \inf\{\gamma \geq 0 : \|y\|_2 \leq \gamma\|u\|_2,\ \forall u \in \mathcal{L}_2,\ \sigma \in \mathcal{S}\},$$

where y is computed along solutions to (2) with $x(0) = 0$. The RMS gain $\mathfrak{g}_{\mathcal{S}}$ can be viewed as a "worst-case" energy amplification gain for the switched system, *over all possible input and switching signals.*

When (2) is uniformly exponentially stable over $\mathcal{S}$, the RMS gain $\mathfrak{g}_{\mathcal{S}}$ is known to be finite. Since both slow dwell-time switching and slow-switching on-the-average result in exponential stability, both these forms of slow switching result in a finite RMS gain $\mathfrak{g}_{\mathcal{S}}$. In [16, 12, 17] one can find upper bounds on RMS gains over appropriately defined sets of switching signals that satisfy the slow-switching properties mentioned above. However, these bounds are typically very conservative and prompted us to compute the precise value of the RMS gain under slow-switching.

The RMS gain over the set of switching signals $\mathcal{S}[\tau_D]$ with dwell-time $\tau_D > 0$ is a monotone decreasing function of τ_D. This is simply because, given two dwell-times $\tau_{D_1} \leq \tau_{D_2}$, we have that $\mathcal{S}[\tau_{D_1}] \supset \mathcal{S}[\tau_{D_2}]$. One can then ask: *What is the smallest RMS gain that can be obtained for* (2) *by increasing the dwell-time?* To answer this question we define the *slow-switching RMS gain*, by

$$\mathfrak{g}_{\text{slow}} := \inf_{\tau_D > 0} \mathfrak{g}_{\mathcal{S}[\tau_D]}. \tag{4}$$

Since every $\mathcal{S}[\tau_D]$, $\tau_D > 0$ contains the constant switching signals $\sigma(t) = p$, $t \geq 0$, $p \in \mathcal{P}$, $\mathfrak{g}_{\text{slow}}$ is necessarily larger or equal to the RMS gains of all the "unswitched systems," i.e.,

$$\mathfrak{g}_{\text{slow}} \geq \mathfrak{g}_{\text{static}} := \sup_{p \in \mathcal{P}} \|C_p(sI - A_p)^{-1}B_p + D_p\|_\infty, \tag{5}$$

where $\|T(s)\|_\infty$ denotes the $\mathcal{H}_\infty$-norm of a transfer matrix $T(s)$. We recall that $\|T(s)\|_\infty$ is numerically equal to the RMS gain of any linear time-invariant system with transfer matrix $T(s)$. Until recently, it was believed that the inequality in (5) was actually an equality. This would mean that, by switching sufficiently slow, one would recover the RMS gains of the "unswitched systems," which are realization independent. It turns out that this is not true and the above inequality is often strict. The following simple example illustrates this and also provides some intuition on the difficulties introduced by switching: Consider a SISO switched system

$$\dot{x} = A_\sigma x + b_\sigma u, \qquad\qquad y = c_\sigma x, \tag{6}$$

for which the switching signal σ takes values in the set $\mathcal{P} := \{1, 2\}$ and both unswitched systems have $\mathcal{H}_\infty$-norm (and therefore RMS gain) equal to one, i.e.,

$$\|c_p(sI - A_p)^{-1}b_p\|_\infty = 1, \qquad\qquad p \in \mathcal{P}. \tag{7}$$

Consider also the switching signal

$$\sigma(t) := \begin{cases} 1 & t < 1 \\ 2 & t \geq 1 \end{cases} \qquad\qquad t \geq 0. \tag{8}$$

Since σ has a single discontinuity, it belongs to every family of switching signals $\mathcal{S}[\tau_D]$, $\tau_D > 0$. Take now the following probe input signal with $\mathcal{L}_2$-norm equal to one:

$$u(t) := \begin{cases} 1 & t < 1 \\ 0 & t \geq 1 \end{cases} \qquad\qquad t \geq 0. \tag{9}$$

The output of (6) corresponding to the switching signal (8) and the input (9) is given by

$$y(t) = \begin{cases} c_1\big(\int_0^t e^{A_1(t-\tau)}d\tau\big)b_1 & 0 \leq t < 1 \\ c_2\big(\int_0^1 e^{A_1(t-\tau)}d\tau\big)b_1 & t \geq 1 \end{cases} \qquad\qquad t \geq 0.$$

