Conditions for Saddle-Point Equilibria in Output-Feedback MPC with MHE

David A. Copp and João P. Hespanha

***Abstract*— A new method for solving output-feedback model predictive control (MPC) and moving horizon estimation (MHE) problems simultaneously as a single min-max optimization problem was recently proposed. This method allows for stability analysis of the joint output-feedback control and estimation problem. In fact, under the main assumption that a saddle-point solution exists for the min-max optimization problem as well as standard observability and controllability assumptions, practical stability can be established in the presence of noise and disturbances. In this paper we derive sufficient conditions for the existence of a saddle-point solution to this min-max optimization problem. For the specialized linear-quadratic case, we show that a saddle-point solution exists if the system is observable and weights in the cost function are chosen appropriately. A numerical example is given to illustrate the effectiveness of this combined control and estimation approach.**

I. Introduction

Classical MPC has been a prominent control technique in academia and industrial applications for decades because of its ability to handle complex multivariable systems and hard constraints. MPC, which is classically formulated with state-feedback and involves the repeated solution of an open-loop optimal control problem online in order to find a sequence of future control inputs, has a well-developed theory as evidenced by [1–3], and has been shown to be effective in practice [4]. However, with more efficient methods and continued theoretical work, more recent advances in MPC include the incorporation of disturbances, uncertainties, faster dynamics, distributed systems, and output-feedback.

Related to these advances is the recent work combining nonlinear output-feedback MPC with MHE into a single min-max optimization problem [5]. This combined approach simultaneously solves an MHE problem, which involves the repeated solution of a similar optimization problem over a finite-horizon of past measurements in order to find an estimate of the current state [6, 7], and an MPC problem. In order to be robust to "worst-case" disturbances and noise, this approach involves the solution of a min-max optimization where an objective function is minimized with respect to control input variables and maximized with respect to disturbance and noise variables, similar to game-theoretic approaches to MPC considered in [8, 9].

The motivation for the combined MPC/MHE approach is proving joint stability of the combined estimation and control problems, and the results in [5] guarantee boundedness of the state, bounds on the tracking error for trajectory tracking problems, and practical stability in the presence of noise and disturbances. Besides standard assumptions regarding observability and controllability of the nonlinear process, the main assumption required for these results to hold is that there exists a saddle-point solution to the min-max optimization problem at every time step.

The analysis of the min-max problem that appears in the forward horizon of the combined MPC/MHE approach is closely related to the analysis of two-player zero-sum dynamic games as in [10] and to the dynamic game approach to H^∞ optimal control as in [11]. In these analyses the control is designed to guard against the worst-case unknown disturbances and model uncertainties, and in both of these references, saddle-point equilibria and conditions under which they exist are analyzed. The problem proposed in [5] differs, however, in that a backwards finite horizon is also considered in order to incorporate the simultaneous solution of an MHE problem, which also allows the control to be robust to worst-case estimates of the initial state.

In this paper we derive conditions under which a saddle-point solution exists for the combined MPC/MHE min-max optimization problem proposed in [5] and specialize those results for discrete linear time-invariant (DLTI) systems and quadratic cost functions. We show that in the linear-quadratic case, if the system is observable, simply choosing appropriate weights in the cost function is enough to ensure that a saddle-point solution exists. A numerical example discussed at the end of the paper shows that, even for unconstrained linear-quadratic problems, better regulation performance may be achieved using this MPC/MHE approach with

D. A. Copp and J. P. Hespanha are with the Center for Control, Dynamical Systems, and Computation, University of California, Santa Barbara, CA 93106 USA. dacopp@engr.ucsb.edu,hespanha@ece.ucsb.edu

shorter finite horizons.

The paper is organized as follows. In Section II, we formulate the control problem and discuss the main stability assumption regarding the existence of a saddle-point. In Section III, we describe a method that can be used to compute a saddle-point solution and give conditions under which this method succeeds. A numerical example is presented in Section IV. Finally, we provide some conclusions in Section V.

II. Problem Formulation

As in [5], we consider the control of a time-varying discrete-time process of the form

$$x_{t+1} = f_t(x_t, u_t, d_t), \qquad y_t = g_t(x_t) + n_t \tag{1}$$

$\forall t \in \mathbb{Z}_{\geqslant 0}$, with *state* x_t taking values in a set $\mathcal{X} \subset \mathbb{R}^{n_x}$. The inputs to this system are the *control input* u_t that must be restricted to a set $\mathcal{U} \subset \mathbb{R}^{n_u}$, the *unmeasured disturbance* d_t that is known to belong to a set $\mathcal{D} \subset \mathbb{R}^{n_d}$, and the *measurement noise* $n_t \in \mathbb{R}^{n_n}$. The signal $y_t \in \mathcal{Y} \subset \mathbb{R}^{n_y}$ denotes the *measured output* that is available for feedback. The *control objective* is to select the control signal $u_t \in \mathcal{U}$, $\forall t \in \mathbb{Z}_{\geqslant 0}$, so as to minimize a finite-horizon criterion of the form[1]

