Disturbance Attenuation for Linear Systems: Optimal Solutions and Nonlinear Control

Davide Mannini, Joao P. Hespanha, James B. Rawlings

Abstract—This study presents the optimal solutions to the state feedback finite horizon disturbance attenuation problem using a game theory approach. The proposed solution method is valid for linear dynamical systems and quadratic objective functions, under bounded disturbances. Two solution regions in the space of initial states define the optimal value of the controller. The first region, which contains the zero initial state, features the linear optimal H_∞ controller, whereas the second region is characterized by a nonlinear optimal control, which converges to the linear quadratic regulator (LQR) for large initial states. The transition between the two regions provides a unified framework that spans from H_∞ control to LQR control, depending on the relative sizes of the initial state and the bound on the disturbance. A novel solution algorithm is introduced, reducing the optimization of the disturbance attenuation problem to a linear algebra problem for the first region and a single, scalar nonlinear program for the second region. The performance of the algorithm is tested on a representative set of numerical examples. This paper enhances the versatility of disturbance attenuation state feedback controllers by introducing an efficient optimal solution strategy applicable to both zero and nonzero initial states. One consequence of these results is that optimal disturbance attenuation of even linear systems requires nonlinear control.

I. INTRODUCTION

The disturbance attenuation problem, also known as the sensitivity problem, involves designing a state feedback control policy that ensures disturbances on a dynamic system are attenuated to a tolerable extent. As a form of robust control, disturbance attenuation is critical for systems with highly coupled controls, operational state-dependent dynamics, or substantial modeling uncertainties [1]. Applications span across diverse fields including aerospace engineering (particularly helicopter control) [1, 2], mechanical engineering (engine and submersibles control), chemical engineering (distillation column operations), and nuclear engineering (pressurized water reactor control) [3–7].

The design of minmax controllers using game-theoretic approaches dates back to the 1960s, with early work by Dorato and Drenick [8] examining the sensitivity problem in the time domain. Zames [9] popularized the disturbance attenuation problem for linear systems by formulating it in the frequency domain, aiming to minimize the H_∞ norm of the closed-loop transfer function from bounded disturbance input to performance output. The complexity of solving the problem in the frequency domain motivated Glover and Doyle [10] to develop time domain solutions. Basar [11] further expanded this approach, facilitating a reconciliation with previous dynamic game theory approaches by solving the finite and infinite horizon problems recursively using dynamic programming for zero initial states. This framework casts the disturbance attenuation problem as a dynamic noncooperative sequential zero-sum game with two participants: the control (leader) and the disturbance (follower).

Existing time domain solutions typically assume zero initial states or consider small regions around the origin. Didinsky and Basar [12] addressed nonzero initial states by delineating distinct solution regions in the initial state space, while Basar [13] obtained a nonlinear controller for a scalar example near the origin. Despite developments in both recursive and non-recursive solution methods, a comprehensive, efficient algorithm spanning all potential initial states and bounded disturbances remains elusive.

In this paper, we broaden disturbance attenuation by formulating a unified framework capable of handling both zero and nonzero initial states. We adopt a finite-horizon, zero-sum Stackelberg game approach with bounded disturbances, leading to an optimal state-feedback controller that is inherently nonlinear. The main contributions are:

- A scalar-parameter (Lagrange multiplier) optimization for linear systems and quadratic costs, enabling an efficient way to solve the disturbance attenuation problem under any initial condition.
- A characterization of two solution regions in the initial-state space: (i) a region containing the origin, which features the classic H_∞ solution, a linear control policy, and (ii) a region characterized by a nonlinear state-dependent control policy, which converges to the linear quadratic regulator (LQR) solution for large initial states.
- A practical algorithm combining linear algebra operations (in the linear region) with a single scalar nonlinear program (in the nonlinear region).

The paper is structured as follows: Section II provides background on previous disturbance attenuation approaches and introduces game theory preliminaries. Section III presents key theoretical results for solving

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the disturbance attenuation problem. Section IV examines static disturbance attenuation to illustrate solution structures and sequential game properties. Section V formulates the finite horizon disturbance attenuation problem for linear systems under bounded disturbances. Section VI characterizes the solution regions and presents an efficient algorithm. Section VII demonstrates the proposed approach through numerical examples, showing the nonlinear nature of optimal control and computational performance across different systems. The paper concludes with a summary in Section VIII, and mathematical proofs provided in the Appendix.

Notation: Let $\mathbb{I}$ and $\mathbb{R}$ denote the integers and reals. $|x|$ denotes the Euclidean norm of vector x; $|M|$ denotes the induced 2-norm of matrix M. The norm of a signal, defined as vector-valued function of scalar-valued time t, $x(t)$, is defined as $\|x(t)\|_2 := (\sum_{k=0}^{\infty} x(k)^* x(k))^{1/2}$. Let $\mathbf{y} \in \mathbb{Y}^N$ be a vector sequence defined as $\mathbf{y} := (y(0), y(1), \ldots, y(N-1))$. Let K_∞ and K_{LQR} denote the optimal control state feedback gain obtained from the H_∞ control problem and the LQR problem, respectively. The column space (range) and nullspace of a matrix M are denoted by $\mathscr{R}(M)$ and $\mathscr{N}(M)$, respectively. The pseudoinverse of a matrix M is denoted as M^+. A stationary point is a point in the domain of a function at which the gradient is zero.

II. BACKGROUND

A. Previous Disturbance Attenuation Work

Previous approaches to disturbance attenuation with nonzero initial conditions have provided valuable insights but have some limitations. Didinsky and Basar [12] identified that the space of initial conditions can be partitioned into two distinct regions: one admitting pure-strategy saddle points and another requiring mixed policies. Their analysis did not yield explicit structure for the optimal controller in all regions, however.

Khargonekar et al. [14] formulated an H_∞ control problem where, unlike the approach proposed here, the initial state is treated as an additional disturbance input rather than as a parameter for feedback. Their formulation maximizes over both the initial state and the exogenous disturbance, effectively seeking robustness against both sources of uncertainty simultaneously. This differs fundamentally from the ideas of interest here, where the initial state is a measured (or estimated) parameter used to compute the feedback control.

This paper solves for the optimal control policy across the entire state space, explicitly revealing how the nonlinear control structure emerges from the underlying game-theoretic minmax problem, and provides a seamless transition between H_∞ and LQR control as the initial state varies.

Furthermore, previous work considered open constraint sets and thus formulated the problem by taking infimum, the current approach extends the formulation to closed constraint sets. This modification allows the problem statement to use minimization and guarantees the existence of solutions without requiring limiting processes.

B. Game Theory Preliminaries

The disturbance attenuation problem can be formulated as a two-player, zero-sum game. We introduce concepts related to static Stackelberg games, which feature sequential moves with one player (the leader) moving first and the other (the follower) responding.

Consider a function $f : \mathbb{U} \times \mathbb{W} \to \mathbb{R}$, where $(u, w) \in \mathbb{U} \times \mathbb{W}$, with $\mathbb{U} \subseteq \mathbb{R}^n$ and $\mathbb{W} \subseteq \mathbb{R}^m$. When control u is the leader and disturbance w is the follower, the Stackelberg game formulation is

$$\overline{\mathscr{P}}: \quad \min_u \max_w f(u, w) = f(u^*, w^0(u^*))$$

where $w^0(\cdot)$ denotes the optimal solution to the inner problem $\max_w f(\cdot)$, when it exists.

Definition 1 (Stackelberg equilibrium for $\overline{\mathscr{P}}$). The point $(u^*, w^0(u^*)) \in \mathbb{U} \times \mathbb{W}$ is a Stackelberg equilibrium of problem $\overline{\mathscr{P}}$ if

$$f(u^*, w) \le f(u^*, w^0(u^*)) \le f(u, w^0(u)) \quad u \in \mathbb{U}, w \in \mathbb{W}$$

When disturbance w is the leader and control u is the follower, the formulation becomes

$$\underline{\mathscr{P}}: \quad \max_w \min_u f(u, w) = f(u^0(w^*), w^*)$$

where $u^0(\cdot)$ denotes the optimal solution to the inner problem $\min_u f(\cdot)$.

Definition 2 (Stackelberg equilibrium for $\underline{\mathscr{P}}$). The point $(u^0(w^*), w^*) \in \mathbb{U} \times \mathbb{W}$ is a Stackelberg equilibrium of problem $\underline{\mathscr{P}}$ if

$$f(u^0(w), w) \le f(u^0(w^*), w^*) \le f(u, w^*) \quad u \in \mathbb{U}, w \in \mathbb{W}$$

Remark 3. While nonzero-sum Stackelberg games typically feature a first-mover advantage, zero-sum Stackelberg games often exhibit a second-mover advantage. Generally, strong duality does not hold for Stackelberg games, so that

$$\min_u \max_w f(u, w) > \max_w \min_u f(u, w)$$

When strong duality does hold in zero-sum Stackelberg games, the solutions are saddle points.

Definition 4 (Saddle point for zero-sum games). The point $(u^*, w^*) \in \mathbb{U} \times \mathbb{W}$ is a saddle point (also called Nash equilibrium) for zero-sum games with function $f(\cdot)$ if

$$f(u^*, w) \le f(u^*, w^*) \le f(u, w^*) \quad u \in \mathbb{U}, w \in \mathbb{W}$$

For saddle points, the order of players' actions does not impact the outcome of the game, i.e., $\min_u \max_w f(u, w) = \max_w \min_u f(u, w)$. Moreover, saddle points for zero-sum games with differentiable functions are always stationary points, while Stackelberg equilibria are not necessarily stationary points [15].

The disturbance attenuation problem is formulated as a zero-sum Stackelberg dynamic game with a second-mover advantage. Robust control designs a controller resilient against worst-case bounded disturbances, mathematically interpreted as a minmax sequential game where the

disturbance (follower) holds an advantageous position. The dual problem, placing control in the follower position, represents finding the worst-case disturbance of a feedforward controller.

Although the optimal control is obtained by solving a sequential game between control and disturbance, in actual applications the disturbances entering the closed-loop system are typically unknown. We may consider a variety of scenarios, including how the controller performs when faced with the computed worst-case disturbances, or how fragile the controller is when its assumptions about the disturbance class are violated.