It turns out that c_2 can be chosen arbitrarily large without violating (7), provided that b_2 is sufficiently small. Therefore, by appropriate choice of c_2, the $\mathcal{L}_2$-norm of y can actually be very large (in particular larger than 1) for a unit-norm input, *even though we are switching between two systems with RMS gain equal to one.* The intuition behind this example is that for $t < 1$ the input energy is "stored" in the system through one realization (perhaps through a "large" b_1) and then it is "released" to the output for $t \geq 1$ through a different realization (perhaps through a "large" c_2). From this example, we conclude that even for very slow switching (in the extreme a single switch), the RMS gain of a switched system can be arbitrarily larger than the RMS gains of the "unswitched systems." Moreover, the RMS gain of a switched system is realization dependent and cannot be determined just from the transfer functions of the systems being switched. The goal of this paper is then to compute the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ defined by (4).

For linear time-invariant systems, the RMS gain can be determined by solving algebraic Riccati equations. In particular, it is well known (*Bounded-real Lemma*) that the RMS gain is smaller than a given constant $\gamma > 0$ if and only if an algebraic Riccati equation of the form

$$S(P; \gamma) := A'_\gamma P + PA_\gamma + PR_\gamma P + Q_\gamma = 0, \tag{10}$$

has a positive definite solution P, where A_γ, Q_γ, and R_γ are appropriately defined γ-dependent $n \times n$ real matrices with Q_γ and R_γ symmetric and positive semidefinite (cf. Theorem 1 in Sect. 3 for details). This allows one to determine the precise value of the RMS gain of a system using a bisection algorithm to find the smallest value of γ for which a positive definite solution to (10) exists [18]. Although in general algebraic Riccati equations have more than one solution, in order to determine the RMS gain of a linear system it is sufficient to find one of these solutions. However, to determine the RMS gain of a switched system, we have to pay special attention to the multiple solutions to the algebraic Riccati equations involved. We shall see below that the condition for the slow-switching RMS gain of a switched system to be smaller than γ requires a separation between *all* the stabilizing and *all* the antistabilizing solutions to the algebraic Riccati equations of the systems being switched.

The remaining of this paper is organized as follows: In Sect. 2 we derive a general formula for the RMS gain of a switched system in terms of the solution to a differential Riccati equation. This prompts us to investigate the solutions to differential Riccati equations, which is done in Sect. 3. In Sect. 4 we determine upper and lower bounds on the RMS gain of a switched system over the set $\mathcal{S}_1$ of switching signals with no more than one discontinuity. We then show in Sect. 5 that these same bounds are also valid for the slow-switching RMS gain. A bisection-type algorithm to compute it is given in Sect. 6. Section 7 contains final conclusions and directions for future research.

2 Computation of RMS gains

To determine the RMS gain of (2) over $\mathcal{S}$ we consider the following optimization problem:

$$J(\gamma,\sigma) := \lim_{T\to\infty} \sup_{u\in\mathcal{L}_2} \int_0^T \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau, \qquad \gamma \geq 0,\ \sigma\in\mathcal{S},$$

where y is computed along solutions to (2) with $x(0)=0$. The RMS gain of (2) over $\mathcal{S}$ is then given by

$$\mathfrak{g}_{\mathcal{S}} := \inf\{\gamma \geq 0 : J(\gamma,\sigma) \leq 0,\ \forall \sigma\in\mathcal{S}\}. \tag{11}$$

Note that because the system is linear, when $J(\gamma,\sigma)>0$ we must actually have $J(\gamma,\sigma)=+\infty$, which means that $J(\gamma,\sigma)\leq 0$ in (11) could be replaced by $J(\gamma,\sigma)<\infty$. It is also sufficient to consider the case $\gamma > \|D_p\|$, $p\in\mathcal{P}$ as the RMS gain is always larger or equal to any of the high frequency gains $\|D_p\|$ because of (5).