$$\begin{aligned} J_t(x_{t-L}, u_{t-L:t+T-1}, d_{t-L:t+T-1}, y_{t-L:t}) \\ \coloneqq \sum_{k=t}^{t+T-1} c_k(x_k, u_k) + q_{t+T}(x_{t+T}) \\ - \sum_{k=t-L}^{t} \eta_k(n_k) - \sum_{k=t-L}^{t+T-1} \rho_k(d_k) \end{aligned} \tag{2}$$

for worst-case values of the unmeasured disturbance $d_t \in \mathcal{D}$, $\forall t \in \mathbb{Z}_{\geqslant 0}$ and the measurement noise $n_t \in \mathbb{R}^{n_n}$, $\forall t \in \mathbb{Z}_{\geqslant 0}$. The functions $c_k(\cdot)$, $\eta_k(\cdot)$, and $\rho_k(\cdot)$ in (2) are all assumed to take non-negative values. One can view the terms $\rho_t(\cdot)$ and $\eta_t(\cdot)$ as measures of the likelihood of specific values for d_t and n_t. Then, the negative signs in front of $\rho_t(\cdot)$ and $\eta_t(\cdot)$ penalize the maximizer for using low likelihood values for the disturbances and noise (low likelihood meaning very large values for $\rho_t(\cdot)$ and $\eta_t(\cdot)$).

The optimization criterion includes $T \in \mathbb{Z}_{\geqslant 1}$ terms of the running cost $c_t(x_t, u_t)$, which recede as the current time t advances, $L+1 \in \mathbb{Z}_{>1}$ terms of the measurement cost $\eta_t(n_t)$, and $L+T \in \mathbb{Z}_{>1}$ terms of the cost on the disturbances $\rho_t(d_t)$. We also include a terminal cost $q_{t+T}(x_{t+T})$ to penalize the "final" state at time $t+T$.

[1]Given a discrete-time signal $z : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}^n$, and two times $t_0, t \in \mathbb{Z}_{\geqslant} 0$ with $t_0 \leqslant t$, we denote by $z_{t_0:t}$ the sequence $\{z_{t_0}, z_{t_0+1}, \ldots, z_t\}$.

Just as in a two-player zero-sum dynamic game, player 1 (the controller) desires to minimize this criterion while player 2 (the noise and disturbance) would like to maximize it. This leads to a control input that is designed for the worst-case disturbance input, measurement noise, and initial state. This motivates the following finite-dimensional optimization

$$\min_{\hat{u}_{t:t+T-1} \in \mathcal{U}} \quad \max_{\hat{x}_{t-L} \in \mathcal{X}, \hat{d}_{t-L:t+T-1} \in \mathcal{D}} \\ J_t(\hat{x}_{t-L}, u_{t-L:t-1}, \hat{u}_{t:t+T-1}, \hat{d}_{t-L:t+T-1}, y_{t-L:t}). \tag{3}$$

The measurement noise variables $n_{t-L:t}$ do not explicitly show up in (3) because they are not independent optimization variables as they are uniquely defined by the remaining optimization variables and the output equation (1). In this formulation, we use a control law of the form

$$u_t = \hat{u}_t^*, \quad \forall t \geqslant 0, \tag{4}$$

where $\hat{u}_t^*$ denotes the first element of the sequence $\hat{u}_{t:t+T-1}^*$ computed at each time t that minimizes (3).

For the implementation of the control law (4), the outer minimizations in (3) must lead to finite values for the optima that are achieved at specific sequences $\hat{u}_{t:t+T-1}^* \in \mathcal{U}$, $t \in \mathbb{Z}_{\geqslant 0}$. However, for the stability results given in [5], we actually ask for the existence of a saddle-point solution to the min-max optimization in (3) as follows:

Assumption 1 (Saddle-point [5, Assumption 1]): The min-max optimization (3) with cost given as in (2) always has a saddle-point solution for which the min and max commute. Specifically, for every time $t \in \mathbb{Z}_{\geqslant 0}$, past control input sequence $u_{t-L:t-1} \in \mathcal{U}$, and past measured output sequence $y_{t-L:t} \in \mathcal{Y}$, there exists a finite scalar $J_t^* \in \mathbb{R}$, an initial condition $\hat{x}_{t-L}^* \in \mathcal{X}$, and sequences $\hat{u}_{t:t+T-1}^* \in \mathcal{U}$, $\hat{d}_{t-L:t+T-1}^* \in \mathcal{D}$ such that

$$\begin{aligned} J_t^* &= J_t(\hat{x}_{t-L}^*, u_{t-L:t-1}, \hat{u}_{t:t+T-1}^*, \hat{d}_{t-L:t+T-1}^*, y_{t-L:t}) \\ &= \max_{\hat{x}_{t-L} \in \mathcal{X}, \hat{d}_{t-L:t+T-1} \in \mathcal{D}} \\ &\quad J_t(\hat{x}_{t-L}, u_{t-L:t-1}, \hat{u}_{t:t+T-1}^*, \hat{d}_{t-L:t+T-1}, y_{t-L:t}) \qquad \text{(5a)} \\ &= \min_{\hat{u}_{t:t+T-1} \in \mathcal{U}} \\ &\quad J_t(\hat{x}_{t-L}^*, u_{t-L:t-1}, \hat{u}_{t:t+T-1}, \hat{d}_{t-L:t+T-1}^*, y_{t-L:t}). \qquad \text{(5b)} \end{aligned}$$