C. Mathematical Problem Statement

This paper addresses the finite horizon disturbance attenuation problem for discrete-time linear systems with quadratic costs and bounded disturbances. The system dynamics are

$$x^+ = Ax + Bu + w \qquad x(0) = x_0$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^n$ is the disturbance. The objective function is

$$V_N(x_0; \mathbf{u}, \mathbf{w}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

with stage cost $\ell(x,u) = (1/2)(x'Qx + u'Ru)$, terminal cost $V_f(x) = (1/2)x'P_f x$, and sequences $\mathbf{u} := (u(0), \ldots, u(N-1))$, $\mathbf{w} := (w(0), \ldots, w(N-1))$. The disturbance is constrained to $\|\mathbf{w}\|_2^2 \le \alpha$ for given bound $\alpha > 0$, and the weighting matrices satisfy $Q \ge 0$, $R > 0$, and $P_f \ge 0$.

The disturbance attenuation problem is formulated as the zero-sum Stackelberg game

$$\min_{\mathbf{u}} \max_{\mathbf{w}} V_N(x_0; \mathbf{u}, \mathbf{w}) \quad \text{s.t. } \|\mathbf{w}\|_2^2 \le \alpha$$

where control acts as the leader and disturbance as the follower. This paper derives and characterizes the optimal state feedback control policy $\kappa_N(x_0)$ across different regions of the initial state space.

III. GAME THEORY RESULTS

This section introduces a proposition to solve the disturbance attenuation problem, a minmax optimization, and the worst-case feedforward control problem, a maxmin optimization. The following Schur complements of the matrix $M(\lambda)$ are useful for expressing the solutions

$$M(\lambda) := \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} - \lambda I \end{bmatrix}$$

$$\tilde{M}_{11}(\lambda) := (M_{22} - \lambda I) - M_{12}' M_{11}^+ M_{12}$$

$$\tilde{M}_{22}(\lambda) := M_{11} - M_{12}(M_{22} - \lambda I)^+ M_{12}'$$

Proposition 5 (Minmax and maxmin of constrained quadratic functions). Consider quadratic function $V(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}$, compact constraint set $\mathbb{W} := \{w \mid w'w = 1\}$, and Lagrangian function $L(\cdot) : \mathbb{R}^{m+n+1} \to \mathbb{R}$

$$V(u,w) = (1/2)\begin{bmatrix} u \\ w \end{bmatrix}' M(0) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d$$

$$L(u,w,\lambda) = (1/2)\begin{bmatrix} u \\ w \end{bmatrix}' M(\lambda) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

where $L := V - (\lambda/2)(w'w - 1)$. We consider the two constrained optimization problems

$$\min_{u} \max_{w \in \mathbb{W}} V(u,w) \quad \text{robust control} \tag{1}$$

$$\max_{w \in \mathbb{W}} \min_{u} V(u,w) \quad \text{worst-case feedforward control} \tag{2}$$

Assume $M(0) \ge 0$ and $M_{11} > 0$. For $d \in \mathscr{R}(M(\lambda))$ denote stationary points by $(u^*(\lambda), w^*(\lambda))$ and evaluated Lagrangian function

$$\begin{bmatrix} u^*(\lambda) \\ w^*(\lambda) \end{bmatrix} \in -M^+(\lambda)d + \mathscr{N}(M(\lambda))$$

$$\begin{aligned} L(\lambda) &= V(u^*, w^*) - (1/2)\lambda((w^*)'w^* - 1) \\ &= -(1/2)d'M^+(\lambda)d + \lambda/2 \end{aligned}$$

We then have the following results.

1) The solution to problem (1) exists for all $d \in \mathbb{R}^{m+n}$ and is given by

$$u_r^0 = u^*(\lambda_r^0) \qquad w_r^0 = \overline{w}^0 \cap \mathbb{W}$$

where λ_r^0 denotes the solution to the following optimization, which exists for all $d \in \mathbb{R}^{m+n}$

$$\lambda_r^0 = \arg \min_{\lambda \ge |M_{22}|} L(\lambda) \tag{3}$$

and $\overline{w}^0$ is all solutions to

$$M_{12}' u^*(\lambda_r^0) + (M_{22} - \lambda_f^0 I)\overline{w}^0 = -d_1$$

The optimal cost is given by $V(u_r^0, w_r^0) = L(\lambda_r^0)$.

2) The solution to problem (2) exists for all $d \in \mathbb{R}^{m+n}$ and is given by

$$u_f^0 = \underline{u}^0 \qquad w_f^0 = w^*(\lambda_f^0) \cap \mathbb{W}$$

where λ_f^0 denotes the solution to the following optimization, which exists for all $d \in \mathbb{R}^{m+n}$

$$\lambda_f^0 = \arg \min_{\lambda \ge |\tilde{M}_{11}|} L(\lambda) \tag{4}$$

and $\underline{u}^0$ is all solutions to

$$M_{11}\underline{u}^0 + M_{12} w^*(\lambda_f^0) = -d_2$$

The optimal cost is given by $V(u_f^0, w_f^0) = L(\lambda_f^0)$.

The proof of Proposition 5 is given in the Appendix.

Remark 6 (Proposition 5 with $d = 0$). When $d = 0$, the optimal solutions are

$$\lambda_r^0 = |M_{22}| \qquad \lambda_f^0 = |\tilde{M}_{11}|$$

and the stationary points are

$$\begin{bmatrix} u^*(\lambda) \\ w^*(\lambda) \end{bmatrix} \in \mathscr{N}(M(\lambda)).$$

More details are available in [16, 17].

Furthermore, to facilitate the definition of the solution regions in Section VI, we present the following proposition, which provides a solution to the constrained quadratic maximization problem.

Proposition 7 (Constrained quadratic optimization). Define the convex quadratic function, $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and compact constraint set $\mathbb{W}$

$$V(w) := (1/2)w'Dw + w'd \qquad \mathbb{W} := \{w \mid w'w = 1\}$$

with $D \in \mathbb{R}^{n\times n} \geq 0$. Consider the constrained maximization problem

$$\max_{w\in\mathbb{W}} V(w) \tag{5}$$

Define the Lagrangian function

$$L(w,\lambda) = V(w) - (1/2)\lambda(w'w - 1)$$

and the (unconstrained) Lagrangian problem

$$\max_{w}\min_{\lambda} L(w,\lambda) \tag{6}$$

and the (unconstrained) dual Lagrangian problem

$$\min_{\lambda}\max_{w} L(w,\lambda) \tag{7}$$

1) Solutions to all three problems (5), (6), and (7) exist for all $D \geq 0$ and $d \in \mathbb{R}^n$ with optimal value

$$V^0 = L^0 = -(1/2)d'(D - \lambda_P I)^+ d + \lambda_P/2$$

where

$$\lambda_P := \text{ largest real eigenvalue of } P, \quad P := \begin{bmatrix} D & I \\ dd' & D \end{bmatrix}$$

2) Problems (6) and (7) satisfy strong duality, and the function $L(w,\lambda)$ has saddle points (sets) (w^*,λ^*) given by

$$w^* = \begin{cases} \left(-(D-\lambda_P I)^+ d + \mathscr{N}(D-\lambda_P I) \right) \cap W, & \lambda_P = |D| \\ -(D-\lambda_P I)^{-1} d, & \lambda_P > |D| \end{cases}$$

$$\lambda^* = \lambda_P$$

3) The optimizer of (5) is given by $w^0 = w^*$.
4) The optimizer of (6) is given by

$$w^0 = w^* \qquad \underline{\lambda}^0(w^0) = \mathbb{R}$$

5) The optimizer of (7) is given by

$$\overline{w}^0(\lambda^0) = \begin{cases} -(D-\lambda_P I)^+ d + \mathscr{N}(D-\lambda_P I), & \lambda_P = |D| \\ -(D-\lambda_P I)^{-1} d, & \lambda_P > |D| \end{cases}$$

$$\lambda^0 = \lambda_P$$

6) Additionally $\lambda_P = |D|$ if and only if (i) $d \in \mathscr{R}(D - |D|I)$ and (ii) $|(D-|D|I)^+ d| \leq 1$. If (i) or (ii) do not hold, then $\lambda_P > |D|$ and $\left|(D-\lambda_P I)^{-1} d\right| = 1$.

The proof for Proposition 7 is given in [16, 17].

IV. STATIC DISTURBANCE ATTENUATION

Before examining the disturbance attenuation problem for dynamic systems, we analyze the simpler steady-state case to illustrate the solution structure of sequential games. We consider both the disturbance attenuation problem (minmax optimization) and the worst-case feedforward control problem (maxmin optimization).

A. Minmax Optimization: Disturbance Attenuation

The static disturbance attenuation problem is formulated as

$$\min_{u_s}\max_{w_s} V_s(u_s,w_s) \qquad \text{s.t. } |w_s|^2 \leq \alpha \tag{8}$$

with objective function

$$V_s(u_s,w_s) = (1/2)(x_s'Qx_s + u_s'Ru_s)$$

where $Q \geq 0$ and $R > 0$. The steady-state model is defined as

$$x_s = Ax_s + Bu_s + Gw_s \qquad x_s = Y_u u_s + Y_w w_s$$

with $Y_u := (I-A)^{-1}B$ and $Y_w := (I-A)^{-1}G$. Assume the system lacks an integrator ($1 \notin \text{eig}(A)$, i.e., matrix A has no eigenvalues equal to 1), and $x_s \in \mathbb{R}^n$, $u_s \in \mathbb{R}^m$, and $w_s \in \mathbb{R}^n$ represent steady-state values.

We first establish the equivalence of the inequality and equality constraints in the optimization problem.

Proposition 8. Problem (8) can be equivalently formulated with the constraint $|w_s|^2 = \alpha$ rather than $|w_s|^2 \leq \alpha$, yielding

$$\min_{u_s}\max_{w_s} \frac{V_s(u_s,w_s)}{\alpha} \qquad \text{s.t. } w_s \in \mathbb{W}_s \tag{9}$$

where $\mathbb{W}_s := \{w_s \mid |w_s|^2 = \alpha\}$.