We shall use dynamic programming to compute $J(\gamma,\sigma)$ for a given $\sigma\in\mathcal{S}$ and $\gamma > \|D_p\|$, $p\in\mathcal{P}$. To this effect let us define the following finite-horizon "cost-to-go:"

$$V(x_0,t;T,\gamma,\sigma) := \sup_{u\in\mathcal{L}_2,\, x(t)=x_0} \int_t^T \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau.$$

The corresponding dynamic-programming equation is given by[2]

$$
\begin{aligned}
-V_t &= \sup_{u \in \mathbb{R}} \|Cx_0 + Du\|^2 - \gamma^2 \|u\|^2 + V_x\big(Ax_0 + Bu\big) \\
&= V_x A x_0 + x_0' C' C x_0 + \Big(\frac{1}{2} V_x B + x_0' C' D\Big) X^{-1} \Big(\frac{1}{2} B' V_x' + D' C x_0\Big), \quad (12)
\end{aligned}
$$

where $X_p := \gamma^2 I - D_p' D_p > 0$. Since for a fixed σ we are dealing with a linear time-varying system, we can restrict our attention to costs-to-go of the form $V(x_0, t; T, \gamma, \sigma) = x_0' P(t) x_0$, $t \geq 0$ with $P(t)$ real symmetric. Replacing this in (12) we obtain

$$
0 = x_0' \big(\dot{P} + P A_\sigma + A_\sigma' P + C_\sigma' C_\sigma + (P B_\sigma + C_\sigma' D_\sigma) X_\sigma^{-1} (B_\sigma' P + D_\sigma' C_\sigma)\big) x_0,
$$

for every x_0, which (because of symmetry) actually implies that

$$
\begin{aligned}
-\dot{P} = P A_\sigma &+ A_\sigma' P + C_\sigma' C_\sigma \\
&+ (P B_\sigma + C_\sigma' D_\sigma) X_\sigma^{-1} (B_\sigma' P + D_\sigma' C_\sigma) = S_\sigma(P; \gamma), \quad (13)
\end{aligned}
$$

where

$$
S_p(P; \gamma) := P \bar{A}_p + \bar{A}_p P + Q_p + P R_p P, \qquad p \in \mathcal{P}, \tag{14}
$$

with

$$
\bar{A}_p := A_p + B_p X_p^{-1} D_p' C_p, \quad R_p := B_p X_p^{-1} B_p' \quad Q_p := C_p' (I + D_p X_p^{-1} D_p') C_p.
$$

Moreover,

$$
P(T) = 0, \tag{15}
$$

since $V(x_0, T; T, \gamma, \sigma) = 0$ for every x_0. Since (13) is locally Lipschitz, we conclude that $J(\gamma, \sigma) < \infty$ if and only if the unique solution to the final value problem (13)–(15) exists on every interval $[0, T)$, $T > 0$. Therefore

$$
\mathfrak{g}_{\mathcal{S}} = \inf \Big\{ \gamma \geq 0 : \text{solution to (13)–(15) exists on } [0, T),\ \forall T > 0,\ \sigma \in \mathcal{S} \Big\}. \tag{16}
$$

To compute the RMS gain of a switched system we must then study the existence of solution to a differential Riccati equation. We do this in the following section.

3 Differential Riccati equations

We start by considering the following *algebraic Riccati equation*

$$
S(P) := A' P + P A + P R P + Q = 0, \tag{17}
$$

[2] For simplicity we dropped the subscript $\sigma(t)$ from all matrices and the dependence of $(x_0, t; T, \gamma, \sigma)$ on V. We denote the partial derivatives of V with respect to t and x by V_t and V_x, respectively.

where A, Q, and R are $n \times n$ real matrices with Q and R symmetric. Associated with this equation one defines a $2n \times 2n$ *Hamiltonian matrix* by

$$H := \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix}.$$

In the sequel, we say that a matrix P^- is a *stabilizing* solution to the algebraic Riccati equation (17) if P^- is real, symmetric, positive definite, and $A + RP^-$ is asymptotically stable. We say that P^+ is an *antistabilizing* solution to (17) if P_+ is real, symmetric, positive definite, and $-(A + RP^+)$ is asymptotically stable. The following theorem is a consequence of several well known results on algebraic Riccati equations.

Theorem 1. *Consider a minimal realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ with $\bar{A}$ asymptotically stable and*

$$\|\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}\|_\infty < \gamma \tag{18}$$

for some $\gamma > 0$. Then $\|D\| < \gamma$ and there exist stabilizing and antistabilizing solutions P^- and P^+, respectively, to the algebraic Riccati equation (17) *with*

$$A := \bar{A} + \bar{B}\bar{X}^{-1}\bar{D}'\bar{C}, \qquad R := \bar{B}\bar{X}^{-1}\bar{B}',$$
$$Q := \bar{C}'(I + \bar{D}\bar{X}^{-1}\bar{D}')\bar{C}, \qquad X := \gamma^2 I - \bar{D}'\bar{D}.$$