□

In general, J_t^* depends on the past control and outputs, so we can alternatively write $J_t^*(u_{t-L:t-1}, y_{t-L:t})$.

In the next section we derive conditions under which a saddle-point solution exists for the general nonlinear case and then specialize those results for DLTI systems and quadratic cost functions.

III. Main Results

Before presenting the main results, for convenience we define the following sets of time sequences for the forward and backward horizons, respectively, $\mathcal{T} \coloneqq \{t, t+1, ..., t+T-1\}$ and $\mathcal{L} \coloneqq \{t-L, t-L+1, ..., t-1\}$ and use them in the sequel.

A. Nonlinear systems

Theorem 1: (Existence of saddle-point)

Suppose there exist recursively computed functions $V_k(\cdot)$, for all $k \in \mathcal{T}$, and $V_j(\cdot)$, for all $j \in \mathcal{L}$, such that for all $y_{t-L:t} \in \mathcal{Y}$, and $u_{t-L:t-1} \in \mathcal{U}$,

$$V_{t+T}(x_{t+T}) \coloneqq q_{t+T}(x_{t+T}), \tag{6a}$$

$$\begin{aligned} V_k(x_k) &\coloneqq \min_{\hat{u}_k \in \mathcal{U}} \max_{\hat{d}_k \in \mathcal{D}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k) + V_{k+1}(x_{k+1})\big) \\ &= \max_{\hat{d}_k \in \mathcal{D}} \min_{\hat{u}_k \in \mathcal{U}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k) + V_{k+1}(x_{k+1})\big), \\ &\qquad \forall k \in \mathcal{T} \backslash t, \end{aligned} \tag{6b}$$

$$\begin{aligned} V_t(x_t, y_t) &\coloneqq \min_{\hat{u}_t \in \mathcal{U}} \max_{\hat{d}_t \in \mathcal{D}} \big(l_t(x_t, \hat{u}_t, \hat{d}_t, y_t) + V_{t+1}(x_{t+1})\big) \\ &= \max_{\hat{d}_t \in \mathcal{D}} \min_{\hat{u}_t \in \mathcal{U}} \big(l_t(x_t, \hat{u}_t, \hat{d}_t, y_t) + V_{t+1}(x_{t+1})\big), \end{aligned} \tag{6c}$$

$$\begin{aligned} V_j(x_j, u_{j:t-1}, y_{j:t}) &\coloneqq \max_{\hat{d}_j \in \mathcal{D}} \big(l_j(x_j, u_j, \hat{d}_j, y_j) \\ &+ V_{j+1}(x_{j+1}, u_{j+1:t-1}, y_{j+1:t})\big), \ \forall j \in \mathcal{L} \backslash t-L, \end{aligned} \tag{6d}$$

$$\begin{aligned} V_{t-L}(u_{t-L:t-1}, y_{t-L:t}) &\coloneqq \max_{\hat{x}_{t-L} \in \mathcal{X}, \hat{d}_{t-L} \in \mathcal{D}} \\ &\big(l_{t-L}(\hat{x}_{t-L}, u_{t-L}, \hat{d}_{t-L}, y_{t-L}) \\ &+ V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t})\big), \end{aligned} \tag{6e}$$

where

$$l_k(x_k, \hat{u}_k, \hat{d}_k) \coloneqq c_k(x_k, \hat{u}_k) - \rho_k(\hat{d}_k), k \in \mathcal{T} \backslash t, \tag{7a}$$

$$l_t(x_t, \hat{u}_t, \hat{d}_t, y_t) \coloneqq c_t(x_t, \hat{u}_t) - \eta_t(n_t) - \rho_t(\hat{d}_t), \tag{7b}$$

$$l_j(x_j, u_j, \hat{d}_j, y_j) \coloneqq -\eta_j(n_j) - \rho_j(\hat{d}_j), \quad j \in \mathcal{L}. \tag{7c}$$

Then the solutions $\hat{u}_k^*$, $\hat{d}_k^*$, $\hat{d}_j^*$, and $\hat{x}_{t-L}^*$ defined as follows, for all $k \in \mathcal{T}$ and $j \in \mathcal{L}$, satisfy the saddle-point Assumption 1.