Proof. Consider arbitrary fixed u_s. If $Y_w'QY_w = 0$, then V_s does not depend on w_s, so assume $Y_w'QY_w \neq 0$. Suppose for contradiction that $\max_{w_s} V_s(u_s,w_s)$ has an unconstrained optimum when $|w_s|^2 < \alpha$.

The quadratic term in $V_s(u_s,w_s)$ with respect to w_s is $(1/2)w_s'(Y_w'QY_w)w_s$. An unconstrained maximum exists only if $Y_w'QY_w \leq 0$. However, since $Q \geq 0$, $Y_w'QY_w \geq 0$. Therefore $Y_w'QY_w = 0$, which contradicts our assumption.

Thus, $\max_{w_s} V_s(u_s,w_s)$ does not have an unconstrained optimum, and the constraint must be active at the solution, giving $|w_s|^2 = \alpha$. □

Thus problem (8) is equivalent to (9)

$$\min_{u_s}\max_{w_s} \frac{V_s(u_s,w_s)}{\alpha} \qquad \text{s.t. } w_s \in \mathbb{W}_s$$

The objective function can be rewritten as

$$V_s(u_s,w_s) = (1/2)\begin{bmatrix} u_s \\ w_s \end{bmatrix}' \begin{bmatrix} Y_u'QY_u + R & Y_u'QY_w \\ Y_w'QY_u & Y_w'QY_w \end{bmatrix} \begin{bmatrix} u_s \\ w_s \end{bmatrix}$$

Applying Proposition 5, the optimal solutions are

$$u_s^0 = 0 \qquad w_s^0(u_s^0) \in \mathscr{N}(Y_w'QY_w - \lambda^0 I) \cap \mathbb{W}_s$$

with optimal objective value

$$V_s(u_s^0, w_s^0(u_s^0)) = (1/2)\alpha\lambda^0$$

where $\lambda^0 = |Y_w'QY_w|$.

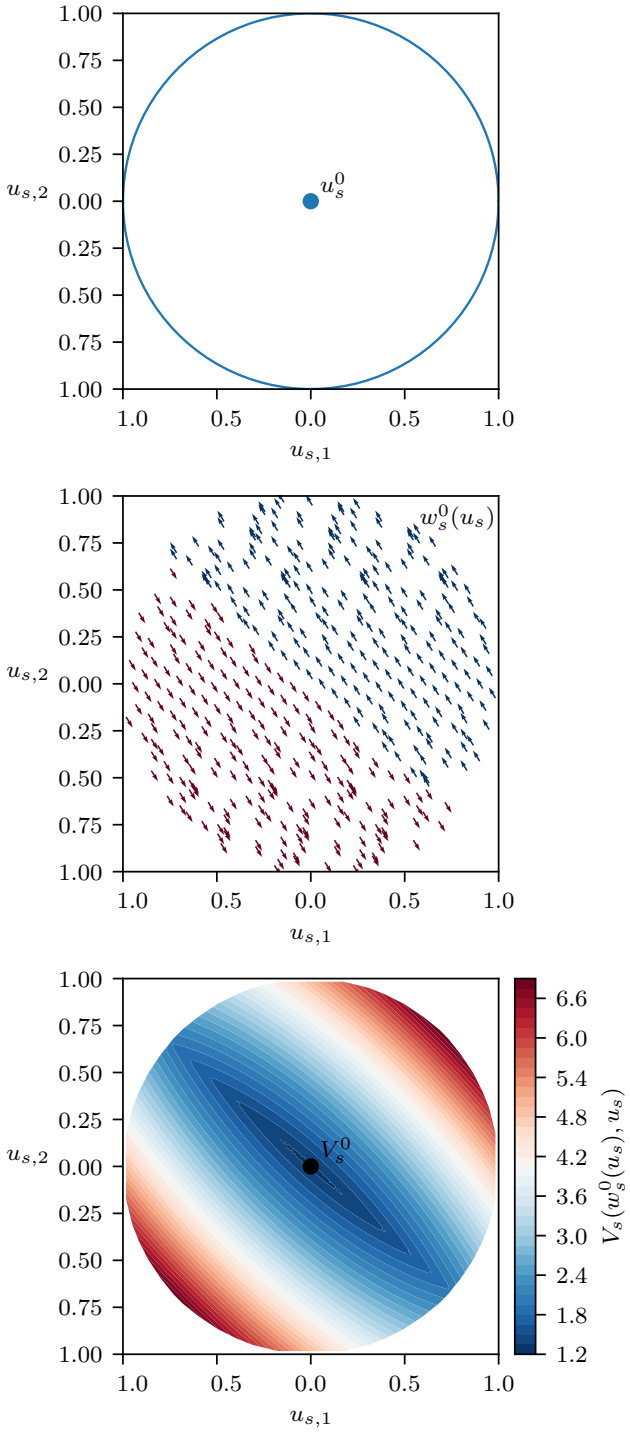

Fig. 1. Minmax problem from (8). Top: optimal control u_s^0 within the unit circle. Middle: vector plot of $w_s^0(u_s)$ as a function of u_s. Bottom: Contour plot of the objective function $V_s(w_s^0(u_s), u_s)$ as a function of u_s, including the optimal objective value V_s^0.

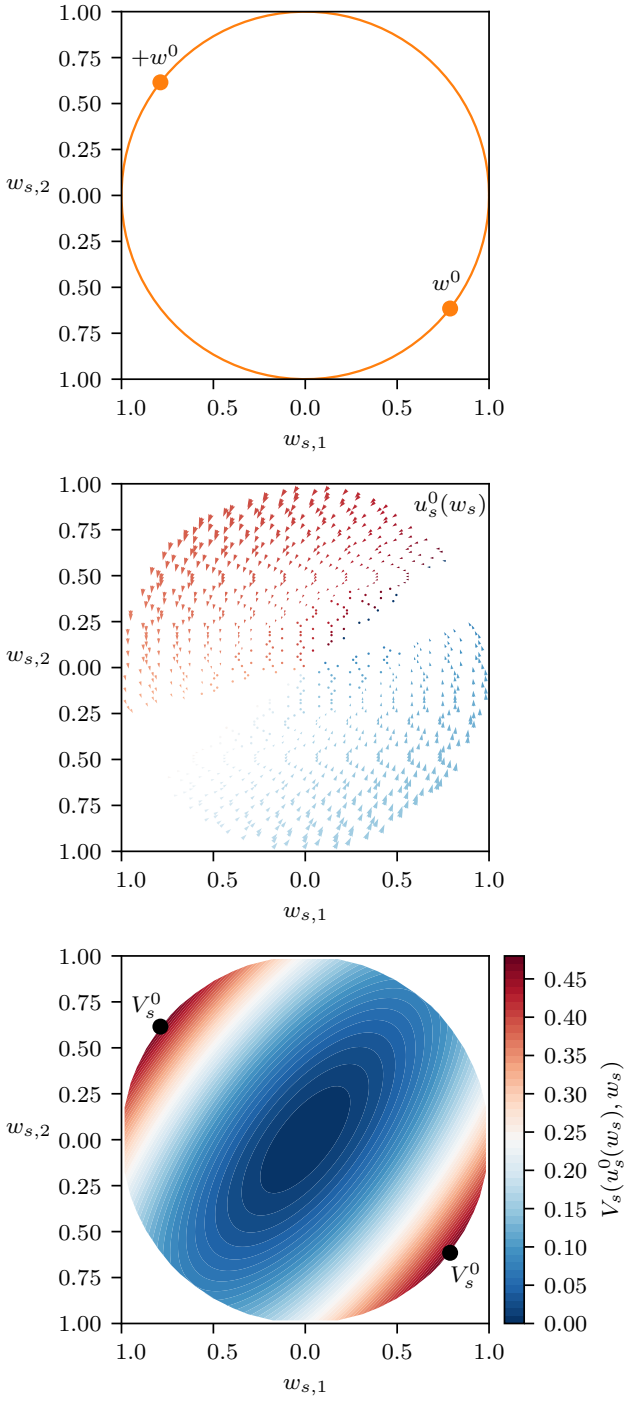

Fig. 2. Maxmin problem from (10). Top: optimal disturbance w_s^0 on the unit circle. Middle: vector plot of $u_s^0(w_s)$ as a function of w_s. Bottom: Contour plot of the objective function $V_s(u_s^0(w_s), w_s)$ as a function of w_s, including the optimal objective value V_s^0.

B. Maxmin Optimization: Worst-case Feedforward Control

The dual problem to (9) is

$$\max_{w_s} \min_{u_s} \frac{V_s(u_s, w_s)}{\alpha} \qquad \text{s.t. } w_s \in \mathbb{W}_s \tag{10}$$

This formulation finds the worst-case disturbance for a feedforward controller. From Proposition 5, the solutions are

$$w_s^0 = w^* \cap \mathbb{W}_s \qquad u_s^0(w_s^0) = -(Y_u' Q Y_u + R)^{-1} Y_u' Q Y_w w_s^0$$

where

$$\begin{bmatrix} u^* \\ w^* \end{bmatrix} \in \mathscr{N}\left(\begin{bmatrix} Y_u' Q Y_u + R & Y_u' Q Y_w \\ Y_w' Q Y_u & Y_w' Q Y_w - \lambda^0 I \end{bmatrix} \right)$$

with optimal objective value

$$V_s(u_s^0(w_s^0), w_s^0) = (1/2)\alpha\lambda^0$$

where $\lambda^0 = \left| Y_w' Q Y_w - Y_w' Q Y_u (Y_u' Q Y_u + R)^{-1} Y_u' Q Y_w \right|$.

Remark 9. Comparing the optimal values:

$$V^0_{s,\min\max} = (1/2)\alpha \left| Y_w' Q Y_w \right|$$
$$V^0_{s,\max\min} = (1/2)\alpha \left| Y_w' Q Y_w - Y_w' Q Y_u (Y_u' Q Y_u + R)^{-1} Y_u' Q Y_w \right|$$

Since the matrix $Y_w' Q Y_u (Y_u' Q Y_u + R)^{-1} Y_u' Q Y_w \geq 0$ is typically positive definite, a nonzero duality gap exists with $V^0_{s,\min\max} > V^0_{s,\max\min}$.