Moreover, $P_+ - P_- > 0$ [3] *and these solutions can be computed by*

$$P^- = P_2^-(P_1^-)^{-1}, \qquad P^+ = P_2^+(P_1^+)^{-1}, \tag{19}$$

where the columns of $[\, P_1^{-\prime} \ P_2^{-\prime} \,]'$ and $[\, P_1^{+\prime} \ P_2^{+\prime} \,]'$ are chosen to form bases for the stable and antistable invariant[4] *subspaces of H, respectively. The matrices P_1^- and P_1^+ above are guaranteed to be nonsingular.*

Consider now the final-value solution to the following differential Riccati equation

$$-\dot{P} = S(P) = A'P + PA + PRP + Q, \qquad P(t_0) = P_0, \qquad t \le t_0. \tag{20}$$

The following theorem is inspired by results from [19] and [20, Chapter 3].

[3] Given a symmetric matrix Q we write $Q > 0$, $Q \geq 0$, $Q < 0$, and $Q \leq 0$ when Q is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively. We write $Q \not\geq 0$ when Q is nonsingular and not positive semidefinite (therefore Q must have at least one strictly negative eigenvalue). Similarly, $Q \not\leq 0$ means that Q is nonsingular and not negative semidefinite.

[4] Let H be a matrix with characteristic polynomial $\alpha(s) = \alpha^-(s)\alpha^0(s)\alpha^+(s)$, where all roots of the polynomials $\alpha^-(s)$, $\alpha^+(s)$, and $\alpha^0(s)$ have negative, positive, and zero real parts, respectively. The *stable* and *antistable* invariant subspaces of H are defined to be $\operatorname{Ker} \alpha^-(H)$ and $\operatorname{Ker} \alpha^+(H)$, respectively.

Theorem 2. *Suppose that there exist stabilizing and antistabilizing solutions to* (17) *with* $P^+ - P^- > 0$.

(i) When $P_0 - P^+$ *is nonsingular, the solution* $\Pi(t; P_0)$ *to* (20) *is given by* $\Pi(t; P_0) = P^+ + \Lambda(t)^{-1}$, $t \leq t_0$, *with*

$$\Lambda(t) := e^{(A+RP^+)(t-t_0)}\Big((P_0 - P^+)^{-1} + (P^+ - P^-)^{-1}\Big) e^{(A+RP^+)'(t-t_0)} - (P^+ - P^-)^{-1} \quad (21)$$

on the interval $\mathcal{I}$ *on which* Λ *is nonsingular.*

(ii) When $R \geq 0$ *and* $P_0 - P^+ < 0$ *the solution* $\Pi(t; P_0)$ *to* (20) *exists globally for* $t \leq t_0$ *and* $\Pi(t; P_0) \to P^-$ *as* $t \to -\infty$. *The same result holds when* $R \leq 0$ *and* $P_0 - P^+ > 0$.

(iii) When $R \geq 0$ *and* $P_0 - P^+ \not\leq 0$, *the solution* $\Pi(t; P_0)$ *to* (20) *has finite escape time. The same result holds when* $R \leq 0$ *and* $P_0 - P^+ \not\geq 0$.

The proof of this Theorem can be found in [21].

4 RMS gain of single-switch systems

In this section we consider the set $\mathcal{S}_1$ of switching signals with no more than one discontinuity. Using the results from the previous section we compute bounds on the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ of (2) over $\mathcal{S}_1$:

Theorem 3. *Assume that the realizations in* (1) *are minimal. Given any* $\gamma > \mathfrak{g}_{\mathrm{static}}$,

$$\exists p, q \in \mathcal{P} \quad P_q^+ - P_p^- \not\geq 0 \quad \Rightarrow \quad \mathfrak{g}_{\mathcal{S}_1} \geq \gamma \quad (22)$$

$$\forall p, q \in \mathcal{P} \quad P_q^+ - P_p^- > 0 \quad \Rightarrow \quad \mathfrak{g}_{\mathcal{S}_1} \leq \gamma, \quad (23)$$

where P_p^+ *and* P_p^-, $p \in \mathcal{P}$ *respectively denote the stabilizing and antistabilizing solutions to the algebraic Riccati equation* $S_p(P; \gamma) = 0$, *with* S_p *defined by* (14).