$$\hat{u}_k^* \coloneqq \arg\min_{\hat{u}_k \in \mathcal{U}} \max_{\hat{d}_k \in \mathcal{D}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k, y_k) + V_{k+1}(x_{k+1})\big), \tag{8a}$$

$$\hat{d}_k^* \coloneqq \arg\max_{\hat{d}_k \in \mathcal{D}} \min_{\hat{u}_k \in \mathcal{U}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k, y_k) + V_{k+1}(x_{k+1})\big), \tag{8b}$$

$$\begin{aligned} \hat{d}_j^* \coloneqq \arg\max_{\hat{d}_j \in \mathcal{D}} &\big(l_j(x_j, u_j, \hat{d}_j, y_j) \\ &+ V_{j+1}(x_{j+1}, u_{j+1:t-1}, y_{j+1:t})\big), \end{aligned} \tag{8c}$$

$$\begin{aligned} \hat{x}_{t-L}^* \coloneqq \arg\max_{\hat{x}_{t-L} \in \mathcal{X}} &\big(l_{t-L}(\hat{x}_{t-L}, u_{t-L}, \hat{d}_{t-L}, y_{t-L}) \\ &+ V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t})\big). \end{aligned} \tag{8d}$$

Moreover, the saddle-point value is equal to

$$J_t^*(u_{t-L:t-1}, y_{t-L:t}) = V_{t-L}(u_{t-L:t-1}, y_{t-L:t}). \quad \square$$

Proof. We begin by proving equation (5b) in Assumption 1. Let $\hat{u}_k^*$ be defined as in (8a), and let $\bar{u}_k$ be another arbitrary control input. To prove optimality, we need to show that the latter trajectory cannot lead to a cost lower than the former. Since $V_k(x_k)$ satisfies (6b) and $\hat{u}_k^*$ achieves the minimum in (6b), for every $k \in \mathcal{T} \backslash t$,

$$\begin{aligned} V_k(x_k) &= \min_{\hat{u}_k \in \mathcal{U}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k^*) + V_{k+1}(x_{k+1})\big) \\ &= l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) + V_{k+1}(x_{k+1}). \end{aligned} \tag{9}$$

However, since $\bar{u}_k$ does not necessarily achieve the minimum, we also have that

$$\begin{aligned} V_k(x_k) &= \min_{\hat{u}_k \in \mathcal{U}} \big(l_k(x_k, \hat{u}_k, \hat{d}_k^*) + V_{k+1}(x_{k+1})\big) \\ &\leqslant l_k(x_k, \bar{u}_k, \hat{d}_k^*) + V_{k+1}(x_{k+1}). \end{aligned} \tag{10}$$

Summing both sides of (9) from $k = t+1$ to $k = t+T-1$, we conclude that

$$\begin{aligned} &\sum_{k=t+1}^{t+T-1} V_k(x_k) = \\ &\quad \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) + \sum_{k=t+1}^{t+T-1} V_{k+1}(x_{k+1}) \\ &\quad \iff V_{t+1}(x_{t+1}) = \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*). \end{aligned}$$

Next, summing both sides of (10) from $k = t+1$ to $k = t+T-1$, we conclude that

$$\begin{aligned} &\sum_{k=t+1}^{t+T-1} V_k(x_k) \leqslant \\ &\quad \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, \hat{d}_k^*) + \sum_{k=t+1}^{t+T-1} V_{k+1}(x_{k+1}) \\ &\quad \iff V_{t+1}(x_{t+1}) \leqslant \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, \hat{d}_k^*), \end{aligned}$$

from which we conclude that

$$V_{t+1}(x_{t+1}) = \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) \leqslant \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, d_k^*). \quad (11)$$

Similarly, since $V_t(x_t, y_t)$ satisfies (6c) and $\hat{u}_t^*$ achieves the minimum in (6c), we can conclude that

$$\begin{aligned} V_t(x_t, y_t) &= l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) + V_{t+1}(x_{t+1}) \\ &\leqslant l_t(x_t, \bar{u}_t, \hat{d}_t^*, y_t) + V_{t+1}(x_{t+1}). \quad (12) \end{aligned}$$

Then from (11) and (12), we conclude that

$$\begin{aligned} V_t(x_t, y_t) &= l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) + \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) \\ &\leqslant l_t(x_t, \bar{u}_t, \hat{d}_t^*, y_t) + \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, d_k^*). \quad (13) \end{aligned}$$

Next, since $V_j(x_j, u_{j:t-1}, y_{j:t})$ satisfies (6d) and $\hat{d}_j^*$ achieves the maximum in (6d), we can conclude that

$$\begin{aligned} V_j(x_j, u_{j:t-1}, y_{j:t}) &= l_j(x_j, u_j, \hat{d}_j^*, y_j) \\ &\quad + V_{j+1}(x_{j+1}, u_{j:t-1}, y_{j:t}). \quad (14) \end{aligned}$$