C. Numerical Example

To visualize the solutions, we examine a system defined by

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix}$$

with $G = I$, $Q = R = I$, and $\alpha = 1$. Fig. 1 shows the optimal solutions for the minmax problem (disturbance attenuation), depicting u_s^0, $w_s^0(u_s^0)$, and objective function contours for u_s within the unit circle. Similarly, Fig. 2 illustrates the maxmin problem (worst-case feedforward control) solutions.

The quantitative difference between the two formulations is evident in their duality gap:

$$V^0_{s,\min\max} - V^0_{s,\max\min} = 1.31 - 0.48 = 0.83$$

The minmax problem exhibits a unique Stackelberg equilibrium with multiple possible disturbance outcomes:

$$u_s^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad w_s^0(u_s^0) = \pm \begin{bmatrix} -0.53 \\ 0.85 \end{bmatrix}$$

yielding $V_s(u_s^0, w_s^0(u_s^0)) = 1.31$. Notably, $(u_s^0, w_s^0(u_s^0))$ is not a stationary point, and $V(\cdot)$ is not differentiable at this solution. Conversely, the maxmin problem features two distinct Stackelberg equilibria:

$$w_s^0 = \pm \begin{bmatrix} -0.79 \\ 0.62 \end{bmatrix} \qquad u_s^0(w_s^0) = \mp \begin{bmatrix} -0.13 \\ -0.59 \end{bmatrix}$$

with objective value $V_s(u_s^0(w_s^0), w_s^0) = 0.48$. Unlike the minmax case, each maxmin solution $(u_s^0(w_s^0), w_s^0)$ is a stationary point.

This example clearly demonstrates how the sequential nature of the game significantly impacts both the solution properties and optimal values.

V. DYNAMIC DISTURBANCE ATTENUATION

The disturbance attenuation problem in the frequency domain seeks a feedback controller, $K(s)$, that minimizes the H_∞ norm of the transfer function from disturbance input to performance output [9]. This norm is defined as

$$\|T(s)\|_\infty = \sup_{\bar{w}(s) \neq 0} \frac{\|\bar{z}(s)\|_2}{\|\bar{w}(s)\|_2}$$

leading to the optimization problem

$$\min_{K(s)} \sup_{\bar{w}(s) \neq 0} \frac{\|\bar{z}(s)\|_2}{\|\bar{w}(s)\|_2}$$

Applying Plancherel's theorem provides a time domain interpretation

$$\sup_{\bar{w}(s) \neq 0} \frac{\|\bar{z}(s)\|_2}{\|\bar{w}(s)\|_2} = \sup_{w(t) \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

which, with an introduced control $u(t)$ that minimizes $z(t)$, yields

$$\min_{u(t)} \sup_{\|w(t)\|_2 \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

This is equivalently expressed by squaring the objective function

$$\min_{u(t)} \sup_{\|w(t)\|_2^2 \neq 0} \frac{\|z(t)\|_2^2}{\|w(t)\|_2^2}$$

A. General Framework

Based on Section V, we formulate a general disturbance attenuation problem for discrete-time systems

$$x^+ = f(x, u, w) \qquad f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{W} \to \mathbb{R}^n$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input, and $w \in \mathbb{W} \subseteq \mathbb{R}^q$ is the disturbance.

Define the following sequence of vectors for the state, $\mathbf{x} := (x(1), x(2), \ldots, x(N))$, control, $\mathbf{u} := (u(0), u(1), \ldots, u(N-1))$, auxiliary control, $\mathbf{v} := (v(0), v(1), \ldots, v(N-1))$, and disturbance, $\mathbf{w} := (w(0), w(1), \ldots, w(N-1))$.

Parameterize the control input with an embedded feedback structure

$$\mu(v) = \bar{K} x + v$$

where $\bar{K}$ is either an optimization variable or a fixed parameter, and v becomes the primary control variable. This formulation incorporates feedback while optimizing only over control actions. The finite horizon optimal control problem is

$$\overline{\mathscr{P}}_N(x_0): \qquad V_N^0(x_0) = \min_{\mathbf{v} \in \mathscr{V}(x_0)} \max_{\mathbf{w} \in \mathbb{W}_N} V_N(x_0; \mu(\mathbf{v}), \mathbf{w})$$

with cost function

$$V_N(x_0; \mu(\mathbf{v}), \mathbf{w}) := \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

and constraint set

$$\mathscr{V}(x_0) := \{\mathbf{v} \in \mathbb{V} \mid (x(i), u(i)) \in \mathbb{Z}, \quad \forall i \in \mathbb{I}_{0:N-1},\ x(N) \in \mathbb{X}_f,\ \forall \mathbf{w} \in \mathbb{W}_N \subseteq \mathbb{R}^{Nq}\}$$

For horizon $N \in \mathbb{I}_{\geq 0}$, denote the state trajectory by $x(i) := \phi(i, x_0; \mu(\mathbf{v}), \mathbf{w})$ and the control input by $u(i) = \mu(x(i), v(i))$. Let $\mathbf{v}^0(x_0)$ be the minimizing control sequence, $\mu^0(x_0) := \mu(\mathbf{v}^0(x_0))$ the corresponding control policy, and $V_N^0(x_0) := V_N(x_0; \mu^0(x_0))$ the value function. Assume that a solution to $\mathscr{P}_N(x_0)$ exists for all $x_0 \in \mathscr{X}_N(x_0) := \{x_0 \mid \mathscr{V}_N(x_0) \neq \emptyset\}$ and that $\mathscr{X}_N$ is not empty. The optimal control applied to the system is $\kappa_N(x_0) = \bar{K}x_0 + \mathbf{v}^0(0; x_0)$.

Remark 10. In actual closed-loop implementation, the control and disturbance do not interact in a game-theoretic manner; the controller simply applies $\kappa_N(x_0)$ to achieve robustness against unknown bounded disturbances.

Remark 11. The disturbance attenuation problem as formulated here aims to maintain good performance (minimize cost) despite worst-case disturbances, rather than to cancel disturbances. In this game-theoretic framework, the disturbance is chosen adversarially after the control, making disturbance cancellation impossible. Instead, the controller seeks to attenuate the effect of the disturbance on the overall cost, not to counteract the disturbance signal itself. This is fundamentally different from disturbance rejection approaches, such as the dual maxmin optimization, which aim to directly cancel the measured or estimated disturbances in a feedforward fashion.

B. Quadratic Function and Linear System Framework

We implement the optimization framework in Section V for a specific class of systems: linear dynamics with quadratic cost functions and bounded disturbances. The performance output is

$$z(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

with system dynamics and cost function

$$f(x,u,w) = Ax + Bu + w \qquad x(0) = x_0$$
$$|z|^2 = \ell(x,u) = (1/2)(x'Qx + u'Ru) \qquad V_f(x) = (1/2)x'P_f x$$

where $x, w \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $Q \geq 0$, and $R > 0$.

With the control parameterization $u = \bar{K}x + v$, the objective function reduces to

$$V_N(x_0; \mathbf{v}, \mathbf{w}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + V_f(x(N))$$

$$\mathscr{V}_N(x_0) := \{\mathbf{v} \in \mathbb{R}^{mN} \mid (x(i), u(i)) \in \mathbb{Z}, \quad \forall i \in \mathbb{I}_{0:N-1},\ x(N) \in \mathbb{X}_f,\ \forall \mathbf{w} \in \mathbb{W}_N\}$$

where $\mathbb{Z} = \mathbb{R}^n \times \mathbb{R}^m$. The disturbance sequence $\mathbf{w}$ is chosen to lie in a bounded set $\mathbb{W}_\alpha \subseteq \mathbb{W}_N$, defined as

$$\mathbb{W}_\alpha := \{\mathbf{w} \mid \|\mathbf{w}\|_2^2 \leq \alpha\}$$

Where the scalar α is the disturbance bound. By Proposition 8, the inequality constraint can be replaced by an equality constraint

$$\mathbb{W}_\alpha := \{\mathbf{w} \mid \|\mathbf{w}\|_2^2 = \alpha\}$$

The model is solved as a finite horizon non-recursive problem, where the state sequence is

$$\mathbf{x} = \mathscr{A}x_0 + \mathscr{B}\mathbf{v} + \mathscr{G}\mathbf{w}$$

with

$$\mathscr{A} := \begin{bmatrix} A_K \\ A_K^2 \\ A_K^3 \\ \vdots \\ A_K^N \end{bmatrix} \qquad \mathscr{B} := \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ A_K B & B & 0 & \cdots & 0 \\ A_K^2 B & A_K B & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_K^{N-1} B & A_K^{N-2} B & A_K^{N-3} & \cdots & B \end{bmatrix}$$

$$\mathscr{G} := \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ A_K & I & 0 & \cdots & 0 \\ A_K^2 & A_K & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_K^{N-1} & A_K^{N-2} & A_K^{N-3} & \cdots & I \end{bmatrix}$$

where $A_K = A + B\bar{K}$. The control $\mathbf{u}$ given the control parametrization $u = \bar{K}x + v$

$$\mathbf{u} = \mathscr{K}_1 x_0 + \mathscr{K}_2 \mathbf{x} + \mathbf{v}$$

with

$$\mathscr{K}_1 := \begin{bmatrix} \bar{K} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathscr{K}_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \bar{K} & 0 & 0 & 0 & 0 \\ 0 & \bar{K} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{K} & 0 \end{bmatrix}$$

Defining

$$\mathscr{A}_u = \mathscr{K}_1 + \mathscr{K}_2\mathscr{A} \qquad \mathscr{B}_u = \mathscr{K}_2\mathscr{B} + I \qquad \mathscr{G}_u = \mathscr{K}_2\mathscr{G}$$

enables the cost function to be expressed as

$$V_N(x_0; \mathbf{v}, \mathbf{w}) = (1/2)\left|\bar{A}x_0 + \bar{B}\mathbf{v} + \bar{G}\mathbf{w}\right|_{\bar{D}}^2 \tag{11}$$

with

$$\mathscr{Q} := \operatorname{diag}\left(\operatorname{diag}_{N-1}(Q), P_f\right) \qquad \mathscr{R} := \operatorname{diag}_N(R)$$
$$\bar{D} = \operatorname{diag}(Q, \mathscr{Q}, \mathscr{R})$$
$$\bar{A} = \begin{bmatrix} I \\ \mathscr{A} \\ \mathscr{A}_u \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ \mathscr{B} \\ \mathscr{B}_u \end{bmatrix} \qquad \bar{G} = \begin{bmatrix} 0 \\ \mathscr{G} \\ \mathscr{G}_u \end{bmatrix}$$