Before proving Theorem 4, it is interesting to note that the condition in (23) that gives the upper bound on the RMS gain requires a separation between *all* the stabilizing and *all* the antistabilizing solutions to the algebraic Riccati equations of the systems being switched. Note that for $p = q$, the fact that the qth system has RMS gain smaller than γ already guarantees that $P_q^+ - P_q^- > 0$ (cf. Theorem 1). However, this is not enough for $\mathfrak{g}_{\mathcal{S}_1}$ to be smaller or equal to γ. Indeed, we need all the stabilizing solutions P_p^- to be "smaller" than all the antistabilizing solutions P_q^+.

Proof (Theorem 3). To prove (22) we show that when $P_q^+ - P_p^- \not\geq 0$ for some $p, q \in \mathcal{P}$, the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ given by (16) must be larger or equal to γ because the solution to (13)–(15) does not exist for some $T > 0, \sigma \in \mathcal{S}$. To this effect suppose that there are $p, q \in \mathcal{P}$ for which $P_q^+ - P_p^- \not\geq 0$ and take

$$\sigma(t) := \begin{cases} q & t < T_{\text{switch}} \\ p & t \geq T_{\text{switch}} \end{cases} \qquad t \geq 0, \tag{24}$$

for some $T_{\text{switch}} > 0$. Because $R_p \geq 0$ and $-P_p^+ < 0$ (cf. Theorem 1), we conclude from Theorem 2 (ii) that the solution to (13)–(15) exists for every $t \in [T_{\text{switch}}, T)$. Moreover, if σ where equal to p for all times, we would actually have $P(t) \to P_p^-$ as $t \to -\infty$. This means that by choosing T sufficiently large, it is possible to have $P(T_{\text{switch}})$ sufficiently close to P_p^- so that $P_q^+ - P(T_{\text{switch}}) \not\geq 0$. Here, we are using the fact that the set $\{P : P_q^+ - P \not\geq 0\}$ is open and therefore if $P(T_{\text{switch}})$ is sufficiently close to the element P_p^- of this set, it must be inside the set. Once it is known that T is chosen such that $P_q^+ - P(T_{\text{switch}}) \not\geq 0$, by applying Theorem 2 to solve (13)–(15) for $t \leq T_{\text{switch}}$, we conclude from (iii) that the solution is actually only defined in an interval $(T_{\text{escape}}, T_{\text{switch}}]$ of finite length. Because of time invariance, we can choose T sufficiently large so that $T_{\text{escape}} > 0$. So the solution to (13)–(15) does not exist globally. This finishes the proof of (22).

To prove (23) we show directly that when $P_q^+ - P_p^- > 0$ for every $p, q \in \mathcal{P}$, we have $\|y\|_2 \leq \gamma \|u\|_2$ for every input and switching signals. To this effect pick arbitrary $u \in \mathcal{L}_2$, $\sigma \in \mathcal{S}_1$ and define

$$v(t) := V(x(t), t) + \int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau, \qquad t \geq 0,$$

where x and y are evaluated along a solution to (2) and $V(x, t) := x' P(t) x$, $t \geq 0$. Here, $P(t)$ denotes a time-varying continuous real symmetric matrix with the property that on any interval on which σ is constant we have

$$-\dot{P} = S_\sigma(P; \gamma). \tag{25}$$

To construct such $P(t)$ we use the fact that because $\sigma \in \mathcal{S}_1$, the switching signal must be of the form (24) (possibly with $p = q$ if it is actually constant). Suppose then that for $t > T_{\text{switch}}$ we set $P(t)$ constant and equal to P_p^-, and for $t \leq T_{\text{switch}}$, $P(t)$ is defined by the final value problem

$$-\dot{P} = S_q(P; \gamma), \qquad P(T_{\text{switch}}) = P_p^-.$$

The function $P(t)$ is well defined for every $t \leq T_{\text{switch}}$ because $P_p^- - P_q^+ < 0$ (cf. Theorem 2 (ii)). By construction this choice of $P(t)$ does satisfy (25) on every interval on which σ is constant. From (25) and (12)–(13) it is straightforward to conclude that on any such interval we have $\dot{v} \leq 0$. We then conclude that v decreases between discontinuities of σ and is continuous at the switching time T_{switch}. Therefore $v(t) \leq v(0) = x(0)' P(0) x(0)$, $\forall t \geq 0$. For a zero initialization of (2) we then conclude that $v(t) \leq 0$, $\forall t \geq 0$, or equivalently that

$$\int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \leq -V(x(t), t), \qquad \forall t \geq 0.$$

Since $u \in \mathcal{L}_2$, $\lim_{t\to\infty} x(t) = 0$ and therefore

$$\int_0^\infty \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \leq 0.$$

This concludes the proof since it shows that indeed $\|y\|_2 \leq \gamma \|u\|_2$. ■

5 Slow-switching RMS gain

In this section we show that the upper and lower bounds provided by Theorem 3 on the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ of (2) over $\mathcal{S}_1$ also hold for the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ of (2). For simplicity, in this section we assume that the set $\mathcal{P}$ is finite, i.e., that we are switching among a finite number of systems. However, it would be straightforward to replace this assumption by uniformity conditions on the systems being switched, e.g., compactness of the family of realizations $\{A_p, B_p, C_p, D_p : p \in \mathcal{P}\}$.