Summing both sides of (14) from $j = t - L + 1$ to $j = t - 1$, and using (13), we conclude that

$$\begin{aligned} &\sum_{j=t-L+1}^{t-1} V_j(x_j, u_{j:t-1}, y_{j:t}) = \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) \\ &\quad + \sum_{j=t-L+1}^{t-1} V_{j+1}(x_{j+1}, u_{j:t-1}, y_{j:t}) \\ &\iff V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t}) = \\ &\sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) + l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) \\ &+ \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) \leqslant \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) \\ &\quad + l_t(x_t, \bar{u}_t, \hat{d}_t^*, y_t) + \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, \hat{d}_k^*). \end{aligned}$$

Finally, from this and the facts that $V_{t-L}(u_{t-L:t-1}, y_{t-L:t})$ satisfies (6e) and $\hat{d}_{t-L}^*$ and $\hat{x}_{t-L}^*$ achieve the maximum in (6e), we can conclude that

$$\begin{aligned} &V_{t-L}(u_{t-L:t-1}, y_{t-L:t}) = l_{t-L}(\hat{x}_{t-L}^*, u_{t-L}, \hat{d}_{t-L}^*, y_{t-L}) \\ &\quad + V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t}) \\ &= \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) + l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) \\ &\quad + \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) \\ &+ l_{t-L}(\hat{x}_{t-L}^*, u_{t-L}, \hat{d}_{t-L}^*, y_{t-L}) \leqslant \sum_{k=t+1}^{t+T-1} l_k(x_k, \bar{u}_k, \hat{d}_k^*) \\ &\quad + l_t(x_t, \bar{u}_t, \hat{d}_t^*, y_t) + \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) \\ &\quad + l_{t-L}(\hat{x}_{t-L}^*, u_{t-L}, \hat{d}_{t-L}^*, y_{t-L}). \end{aligned}$$

Therefore $\hat{u}_k^*$ is a minimizing policy, for all $k \in \mathcal{T}$, and (5b) is satisfied with $J_t^*(u_{t-L:t-1}, y_{t-L:t}) = V_{t-L}(u_{t-L:t-1}, y_{t-L:t})$.

To prove (5a), let $\hat{d}_k^*$ be defined as in (8b), $\hat{d}_j^*$ be defined as in (8c), and let $\bar{d}_k$ and $\bar{d}_j$ be other arbitrary disturbance inputs. Similarly, let $\hat{x}_{t-L}^*$ be defined as in (8d), and let $\bar{x}_{t-L}$ be another arbitrary initial condition. Then, since $V_k(x_k)$ satisfies (6b), $V_t(x_t, y_t)$ satisfies (6c), $V_j(x_j, u_{j:t-1}, y_{j:t})$ satisfies (6d), and $\hat{d}_k^*$ achieves the maximum in (6b), $\hat{d}_t^*$ achieves the maximum in (6c), and $\hat{d}_j^*$ achieves the maximum in (6d), we can use a similar argument as in the proof of (5b) to conclude that

$$\begin{aligned} &V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t}) = \\ &\quad \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) + l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) \\ &+ \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) \geqslant \sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \bar{d}_j, y_j) \\ &\quad + l_t(x_t, \hat{u}_t^*, \bar{d}_t, y_t) + \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \bar{d}_k). \end{aligned}$$

Finally, (5a) follows from this, (6e), (8c), (8d), and similar arguments as used before (further details of which can be found in [12]), which lead to

$$\begin{aligned} &V_{t-L}(u_{t-L:t-1}, y_{t-L:t}) = l_{t-L}(\hat{x}_{t-L}^*, u_{t-L}, \hat{d}_{t-L}^*, y_{t-L}) \\ &\quad + V_{t-L+1}(x_{t-L+1}, u_{t-L+1:t-1}, y_{t-L+1:t}) \\ &\quad = \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \hat{d}_k^*) + l_t(x_t, \hat{u}_t^*, \hat{d}_t^*, y_t) \\ &+ \sum_{j=t-L}^{t-1} l_j(x_j, u_j, \hat{d}_j^*, y_j) + l_{t-L}(\hat{x}_{t-L}^*, u_{t-L}, \hat{d}_{t-L}^*, y_{t-L}) \\ &\quad \geqslant \sum_{k=t+1}^{t+T-1} l_k(x_k, \hat{u}_k^*, \bar{d}_k) + l_t(x_t, \hat{u}_t^*, \bar{d}_t, y_t) \end{aligned}$$

$$+\sum_{j=t-L+1}^{t-1} l_j(x_j, u_j, \bar{d}_j, y_j) + l_{t-L}(\bar{x}_{t-L}, u_{t-L}, \bar{d}_{t-L}, y_{t-L}).$$

Therefore $\hat{d}_k^*$, for all $k \in \mathcal{T}$, and $\hat{d}_j^*$, for all $j \in \mathcal{L}$, are maximizing policies, $\hat{x}_{t-L}^*$ is a maximizing policy, and (5a) is satisfied with $J_t^*(u_{t-L:t-1}, y_{t-L:t}) = V_{t-L}(u_{t-L:t-1}, y_{t-L:t})$.