The final disturbance attenuation problem is

$$\min_{\mathbf{v}} \max_{\mathbf{w}} \frac{V_N(x_0; \mathbf{v}, \mathbf{w})}{\alpha} \qquad \text{s.t. } \|\mathbf{w}\|_2^2 = \alpha \tag{12}$$

where α represents the disturbance bound. This is a zero-sum Stackelberg game with constrained convex-convex objective. Its dual problem is

$$\max_{\mathbf{w}} \min_{\mathbf{v}} \frac{V_N(x_0; \mathbf{v}, \mathbf{w})}{\alpha} \qquad \text{s.t. } \|\mathbf{w}\|_2^2 = \alpha$$

C. Disturbance Attenuation Problem Solution

The objective function in (11) can be expressed as

$$V_N(x_0;\mathbf{v},\mathbf{w}) = (1/2)\left(\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}' M(0) \begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} + 2\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}' d + x_0'\bar{E}x_0 \right)$$

where

$$M(\lambda) = \begin{bmatrix}\bar{B}'\bar{D}\bar{B} & \bar{B}'\bar{D}\bar{G}\\ \bar{G}'\bar{D}\bar{B} & \bar{G}'\bar{D}\bar{G}-\lambda I\end{bmatrix} \quad d = \begin{bmatrix}\bar{B}'\\ \bar{G}'\end{bmatrix}\bar{D}\bar{A}x_0 \quad \bar{E} = \bar{A}'\bar{D}\bar{A}$$

Defining $\mu_1 = \left|\bar{G}'\bar{D}\bar{G}\right|$ and

$$L(x_0;\lambda) := (1/2)\alpha\lambda + (1/2)(x_0'\bar{E}x_0 - d'M^+(\lambda)d)$$

the solution to (12) from Proposition 5 is

$$\mathbf{v}^0 = \mathbf{v}^*(\lambda^0)$$
$$\mathbf{w}^0 \in -(\bar{G}'\bar{D}\bar{G}-\lambda^0 I)^+(\bar{G}'\bar{D}\bar{B}\mathbf{v}^0 + \bar{G}'\bar{D}\bar{A}x_0) + \mathscr{N}(\bar{G}'\bar{D}\bar{G}-\lambda^0 I)$$

where

$$\begin{bmatrix}\mathbf{v}^*(\lambda)\\ \mathbf{w}^*(\lambda)\end{bmatrix} \in -M^+(\lambda)d + \mathscr{N}(M(\lambda))$$

The optimal objective value is

$$V^0(x_0;\mathbf{v}^0,\mathbf{w}^0) = (1/2)\alpha\lambda^0 + (1/2)(x_0'\bar{E}x_0 - d'M^+(\lambda^0)d)$$

or equivalently

$$V^0(x_0;\mathbf{v}^0,\mathbf{w}^0) = (1/2)\alpha\lambda^0 + (1/2)x_0'\Pi(\lambda^0)x_0$$

where

$$\Pi(\lambda^0) = \bar{E} - \bar{A}'\bar{D}\begin{bmatrix}\bar{B} & \bar{G}\end{bmatrix} M^+(\lambda^0)\begin{bmatrix}\bar{B}'\\ \bar{G}'\end{bmatrix}\bar{D}\bar{A}$$

with λ^0 determined from

$$\lambda^0 = \arg\min_{\lambda\geq\mu_1} L(x_0;\lambda)$$

This approach reduces the constrained minmax optimization to a scalar minimization over a Lagrange multiplier. In a different context, Sayed et al. [18] also solved a class of sequential games, including (12), by reducing the constrained minmax optimization to a scalar minimization over a Lagrange multiplier λ.

VI. SOLUTION STRUCTURE

This section characterizes the two solution regions in the space of initial states. An efficient algorithm for numerically solving the disturbance attenuation problem in each region is then presented.

A. Solution Regions

Optimization problem (12) features two solution regions in the space of initial states, based on the theoretical results from Proposition 7. Applying these results to the proposed disturbance attenuation problem requires establishing a connection between the general conditions in the proposition and the specific problem structure.

From part 6(ii) of Proposition 7, the condition $|(D-|D|I)^+ d| \leq 1$ determines when $\lambda_P = |D|$. For the problem in (12), with $D = \bar{G}'\bar{D}\bar{G}$, $\mu_1 = \left|\bar{G}'\bar{D}\bar{G}\right|$, and $d = [\bar{B};\ \bar{G}]'\bar{D}\bar{A}x_0$, this condition becomes

$$\left|\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix} M^+(\mu_1)d\right|^2 \leq \alpha$$

To facilitate a clear characterization of the solution regions, define the gain matrix $\bar{J}(\mu_1)$ as

$$\bar{J}(\mu_1) = \begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix} M^+(\mu_1)\begin{bmatrix}\bar{B}'\\ \bar{G}'\end{bmatrix}\bar{D}\bar{A}$$

1) Region $\mathscr{X}_L$: Region $\mathscr{X}_L \in \mathbb{R}^n$ consists of initial states where the optimal solution to $\min_{\lambda\geq\mu_1} L(x_0;\lambda)$ is $\lambda^0 = \mu_1$. From part 6(ii) of Proposition 7, this corresponds to

$$\mathscr{X}_L := \{x_0 \mid |\bar{J}(\mu_1)x_0|^2 \leq \alpha\}$$

This region, which contains the origin, is characterized by a linear optimal feedback control policy in the initial state x_0. A unique value of $\lambda^0 = \mu_1$ leads to a unique optimal control sequence $\mathbf{v}^0$ that depends linearly on x_0.

Remark 12. Note that the zero initial state, $x_0 = 0$, is clearly contained in $\mathscr{X}_L$.

2) Region $\mathscr{X}_{NL}$: Region $\mathscr{X}_{NL}$ encompasses initial states where the optimal solution to $\min_{\lambda\geq\mu_1} L(x_0;\lambda)$ is $\lambda^0(\alpha) > \mu_1$.

$$\mathscr{X}_{NL} := \mathbb{R}^n \setminus \mathscr{X}_L$$

In this region, the optimal value $\lambda^0(\alpha)$ depends on the parameter α. Consequently, the optimal control sequence $\mathbf{v}^0(\alpha)$ is a nonlinear function of both x_0 and α, leading to a nonlinear controller.

An important property of region $\mathscr{X}_{NL}$ is that as the initial state magnitude increases, the optimal control approaches the linear quadratic regulator solution, as established next.

Proposition 13 (Convergence to LQR Solution). Fix the disturbance bound $\alpha > 0$. Then as $|x_0| \to \infty$, the optimal solution of the disturbance attenuation problem (12) coincides asymptotically with the linear quadratic regulator (LQR) solution.

Proof. Rescale $V(x_0;\mathbf{v}^0,\mathbf{w}^0)$ by dividing by parameter α, giving

$$\min_{\lambda\geq\mu_1} (1/2)\lambda + (1/2)\left(\frac{x_0}{\sqrt{\alpha}}\right)'\Pi(\lambda)\left(\frac{x_0}{\sqrt{\alpha}}\right)$$

The ratio $|x_0|/\sqrt{\alpha}$ controls the scaling between the size of the disturbance and the size of the initial state. As $|x_0| \to \infty$ with fixed α, this ratio approaches infinity.

When this ratio approaches infinity, the optimal value of λ approaches μ_1, and the system behaves as if the disturbance magnitude α approaches zero. Because the set $\mathbb{W}_\alpha$ is continuous in α [19, pp.152–153], the solution to the inner maximization problem exists and is continuous in α [20, Corollary 5.4.2], [21, Theorem C.28]. Thus, when $\alpha \to 0$, the worst-case disturbance w^0 is forced to 0, reducing the problem to

$$\min_{\mathbf{v}} V(x_0;\mathbf{v},0)$$

which is the standard LQR problem. □

Remark 14. An important distinction between the two regions concerns the uniqueness of the optimal disturbance. In region $\mathscr{X}_L$, the disturbance solution $\mathbf{w}^0(\mathbf{v}^0)$ is not unique, as indicated by Proposition 7. When $\lambda^0 = \mu_1 = \left|\bar{G}'\bar{D}\bar{G}\right|$, the matrix $\bar{G}'\bar{D}\bar{G} - \lambda^0 I$ has a non-zero nullspace, leading to multiple possible worst-case disturbances. In contrast, within region $\mathscr{X}_{NL}$ where $\lambda^0 > \mu_1$, the matrix $\bar{G}'\bar{D}\bar{G} - \lambda^0 I$ is non-singular, resulting in a unique optimal disturbance $\mathbf{w}^0(\mathbf{v}^0)$.

3) Solution Properties: The optimal solutions exhibit markedly different properties in the two regions. In $\mathscr{X}_L$, the optimal solution to (12), $(\mathbf{v}^0, \mathbf{w}^0(\mathbf{v}^0))$, is a Stackelberg equilibrium that is not a stationary point, and strong duality does not hold

$$\min_{\mathbf{v}} \max_{\|\mathbf{w}\|_2^2=\alpha} \frac{V_N(x_0;\mathbf{v},\mathbf{w})}{\alpha} > \max_{\|\mathbf{w}\|_2^2=\alpha} \min_{\mathbf{v}} \frac{V_N(x_0;\mathbf{v},\mathbf{w})}{\alpha}$$

Conversely, in $\mathscr{X}_{NL}$, although the optimization remains a sequential game (where control moves first, followed by disturbance), strong duality holds, and the optimal solution $(\mathbf{v}^0(\alpha), \mathbf{w}^0(\mathbf{v}^0(\alpha)))$ is both a Stackelberg equilibrium and a saddle point

$$\min_{\mathbf{v}} \max_{\|\mathbf{w}\|_2^2=\alpha} \frac{V_N(x_0;\mathbf{v},\mathbf{w})}{\alpha} = \max_{\|\mathbf{w}\|_2^2=\alpha} \min_{\mathbf{v}} \frac{V_N(x_0;\mathbf{v},\mathbf{w})}{\alpha}$$

B. Selection of Control Gain Parameter $\bar{K}$

Basar [11] established the solution to problem (12) for zero initial states, $x_0 = 0$. For the infinite horizon formulation, the resulting optimal control gain constitutes the classical H_∞ controller and can be derived using linear matrix inequality (LMI) methods [22, p.128]. This standard H_∞ gain, denoted as K_∞^0, provides the appropriate stabilizing feedback for the control parameterization

$$\bar{K} = K_\infty^0$$

so that the feedback parameterization is implemented as $u = K_\infty^0 x + v$. When $x_0 \neq 0$, the optimization determines a control sequence $\mathbf{v}^0$ that compensates for this parameterization while still achieving the minimum value of the objective function in (12). This choice of parameter is justified in the following proposition for $x_0 \in \mathscr{X}_L$.