Theorem 4. *Assume that the realizations in* (1) *are minimal and* $\mathcal{P}$ *is finite. Given any* $\gamma > \mathfrak{g}_{\text{static}}$

$$\exists p, q \in \mathcal{P} \quad P_q^+ - P_p^- \not\geq 0 \qquad \Rightarrow \qquad \mathfrak{g}_{\text{slow}} \geq \gamma \tag{26}$$
$$\forall p, q \in \mathcal{P} \quad P_q^+ - P_p^- > 0 \qquad \Rightarrow \qquad \mathfrak{g}_{\text{slow}} \leq \gamma, \tag{27}$$

where P_p^+ *and* P_p^-, $p \in \mathcal{P}$ *respectively denote the stabilizing and antistabilizing solutions to the algebraic Riccati equation* $S_p(P;\gamma) = 0$, *with* S_p *defined by* (14).

Proof (Theorem 4). To prove (26) note that for any $\tau_D > 0$, $\mathcal{S}_1 \subset \mathcal{S}[\tau_D]$ and therefore $\mathfrak{g}_{\mathcal{S}_1} \leq \mathfrak{g}_{\text{slow}}$. From Theorem 3 we then conclude that

$$\exists p, q \in \mathcal{P} \quad P_q^+ - P_p^- \not\geq 0 \qquad \Rightarrow \qquad \mathfrak{g}_{\mathcal{S}_1} \geq \gamma \qquad \Rightarrow \qquad \mathfrak{g}_{\text{slow}} \geq \mathfrak{g}_{\mathcal{S}_1} \geq \gamma.$$

To prove (27) assume that $P_q^+ - P_p^- > 0$ for every $p, q \in \mathcal{P}$ and therefore that we can select a positive constant ϵ and matrices $P_{(p,q)}$, $p, q \in \mathcal{P}$ such that

$$\|P - P_p^-\| \leq \epsilon \quad \Rightarrow \quad P_q^+ > P_{(p,q)} \geq P, \qquad\qquad \forall p, q \in \mathcal{P}. \tag{28}$$

In practice, $P_{(p,q)}$ can be any matrix sufficiently close to P_p^- so that $P_q^+ > P_{(p,q)} > P_p^-$ and ϵ a sufficiently small positive constant so that for any P such that $\|P - P_p^-\| \leq \epsilon$ we still have $P_{(p,q)} \geq P$.

Since $P_{(p,q)} - P_q^+ < 0$, we conclude from Theorem 2 (ii) that the solution to the final value problem

$$-\dot{P} = S_q(P;\gamma), \qquad\qquad P(T) = P_{(p,q)}, \tag{29}$$

exists globally and converges to P_q^- as $t \to -\infty$. Pick now a constant τ_D sufficiently large so that $\|P(t) - P_q^-\| \leq \epsilon$, $\forall t \leq T - \tau_D$ regardless of the choices for

p and q in (29). We show next that the RMS gain over $\mathcal{S}[\tau_D]$ is smaller or equal to γ and therefore so is the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$.

Pick arbitrary $u \in \mathcal{L}_2$, $\sigma \in \mathcal{S}_1$ and define

$$v(t) := V(x(t), t) + \int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau, \qquad t \geq 0,$$

where x and y are evaluated along a solution to (2) and $V(x,t) := x' P(t) x$, $t \geq 0$. Here, $P(t)$ denotes a time-varying symmetric matrix with the property that on any interval (t_1, t_2) on which σ is constant and equal to q we have

$$-\dot{P} = S_q(P; \gamma), \qquad P(t_2) = P_0, \tag{30}$$

where P_0 is chosen equal to P_q^- in case $t_2 = \infty$ and equal to $P_{(p,q)}$ when $t_2 < \infty$, with p being the value of σ immediately after t_2. Note that in the former case (which occurs only when t_1 is the last switching time), $P(t)$ is constant for $t > t_1$. In the latter case, the solution to (30) is well defined for every $t \leq t_2$ because $P_{(p,q)} - P_q^+ < 0$ (cf. Theorem 2 (ii)).