Thus, Assumption 1 is satisfied. □

Next we specialize these results for DLTI systems and quadratic cost functions.

B. DLTI systems and quadratic costs

Consider the following DLTI system, for all $t \in \mathbb{Z}_{\geqslant 0}$,

$$x_{t+1} = Ax_t + Bu_t + Dd_t, \qquad y_t = Cx_t + n_t, \quad (15)$$

with $x_t \in \mathcal{X} = \mathbb{R}^{n_x}$, $u_t \in \mathcal{U} = \mathbb{R}^{n_u}$, $d_t \in \mathcal{D} = \mathbb{R}^{n_d}$, $n_t \in \mathcal{N} = \mathbb{R}^{n_n}$, and $y_t \in \mathcal{Y} = \mathbb{R}^{n_y}$. Also consider the quadratic cost function

$$\begin{aligned} &J_t(x_{t-L}, u_{t-L:t+T-1}, d_{t-L:t+T-1}, y_{t-L:t}) \\ &\quad \coloneqq \sum_{k=t}^{t+T-1} \left(x_k' Q x_k + \lambda_u u_k' u_k\right) + x_{t+T}' Q x_{t+T} \\ &- \sum_{k=t-L}^{t} \lambda_n (y_k - Cx_k)'(y_k - Cx_k) - \sum_{k=t-L}^{t+T-1} \lambda_d d_k' d_k \end{aligned} \quad (16)$$

where $Q = Q' \geqslant 0$ is a weighting matrix, and λ_u, λ_d, and λ_n are positive constants that can be tuned to impose "soft" constraints on the variables u_k, d_k, and n_k, respectively, or to increase or decrease the penalty for choosing low likelihood values for the disturbances and noise. The weights λ_u, λ_d, and λ_n could be replaced with positive-definite matrices, and the following results would still hold with minor adjustments.

Again, the control objective is to solve for a control input u_t^* that minimizes the criterion (16) in the presence of the worst-case disturbance d_t^* and initial state x_{t-L}^*. This motivates solving the optimization problem (3) with the cost given as in (16) subject to the dynamics given in (15). Then the control input as defined in (4) is selected and applied to the plant. The following Theorem gives conditions under which a saddle-point solution exists for problem (3) with cost (16), thereby satisfying Assumption 1.

Theorem 2: (Existence of saddle-point for linear systems with quadratic costs) Let M_k and Λ_k, for all $k \in \mathcal{T}$, and P_j and Z_j, for all $j \in \mathcal{L}$, be matrices of appropriate dimensions defined by[2]

$$M_k = Q + A' M_{k+1} \Lambda_k^{-1} A; \quad M_{t+T} = Q, \quad (17a)$$

$$\Lambda_k \coloneqq I + \Big(\frac{1}{\lambda_u} BB' - \frac{1}{\lambda_d} DD'\Big) M_{k+1}, \quad (17b)$$

$$P_j = A' P_{j+1} A + A' P_{j+1} D Z_j^{-1} D' P_{j+1} A - \lambda_n C' C;$$

$$P_t = M_t - \lambda_n C' C, \quad (17c)$$

$$Z_j \coloneqq \lambda_d I - D' P_{j+1} D. \quad (17d)$$

Then, if the following conditions are satisfied,

$$\lambda_u I + B' M_{k+1} B > 0, \quad (18a)$$

$$\lambda_d I - D' M_{k+1} D > 0, \quad (18b)$$

$$\lambda_d I - D' P_{j+1} D > 0, \quad (18c)$$

$$\begin{bmatrix} \lambda_n C'C - A' P_{t-L+1} A & -A' P_{t-L+1} D \\ -D' P_{t-L+1} A & \lambda_d I - D' P_{t-L+1} D \end{bmatrix} > 0, \quad (18d)$$

the min-max optimization (3) with quadratic costs (16) subject to the linear dynamics (15) admits a unique saddle-point solution that satisfies Assumption 1. □

Proof. This proof can be found in [12]. □

Corollary 1: If the discrete-time linear time-invariant system given in (15) is observable, then the scalar weights λ_n and λ_d can be chosen sufficiently large such that the conditions (18a)-(18d) are satisfied. Therefore, according to Theorem 2, there exists a saddle-point solution for the optimization problem (3) with cost (16). Therefore, also, Assumption 1 is satisfied. □

Proof. Condition (18a) is trivially satisfied for all $k \in \mathcal{T}$ as long as we choose $\lambda_u > 0$ and weighting matrix $Q \geqslant 0$ because $Q \geqslant 0 \implies M_k \geqslant 0, \forall k \in \mathcal{T}$. [3]