Proposition 15 (Optimality of H_∞ Control for $x_0 \in \mathscr{X}_L$). For the disturbance attenuation problem (12) with horizon N, terminal cost $P_f = P_\infty$, and control parametrization $u = \bar{K}x + v$, when $x_0 \in \mathscr{X}_L$, the optimal solution is $\bar{K} = K_\infty^0$ and $v^0 = 0$. Thus, $u^0 = K_\infty^0 x$.

Proof. Consider the cost function from (11)

$$V_N(x_0,\mathbf{v};\mathbf{w}) = (1/2)\left|\bar{A}x_0 + \bar{B}\mathbf{v} + \bar{G}\mathbf{w}\right|_{\bar{D}}^2$$

where $\bar{D} \geq 0$. Given the constrained maximization over $\mathbf{w}$, $\mathbf{w} \in \mathbb{W}_\alpha$

$$\max_{\mathbf{w}} \frac{V_N(x_0,\mathbf{v};\mathbf{w})}{\alpha} \qquad \text{s.t. } \|\mathbf{w}\|_2^2 = \alpha$$

where

$$V_N(x_0,\mathbf{v};\mathbf{w}) = (1/2)\left(\mathbf{w}'\bar{G}'\bar{D}\bar{G}\mathbf{w} + 2\mathbf{w}'\bar{G}'\bar{D}d + x_0'\bar{E}x_0 + d'\bar{D}d\right)$$

with

$$d = \bar{A}x_0 + \bar{B}\mathbf{v} \qquad \bar{E} = \bar{A}'\bar{D}\bar{A}$$

For $x_0 = 0$ and $\mathbf{v} \neq 0$, from Proposition 7

$$V^0(0,\mathbf{v}) = (1/2)\left(-d'(\bar{G}'\bar{D}\bar{G} - \lambda_P I)^+ d + d'\bar{D}d + \lambda_P\alpha\right)$$

where $\lambda_P I \geq \bar{G}'\bar{D}\bar{G}$, and thus $-d'(\bar{G}'\bar{D}\bar{G} - \lambda_P I)^+ d \geq 0$. For $x_0 = 0$ and $\mathbf{v} = 0$, then $d = 0$, and from Proposition 7

$$V^0(0,0) = (1/2)\lambda_P\alpha$$

It follows that

$$V^0(0,0) \leq V^0(0,\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbb{R}^{mN}$$

Thus, when $x_0 = 0$, the optimal value for $\bar{K}$ is K_∞^0 with $\mathbf{v}^0 = 0$.

Since $x_0 = 0 \in \mathscr{X}_L$ and $\mathbf{v}^0 = 0$ at $x_0 = 0$ as shown above, the uniqueness of $\mathbf{v}^0(\lambda^0)$ within $\mathscr{X}_L$ implies that $\mathbf{v}^0(\mu_1) = 0$ for all $x_0 \in \mathscr{X}_L$. Therefore, throughout region $\mathscr{X}_L$, the optimal control is $u^0 = K_\infty^0 x$. □

C. Solution Algorithm

This section introduces an efficient solution algorithm for the disturbance attenuation problem that leverages the properties of regions $\mathscr{X}_L$ and $\mathscr{X}_{NL}$ to reduce computational complexity.

Algorithm 1 Disturbance Attenuation Problem

Require: $x_0, \mu_1, \alpha, K_\infty^0$
Ensure: $\lambda^0, \mathbf{v}^0, \mathbf{w}^0(\mathbf{v}^0)$
  Compute $\bar{J}(\mu_1)$
  if $|\bar{J}(\mu_1)x_0|^2 \leq \alpha$ then
    $\lambda^0 \leftarrow \mu_1$ ▷ Region $\mathscr{X}_L$: linear algebra problem
    Calculate $\mathbf{v}^0$ and $\mathbf{w}^0(\mathbf{v}^0)$ from Proposition 5
  else
    Solve $\lambda^0 = \arg\min_{\lambda \geq \mu_1} L(x_0;\lambda)$ ▷ Region $\mathscr{X}_{NL}$: scalar nonlinear optimization
    Calculate $\mathbf{v}^0$ and $\mathbf{w}^0(\mathbf{v}^0)$ from Proposition 5
  end if

This algorithm efficiently solves a linear algebra problem in region $\mathscr{X}_L$ and a scalar nonlinear optimization problem in region $\mathscr{X}_{NL}$. Efficiency is measured by the reduction from high-dimensional constrained minmax optimization to either matrix operations (in $\mathscr{X}_L$) or single-variable search (in $\mathscr{X}_{NL}$). The optimal gain for the H_∞ control problem, K_∞^0, can be computed offline through standard linear matrix inequalities [23, p.109],[22, p.128]. Computational performance is demonstrated in Section VII, showing modest horizon requirements and convergence with 40-50 function evaluations across different system dimensions.

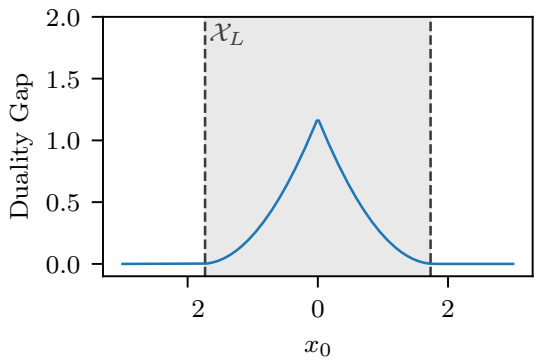

Fig. 3. Duality gap, $V_{\min\max}(x_0;\mathbf{v}^0,\mathbf{w}^0(\mathbf{v}^0)) - V_{\max\min}(x_0;\mathbf{v}^0(\mathbf{w}^0),\mathbf{w}^0)$, as a function of x_0 for $\alpha = 1$.

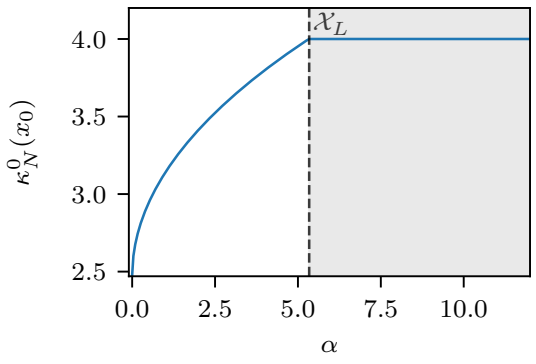

Fig. 4. Optimal control input κ_N^0 as a function of the parameter α for $x_0 = 4$.

VII. RESULTS AND NUMERICAL EXAMPLES

This section presents numerical examples demonstrating the application of Algorithm 1 to the disturbance attenuation problem for both scalar and multivariable systems.

A. Strong Duality and α Dependence

Consider the scalar system from Basar and Bernhard [24, p.92]

$$A = 1 \quad B = 1 \quad R = 1 \quad Q = 1 \tag{13}$$

with horizon $N = 2$, terminal cost $P_f = 2$, and disturbance bound $\alpha = 1$. For this system the region $\mathscr{X}_L$ is

$$\mathscr{X}_L := \{-\sqrt{3} \le x_0 \le \sqrt{3}\}$$

This analysis investigates how the duality gap varies across regions $\mathscr{X}_L$ and $\mathscr{X}_{NL}$ as a function of the initial state. The duality gap is defined as the difference between the optimal objective values of the disturbance attenuation problem $V_{\min\max}(x_0;\mathbf{v}^0,\mathbf{w}^0(\mathbf{v}^0))$ and its dual problem $V_{\max\min}(x_0;\mathbf{v}^0(\mathbf{w}^0),\mathbf{w}^0)$. As shown in Fig. 3, the duality gap is zero in region $\mathscr{X}_{NL}$ where strong duality holds, but positive in region $\mathscr{X}_L$ where strong duality does not hold.

Fig. 4 illustrates the nonlinear dependence of the optimal control input κ_N^0 on parameter α in region $\mathscr{X}_{NL}$ for $x_0 = 4$. In contrast, the optimal control remains independent of α in region $\mathscr{X}_L$.

B. Degenerate Region $\mathscr{X}_L$

Consider another scalar example from Basar and Bernhard [24, p.92]

$$A = 0.5 \quad B = 1 \quad R = 1 \quad Q = 0.2$$

with optimal gains

$$-K_\infty = 0.4 \quad -K_{\text{LQR}} = 0.1$$

and parameters $N = 1500$, $P_f = 0.4$, and $\alpha = 1$. For this system, the region $\mathscr{X}_L$ degenerates to a single point

$$\mathscr{X}_L = \{0\}$$

As shown in Fig. 5, the H_∞ control gain K_∞ is optimal only at the origin. This example requires a large horizon length to achieve convergence to the infinite horizon solution.