As in the proof of Theorem 3, we conclude that $\dot{v} \leq 0$ on any interval on which σ is constant. However, now $v(t)$ is discontinuous at switching times. In particular, if σ switches from q to p at time t_2, we have

$$\lim_{\tau \downarrow t_2} v(\tau) - \lim_{\tau \uparrow t_2} v(\tau) = x(t_2)' \big(\bar{P} - P_{p,q}\big) x(t_2),$$

where $\bar{P}$ is either equal to P_p^- in case t_2 is the last switching time or the solution at time t_2 of a final value problem of the form

$$-\dot{P} = S_p(P; \gamma), \qquad P(t_3) = P_{(\bar{p},p)},$$

for some $\bar{p} \in \mathcal{P}$ and with $t_3 \geq t_2 + \tau_D$. However, we selected τ_D sufficiently large so that $\|P(t_2) - P_p^-\| \leq \epsilon$. Therefore we always have $\|\bar{P} - P_p^-\| \leq \epsilon$ whether or not t_2 is the last switching time. From this and (28) we conclude that $\bar{P} - P_{p,q} \leq 0$ and therefore

$$\lim_{\tau \downarrow t_2} v(\tau) - \lim_{\tau \uparrow t_2} v(\tau) = x(t_2)' \big(\bar{P} - P_{p,q}\big) x(t_2) \leq 0.$$

This means that even at points of discontinuity of v this signal decreases and therefore also here $v(t) \leq v(0)$, $\forall t \geq 0$. As in the proof of Theorem 3 we can then conclude that $\|y\|_2 \leq \gamma \|u\|_2$ for a zero initialization of (2). This finishes the proof since we showed that the RMS gain over $\mathcal{S}[\tau_D]$ is indeed no larger than γ.

■

Remark 1. The proof of (27) actually provides a bound on the dwell-time τ_D to have the RMS gain over $\mathcal{S}[\tau_D]$ smaller or equal to γ. In general, if the separation between the stabilizing and antistabilizing is large, we can choose the ϵ in (28) large and therefore have an RMS gain no larger than γ even for small τ_D.

6 Computation of the slow-switching RMS gain

Theorem 4 suggests the following bisection algorithm to compute the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ of (2):

1. *Select a tolerance ϵ, an upper bound γ_{up} and a lower bound γ_{low} on $\mathfrak{g}_{\text{slow}}$ (e.g., $\gamma_{\text{low}} := \mathfrak{g}_{\text{static}}$ and γ_{up} taken from [12])*
2. *If $\gamma_{\text{up}} - \gamma_{\text{low}} \leq 2\epsilon$ then stop (the RMS gain $\mathfrak{g}_{\text{slow}}$ is away from $\frac{\gamma_{\text{up}}+\gamma_{\text{low}}}{2}$ by less than ϵ)*
3. *Otherwise set $\gamma = \frac{\gamma_{\text{up}}+\gamma_{\text{low}}}{2}$ and*

$$\begin{array}{llll} \textit{If} & \exists p,q \in \mathcal{P} \quad P_q^+ - P_p^- \not> 0 & \textit{then set} & \gamma_{\text{low}} = \gamma \\ \textit{If} & \forall p,q \in \mathcal{P} \quad P_q^+ - P_p^- > 0 & \textit{then set} & \gamma_{\text{up}} = \gamma \end{array}$$

and go back to step 2

The previous algorithm only fails if there is a positive-length interval of values for γ on which all $P_q^+ - P_p^- \geq 0$ but at least one of these matrices has an eigenvalue exactly at the origin. Generically, this should not happen over an interval of positive length since an infinitely small perturbation on γ should make the eigenvalue become either positive or negative. However, even when this happens one can still use algorithms similar to the one above to compute the following upper and lower bounds on $\mathfrak{g}_{\text{slow}}$:

$$\begin{aligned} &\sup\{\gamma > \mathfrak{g}_{\text{static}} : \exists p,q \in \mathcal{P},\ P_q^+ - P_p^- \not> 0\} \\ &\qquad \leq \mathfrak{g}_{\text{slow}} \leq \inf\{\gamma > \mathfrak{g}_{\text{static}} : P_q^+ - P_p^- > 0,\ \forall p,q \in \mathcal{P}\}. \end{aligned}$$

Remark 2. The verification of the matrix-inequality conditions in step 3 is computationally simple when $\mathcal{P}$ is a finite set. Indeed, when $\mathcal{P}$ has n elements, in the worst case the verification of this conditions requires the computation of the stabilizing and anti-stabilizing solutions to n algebraic Riccati equations (e.g., using (19) in Theorem 1) and then verifying if $n(n-1)$ matrices are positive definite. However, if $\mathcal{P}$ is a continuum (i.e., when one can switch among an infinitely large family of systems) the above algorithm may not be practical.