Condition (18b) is satisfied if the scalar weight λ_d is chosen sufficiently large. To show this, we take the limit of the sequence of matrices M_k, as given in (17a), as $\lambda_d \to \infty$ and notice that $M_k \to \bar{M}_k$, where $\bar{M}_k$ is described by

$$\bar{M}_k = Q + A'(\bar{M}_{k+1}[I + \frac{1}{\lambda_u} BB' \bar{M}_{k+1}]^{-1})A;$$
$$\bar{M}_{t+T} = M_{t+T},$$

for all $k \in \mathcal{T}$. Then, as $\lambda_d \to \infty$, λ_d is greater than the largest eigenvalue of $D' \bar{M}_{k+1} D$, and therefore, condition (18b) is satisfied when λ_d is chosen sufficiently large.

Next we prove that conditions (18c) and (18d) are satisfied, for all $j \in \mathcal{L}$, when λ_n and λ_d are chosen

[2] I denotes the identity matrix with appropriate dimensions.

[3] Note that $M_k = M_k'$ due to the fact that $Q = Q'$ and the matrix identity in [13] which says that $A(I + BA)^{-1} = (I + AB)^{-1} A$.

sufficiently large and the system (15) is observable. We first take the limit of the sequence of matrices P_j, as given in (17c), as $\lambda_d \to \infty$ and notice that $P_j \to \bar{P}_j$ where $\bar{P}_j$ is described by

$$\bar{P}_j = -\lambda_n \Theta_j' \Theta_j + A'^{t-j} \bar{M}_t A^{t-j},$$

for all $j \in \mathcal{L} \cup t$, and Θ_j is defined as

$$\Theta_j \coloneqq [C' \quad A'C' \quad A'^2 C' \quad \dots \quad A'^{t-j} C']'.$$

The matrix Θ_j looks similar to the observability matrix, and therefore, $\Theta_j' \Theta_j > 0$ if the system given in (15) is observable.

Then, the scalar weight λ_n can be chosen large enough to ensure that $\lambda_n \Theta_j' \Theta_j > A'^{t-j} \bar{M}_t A^{t-j}$, for all $j \in \mathcal{L}$. It then follows that $\bar{P}_j < 0$ for all $j \in \mathcal{L}$. Therefore, condition (18c) becomes $\lambda_d I - D' \bar{P}_{j+1} D > 0$ in the limit as $\lambda_d \to \infty$ and is trivially satisfied if system (15) is observable and λ_n is chosen sufficiently large.

Finally, consider condition (18d). Using the Schur Complement, condition (18d) is satisfied if, and only if, $\lambda_d I - D' P_{t-L+1} D > 0$ and

$$\lambda_n C'C - A' P_{t-L+1} A$$
$$- A' P_{t-L+1} D (\lambda_d I - D' P_{t-L+1} D)^{-1} D' P_{t-L+1} A > 0.$$

We just proved that $\lambda_d I - D' P_{t-L+1} D > 0$ if the system (15) is observable and the weights λ_d and λ_n are chosen sufficiently large. Then, in the limit as $\lambda_d \to \infty$, the second inequality becomes

$$\lambda_n C'C - A' \bar{P}_{t-L+1} A > 0.$$

Therefore, if the system (15) is observable, λ_n can be chosen sufficiently large such that this inequality is satisfied, and therefore, condition (18d) is satisfied. □

IV. SIMULATION

Various choices for the parameters in the cost function (16) lead to different control inputs that may all satisfy the saddle-point assumption but that produce very different closed-loop performance. For instance, there are examples for which a short finite-horizon approach performs better than a quasi-infinite-horizon approach even for unconstrained linear-quadratic problems. Specifically, better disturbance attenuation can be achieved for the following unconstrained linear-quadratic example, where the system is subjected to impulsive disturbances, using the combined MPC/MHE approach with shorter finite-horizon lengths.

Example 1: (Stabilizing a riderless bicycle.) Consider the following second order continuous-time linearized bicycle model in state-space form:

$$\dot{x}_t = A x_t + B(u_t + d_t), \qquad y_t = C x_t + n_t \qquad (19)$$

The state is given by $x_t = \begin{bmatrix} \phi & \delta & \dot{\phi} & \dot{\delta} \end{bmatrix}'$ where ϕ is the roll angle of the bicycle, δ is the steering angle of the handlebars, and $\dot{\phi}$ and $\dot{\delta}$ are the corresponding angular velocities. The control input u_t is the steering torque applied to the handlebars. The matrices defining the linearized dynamics are, as described in [14],

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 13.67 & 0.225 - 1.319v^2 & -0.164v & -0.552v \\ 4.857 & 10.81 - 1.125v^2 & 3.621v & -2.388v \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ -0.339 \\ 7.457 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where v is the bicycle's forward velocity. Only the roll and steering angles (and not their corresponding angular velocities) are measured and available for feedback. In this example, we fix the forward velocity at $v = 2$ m/s, which results in an unstable system, and discretize the system using a 0.1 second zero-order-hold.