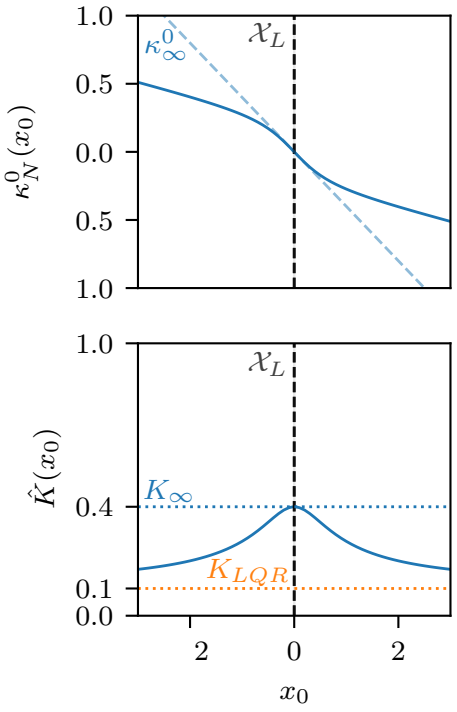

Fig. 5. Top: Optimal control input κ_N^0 as a function of x_0, where $\kappa_N^0(x_0) = \hat{K}x_0$. Bottom: Control gain $\hat{K}$ as a function of x_0, where $\kappa_\infty^0(x_0) = K_\infty x_0$.

C. Optimality for $x_0 \in \mathscr{X}_{NL}$

Returning to the first example system with parameters

$$A = 1 \quad B = 1 \quad R = 1 \quad Q = 1$$

$N = 2$, $P_f = 2$, and $\alpha = 1$, the performance of the proposed method is compared against standard H_∞ control. Fig. 6 shows the difference between the optimal objective value of the proposed solution method, with $u = \kappa_N(x_0) = K_\infty x_0 + \mathbf{v}^0(0;x_0)$, and the objective value obtained from standard state feedback H_∞ control using $u = K_\infty x_0$. The results confirm the suboptimality of H_∞ control when used outside the $\mathscr{X}_L$ region.

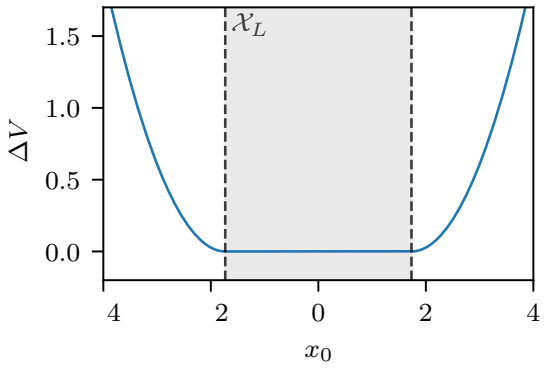

Fig. 6. The cost difference $\Delta V = V(x_0; K_\infty x_0) - V(x_0; \kappa_N(x_0))$ as a function of initial state.

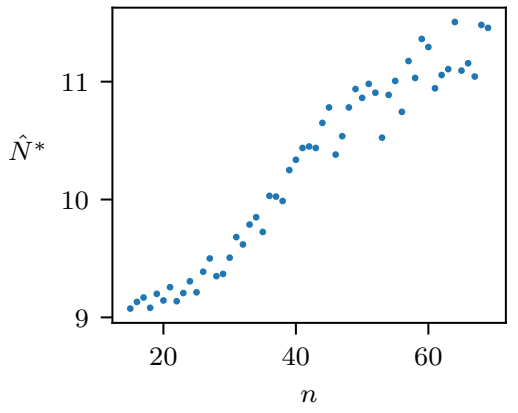

Fig. 7. Average horizon length $\hat{N}^*$ required for solution convergence as a function of system dimension n, averaged over 180 simulations per dimension.

D. Multivariable Systems

Experiments were conducted with multivariable systems using randomly generated matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ where $m = n/2$, with dimensions ranging from $n = 15$ to $n = 65$, over 180 simulations per dimension. In all cases, $Q = I$, $R = I$, and $P_f = 0$, with initial states scaled to fall within the region $\mathscr{X}_{NL}$. The required horizon length for solution convergence remains modest across all system dimensions. As shown in Fig. 7, the average converged horizon length $\hat{N}^*$ ranges from approximately 9 for $n = 15$ to 11.5 for $n = 65$, exhibiting only a weak dependence on system dimension. This characteristic enables efficient handling of large-scale systems without excessive computational requirements. For these experiments, the convergence criterion is $(\lambda^{0-} - \lambda^0)/\lambda^0 \leq 10^{-2}$, where λ^{0-} represents the optimal value from the previous shorter horizon computation. The algorithm consistently achieves convergence with 40-50 function evaluations across all dimensions.

E. Closed-loop Performance

The closed-loop performance of the disturbance attenuation approach is analyzed in a receding horizon implementation, where the optimization problem is solved at each time step. Performance is measured using time-averaged stage cost.

$$\langle l(x(k), u(k)) \rangle = \frac{1}{k} \sum_{i=0}^{k-1} l(x(i), u(i))$$

where $l(x(k), u(k)) = (1/2)(x'(k)Qx(k) + u'(k)Ru(k))$. In this setting, the optimal control $\kappa_N^0(x_0)$ is applied and the state evolution is simulated under the worst-case disturbance w^0. The closed-loop disturbance attenuation problem can be considered as a repeated sequential game.

Definition 16 (Closed-Loop Robust Positive Invariant Set). A set $\mathbb{X} \subseteq \mathbb{R}^n$ is a closed-loop robust positive invariant set for the system $x^+ = f(x, u, w)$ under controller $u = \kappa(x)$, $w \in \mathbb{W}$, if $x \in \mathbb{X}$, then $f(x, \kappa(x), w) \in \mathbb{X}$ for all $w \in \mathbb{W}$.

This investigation focuses on whether region $\mathscr{X}_L$ is robustly positive invariant under the closed-loop dynamics, comparing this approach with standard H_∞ control through two examples. All simulations use a disturbance bound $\alpha = 1$.

Remark 17. The robust positive invariance property of $\mathscr{X}_L$ significantly impacts the long-term performance of disturbance attenuation controllers.

For the first example, consider the system (13) with parameters $N = 2$, $P_f = 2$, and initial state $x_0 = 5 \in \mathscr{X}_{NL}$. Here, region $\mathscr{X}_L$ is

$$\mathscr{X}_L = \{x_0 : -\sqrt{3} \leq x_0 \leq \sqrt{3}\}$$

and is robustly positive invariant for both controllers. While the optimal disturbance is unique when $x \in \mathscr{X}_{NL}$, there are infinitely many possible worst-case disturbances when $x \in \mathscr{X}_L$ for this system. Therefore, disturbance values are randomly selected from $[-1, 1]$ when the state is within $\mathscr{X}_L$. As shown in Fig. 8, which displays the first and last 50 time steps of a 12000-step simulation, both controllers achieve similar performance, with this method providing a temporary 8% cost reduction during the transient phase. This advantage disappears as $t \to \infty$ because both control policies become identical once the state enters $\mathscr{X}_L$.

For the second example, consider the following system

$$A = 0.5 \quad B = 1 \quad R = 1 \quad Q = 0.4 \tag{14}$$

with parameters $N = 10$, $P_f = 0.65$, and initial state $x_0 = 5 \in \mathscr{X}_{NL}$. Here, region $\mathscr{X}_L$ is

$$\mathscr{X}_L = \{x_0 : -\sqrt{0.69} \leq x_0 \leq \sqrt{0.69}\}$$

In this case, $\mathscr{X}_L$ is not robustly positive invariant for either controller. As in the first example, when $x \in \mathscr{X}_L$, disturbance values are randomly selected from $[-1, 1]$, while for $x \in \mathscr{X}_{NL}$, the unique optimal disturbance is applied. As shown in Fig. 9, the approach achieves a

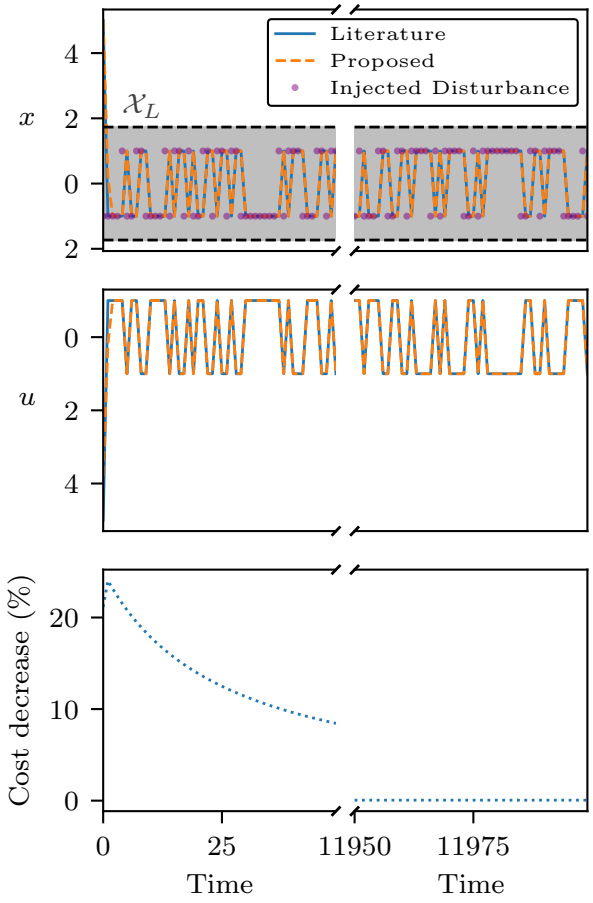

Fig. 8. Closed-loop simulation for the system defined in (13). Top: State x Middle: control u and Bottom: percentage cost savings as a function of time units. The set $\mathscr{X}_L$ is a robust positive invariant set for the closed-loop system. Injected disturbances are shown in purple.