7 Conclusions

We presented a method to compute the slow-switching RMS gain of a switched linear system. The algorithm proposed uses the fact that a given constant γ provides an upper bound on the RMS gain whenever there is a separation between *all* the stabilizing and *all* the antistabilizing solutions to the algebraic Riccati equations of the systems being switched. We are now in the process of expressing this condition in the form of Linear Matrix Inequalities. The motivation is to construct efficient algorithms that can be used to design switching controllers that minimize the RMS gain of the closed-loop.

Acknowledgments

The author would like to thank Dr. G. Zhai for finding an error in previous related work and for subsequent discussions that lead to the results presented here, and also to Dr. D. Liberzon for several comments that found their way into the final version of the paper. The reference [20] was graciously provided by Dr. E. Jonckheere.

References

1. J. P. Hespanha, *Encyclopedia of Life Support Systems*, ch. Stabilization Through Hybrid Control. UNESCO, Feb. 2001. Submitted.
2. A. P. Molchanov and Y. S. Pyatnitskiy, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Syst. & Contr. Lett.*, vol. 13, pp. 59–64, 1989.
3. K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices," *IEEE Trans. on Automat. Contr.*, vol. 39, pp. 2469–2471, Dec. 1994.
4. L. Gurvits, "Stability of linear inclusions—part 2," tech. rep., NECI, Dec. 1996.
5. T. M. Yoshihiro Mori and Y. Kuroe, "A solution to the common Lyapunov function problem for continuous-time systems," in *Proc. of the 36th Conf. on Decision and Contr.*, vol. 3, pp. 3530–3531, Dec. 1997.
6. W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. on Automat. Contr.*, vol. 44, pp. 751–760, Apr. 1999.
7. D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched linear systems: a Lie-algebraic condition," *Syst. & Contr. Lett.*, vol. 37, pp. 117–122, June 1999.
8. R. N. Shorten and K. S. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear time-invariant systems," in *Proc. of the 1999 American Contr. Conf.*, pp. 1410–1414, June 1999.
9. R. N. Shorten and K. S. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for m stable second order linear time-invariant systems," in *Proc. of the 2000 Amer. Contr. Conf.*, pp. 359–363, June 2000.
10. A. A. Agrachev and D. Liberzon, "Lie-algebraic stability criteria for switched systems," *SIAM J. Contr. Optimization*, vol. 40, no. 1, pp. 253–269, 2001.
11. A. S. Morse, "Supervisory control of families of linear set-point controllers—part 1: exact matching," *IEEE Trans. on Automat. Contr.*, vol. 41, pp. 1413–1431, Oct. 1996.
12. J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. of the 38th Conf. on Decision and Contr.*, pp. 2655–2660, Dec. 1999.
13. D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Contr. Syst. Mag.*, vol. 19, pp. 59–70, Oct. 1999.
14. A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. on Automat. Contr.*, vol. 42, pp. 819–830, June 1997.
15. M. Žefran, F. Bullo, and M. Stein, "A notion of passivity for hybrid systems." To appear in the 40th IEEE Conf. on Dec. and Control., Dec. 2001.

16. J. P. Hespanha, *Logic-Based Switching Algorithms in Control*. PhD thesis, Yale University, New Haven, CT, 1998.
17. G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Disturbance attenuation properties of time-controlled switched systems." Submitted to publication., 2001.
18. K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey: Prentice Hall, 1996.
19. J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. on Automat. Contr.*, vol. AC-16, pp. 621–634, Dec. 1971.
20. J. M. Rodriguez-Canabal, *The Geometry of the Riccati Equation*. PhD thesis, University of Southern California, Los Angeles, CA, June 1972.
21. J. P. Hespanha, "Computation of $\mathcal{L}_2$-induced norms of switched linear systems," tech. rep., University of California, Santa Barbara, CA, Oct. 2001.