The control objective is to stabilize the bicycle in the upright position, i.e. around a zero roll angle ($\phi = 0$), by applying a steering torque to the handlebars. The disturbance d_t acts on the input and can be thought of jerks on the steering due to sharp bumps in the bicycle's path or similar environmental perturbations. We solve this problem by solving the optimization given in (3) with cost (16) at each time t and apply the resulting $\hat{u}_t^*$ as the control input. The measurement noise is a random variable $n_t \sim \mathcal{N}(0, 0.001^2)$. The disturbance d_t is nominally a random variable $d_t \sim \mathcal{N}(0, 0.01^2)$ but with occasional large, impulsive values.

Because the system (19) is observable, we are able to choose λ_d and λ_n so that conditions (18) are satisfied. Therefore, a saddle-point solution to (3) with cost (16) exists for the riderless bicycle example according to Theorem 2. However, conditions (18) are only sufficient conditions and may lead to unnecessarily conservative weights. In this example, it is possible to achieve better performance by choosing weights that do not satisfy conditions (18) but that still ensure the existence of a saddle-point which can be verified numerically.

In this example, we compare results for long horizon lengths ($L = T = 200$) to results for short horizon lengths ($L = 2$, $T = 7$) and tune the weights λ_d and λ_n in order to achieve the best performance as determined by minimizing the tracking error $\|\phi\|$. Table I shows four simulation scenarios. Rows #1 and #2 of Table I show the weights λ_d and λ_n that satisfy the conditions (18) and provide the best performance for both the long and short horizon lengths. Rows #3 and #4 of Table I show the best possible weights λ_d and λ_n for performance that still ensure the existence of a saddle-point (verified numerically) but that do not satisfy the conditions (18). For all four scenarios, the weighting matrix Q in the cost (16) is chosen as a 4x4 matrix with the element in upper left corner equal to one and all other elements equal to zero, and λ_u is chosen as 0.001.

TABLE I

TUNING PARAMETERS AND PERFORMANCE

	L	T	λ_d	λ_n	$\|\phi\|$ [deg]	$\|u\|$ [Nm]
#1	200	200	1500	10^7	29.0	19.3
#2	2	7	90	10^7	27.1	19.2
#3	200	200	0.02	15000	22.1	18.0
#4	2	7	0.002	15000	14.6	35.5

Figure 1 shows results for the scenarios in rows #3 and #4 of Table I. The top plot shows the measured output ϕ, the middle plot shows the measured output δ, and the bottom plot shows the applied control input u^* as well as the true disturbance d that is the same for all four of the scenarios in Table I.

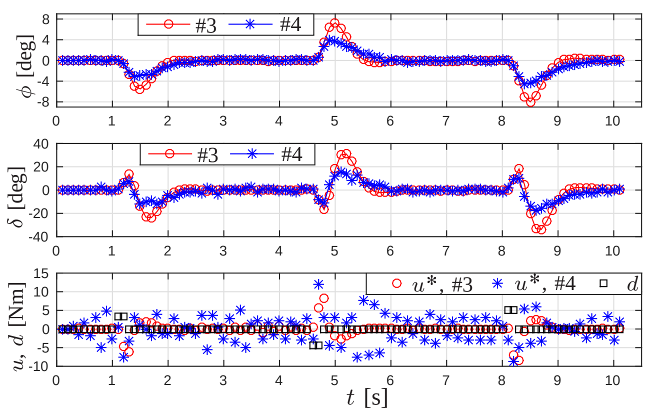

Fig. 1. Comparing results for longer horizons (red o's) versus shorter horizons (blue *'s) with weights given in rows #3 and #4 of Table I.

The control input computed using the shorter horizons is able to regulate the roll angle ϕ back to zero without as much oscillation as the control input computed using the longer horizons. This is because a larger λ_d is required with long horizons to satisfy the saddle-point assumption while a smaller λ_d can be used with short horizons. In this case, a smaller λ_d results in a less conservative control input that better attenuates the large impulsive disturbances. Therefore, it may be beneficial to use the finite-horizon MPC/MHE approach over other standard infinite-horizon control techniques for particular types of unconstrained linear-quadratic problems.

V. CONCLUSION

We discussed the main assumption that a saddle-point solution exists for a new approach to solving MPC and MHE problems simultaneously as a single min-max optimization problem as proposed in [5]. First we gave conditions for the existence of a saddle-point solution when considering a general discrete-time nonlinear system and a general cost function. Next we specialized those results for DLTI systems and quadratic cost functions. For this case, we showed that observability of the linear system and large weights λ_d and λ_n in the cost function are sufficient conditions for a saddle-point solution to exist.

We showed the effectiveness of this control approach in a numerical example of a linearized riderless bicycle system subjected to impulsive disturbances and illustrated the importance of carefully choosing tuning parameters in order to achieve desirable performance.

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