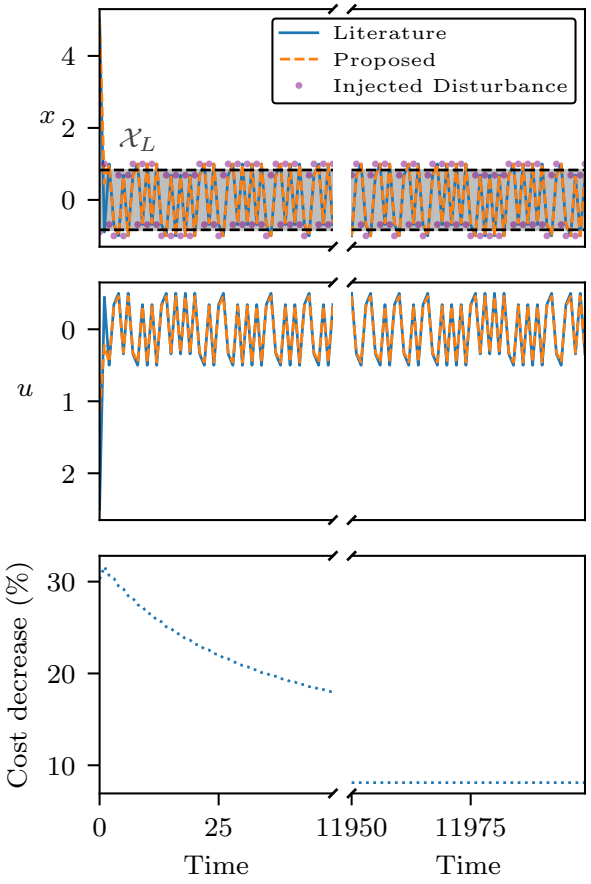

Fig. 9. Closed-loop simulation for the system defined in (14). Top: State x Middle: control u and Bottom: percentage cost savings as a function of time units. The set $\mathscr{X}_L$ is not a robust positive invariant set for the closed-loop system. Injected disturbances are shown in purple.

substantial steady-state performance improvement, with a 8% cost reduction that persists as $t \to \infty$.

These examples demonstrate the following: when $\mathscr{X}_L$ is robustly positive invariant, performance improvements are transient; however, when $\mathscr{X}_L$ is merely a subset of the robust positive invariant set, this approach consistently outperforms standard H_∞ control in steady state.

VIII. SUMMARY

This paper advances the theory of disturbance attenuation by demonstrating that optimal state feedback control for linear systems results in nonlinear control. We establish a unified framework connecting H_∞ control and LQR, formulated as a zero-sum Stackelberg game with bounded disturbances.

Two distinct solution regions in the space of initial states exist. In the first region denoted $\mathscr{X}_L$, which contains the origin, the optimal control coincides with the linear H_∞ controller. In this region, the optimal solution has unique mathematical properties: it forms a Stackelberg equilibrium that is not a stationary point, strong duality does not hold, and the optimal control is independent of the disturbance bound.

In contrast, the second region (denoted $\mathscr{X}_{NL}$) features a nonlinear optimal control policy that varies with the disturbance bound and converges to the LQR solution as the initial state magnitude increases. Here, the optimal solution is both a Stackelberg equilibrium and a saddle point, with strong duality holding throughout this region.

An efficient solution algorithm that exploits this regional structure is introduced, reducing computational complexity to a linear algebra problem in the region and a scalar nonlinear optimization problem in the second region. The numerical studies indicate that modest prediction horizons (10-12 steps) suffice for convergence across a range of system dimensions.

Closed-loop simulations reveal performance improvements over traditional H_∞ control when the first region $\mathscr{X}_L$ is not a robust positive invariant set for the closed-loop system. In such cases, performance gains persist in steady state, with cost reductions of up to 8%.

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IX. Appendix

The proof of Proposition 5 require a few preliminary results, given here as Proposition 18, Proposition 19, and Lemma 20. The proofs of these three preliminary results are given in [16, 17].

Proposition 18 (Positive semidefinite partitioned matrices). The matrix $M \geq 0$ if and only if $M_{11} \geq 0$, $M_{22} - M_{12}'M_{11}^{+}M_{12} \geq 0$, and $\mathscr{R}(M_{12}) \subseteq \mathscr{R}(M_{11})$.

Proposition 19 (Minmax and Maxmin of a Quadratic Function). Consider the Lagrangian function $L(\cdot) : \mathbb{R}^{m+n+1} \to \mathbb{R}$

$$L(u,w,\lambda) \coloneqq (1/2)\begin{bmatrix} u \\ w \end{bmatrix}' M(\lambda) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

with $M_{11} > 0$, $M(0) \geq 0$, and the two problems

$$\min_u \max_w L(u,w,\lambda) \qquad \max_w \min_u L(u,w,\lambda) \tag{15}$$

We characterize existence of solutions as a function of (decreasing) parameter λ.

1) For $\lambda > |M_{22}|$: Solutions to both problems exist for all $d \in \mathbb{R}^{m+n}$.
2) For $\lambda = |M_{22}|$, we have the following two cases:
   a) For $d \in \mathscr{R}(M(|M_{22}|))$: The solutions to both problems exist.
   b) For $d \notin \mathscr{R}(M(|M_{22}|))$: Neither problem has a solution.

   If $M(0)$ is such that $|\tilde{M}_{11}| < |M_{22}|$, then we have the following cases.
3) For $|\tilde{M}_{11}| < \lambda < |M_{22}|$: Only the solution to the $\max_w \min_u L$ problem exists, and it exists for all $d \in \mathbb{R}^{m+n}$.
4) For $\lambda = |\tilde{M}_{11}|$, we have the following two cases:
   a) For $d \in \mathscr{R}(M(|\tilde{M}_{11}|))$: Only the solution to the $\max_w \min_u L$ problem exists.
   b) For $d \notin \mathscr{R}(M(|\tilde{M}_{11}|))$: Neither problem has a solution.
5) For $\lambda < |\tilde{M}_{11}|$: Neither problem has a solution.

If $M(0)$ is such that $|\tilde{M}_{11}| = |M_{22}|$, then cases 3 and 4 do not arise.

For $d \in \mathscr{R}(M(\lambda))$ denote the stationary points $(u^*(\lambda), w^*(\lambda))$ by

$$\begin{bmatrix} u^* \\ w^* \end{bmatrix}(\lambda) \in -M^+(\lambda)d + \mathscr{N}(M(\lambda))$$

When solutions to the respective problems exist, we have that

$$\begin{aligned} u^*(\lambda) &= \arg\min_u \max_w L(u,w,\lambda) \\ w^*(\lambda) &= \arg\max_w \min_u L(u,w,\lambda) \\ L^0(\lambda) &= -(1/2)d'M^+(\lambda)d + \lambda/2 \end{aligned}$$

and the inner optimizations, $\underline{u}^0(w^*(\lambda))$ and $\overline{w}^0(u^*(\lambda))$, are given by all solutions to

$$\begin{array}{cccccc} M_{11} & \underline{u}^0(w^*) & + & M_{12} & w^* & = -d_1 \\ M_{12}' & u^* & + & (M_{22}-\lambda I) & \overline{w}^0(u^*) & = -d_2 \end{array}$$

or, after solving,

$$\begin{array}{ccccc} \underline{u}^0(w^*) & = & -M_{11}^+(M_{12}w^*+d_1) & + & N(M_{11}) \\ \overline{w}^0(u^*) & = & -(M_{22}-\lambda I)^+(M_{12}'u^*+d_2) & + & N(M_{22}-\lambda I) \end{array}$$

Lemma 20 (Saddle points of the Lagrangian of constrained quadratic optimization). The following (w^*, λ^*) are saddle points of $L(w,\lambda) := (1/2)w'Dw + w'd - (1/2)\lambda(w'w-1)$.

$$w^* = \begin{cases} \Big((D-|D|I)^+ d + \mathscr{N}(D-|D|I)\Big) \cap W, & \lambda_P = |D| \\ (D-\lambda_P I)^{-1}d, & \lambda_P > |D| \end{cases}$$
$$\lambda^* = \lambda_P$$

Proof of Proposition 5. First note that $M(0) \geq 0$ implies that $M_{22} \geq 0$ and $\tilde{M}_{11} \geq 0$ as discussed after Proposition 18.

1) First we note that problem (1) is equivalent to $\min_u \max_w \min_\lambda L(u,w,\lambda)$. Because $M_{22} \geq 0$, Proposition 7 applies, and we have strong duality in the inner two optimizations giving the equivalent problems

$$\min_u \min_\lambda \max_w L(u,w,\lambda) = \min_\lambda \min_u \max_w L(u,w,\lambda)$$

Now we apply Proposition 19 and insert the solution to the $\min_u \max_w L$ problem giving

$$\min_{\lambda \geq |M_{22}|} L^0(\lambda) \qquad L^0(\lambda) = -(1/2)d'M^+(\lambda)d + \lambda/2$$

which is (3). Note that the constraint on λ follows because as stated in Proposition 19, a solution to $\min_u \max_w L(u,w,\lambda)$ does not exist for $\lambda < |M_{22}|$. Given the optimal solution λ_r^0, we then evaluate the optimal u_r^0 and w_r^0 from the formulas in Proposition 19, and we have to intersect the inner solution set $\overline{w}^0$ with set $\mathbb{W}$ as in Lemma 20 to obtain the solution to the original constrained problem (1).
2) Proceeding similarly, we note that problem (2) is equivalent to $\max_w \min_\lambda \min_u L(u,w,\lambda)$. But here we must solve the inner $\min_u L$ before we can appeal to Proposition 7 for strong duality of the outer two problems. Since $M_{11} > 0$, the inner problem has a unique solution for all $d \in \mathbb{R}^{m+n}$ with optimal value

$$\begin{aligned} L(u^0,w,\lambda) = (1/2)w'\tilde{M}_{11}w + w'(d_2 - M_{12}'M_{11}^{-1}d_1) \\ -(1/2)d_1'M_{11}^{-1}d_1 - (1/2)\lambda(w'w-1) \end{aligned}$$

Because $\tilde{M}_{11} \geq 0$, Proposition 7 again applies, and we have strong duality in the outer two problems giving the equivalent problem $\min_\lambda \max_w \min_u L(u,w,\lambda)$. From here, we again apply Proposition 19 and proceed as in the previous part. We obtain the optimization problem $\min_{\lambda \geq |\tilde{M}_{11}|} L^0(\lambda)$, which is (4), and the constraint on λ follows because the solution to $\max_w \min_u L(u,w,\lambda)$ does not exist for $\lambda < |\tilde{M}_{11}|$. Finally, in this problem, we must intersect the outer solution w^* of $\max_w \min_u L$ with $\mathbb{W}$ to solve the original constrained problem (2). □

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