Passivity Results for Information Dissemination in Sensor Networks

V. V. Kulkarni J. P. Hespanha B. Jagyasi S. N. Merchant U. B. Desai P. L. Falb

***Abstract*— We derive the largest possible class of linear time-invariant controllers, $\mathcal{M}^{RS}$, to synthesize the information dissemination policies that ensure the closed-loop stability for resource constrained sensor networks. We assume that the sensory information is routed using a tree-topology. Our main contribution is the use of the Zames-Falb multipliers to preserve positivity of repeating monotone nonlinearities that are encountered in a noncooperative game theoretic optimization problem. We show that $\mathcal{M}^{RS}$ fails to ensure the closed-loop incremental stability in general but does ensure it for the special case wherein the setpoints are allowed to vary with time at either only the parent or only the children.**

***Index Terms*— robust stability, multipliers, sensor networks, noncooperative game theory, power control**

I. Introduction

Despite a wide-spread use of data aggregation policies for information dissemination concerning natural phenomena such as forest fires, earthquakes [1], avalanches, and landslides, a complete characterization of the improvement facilitated by feedback remains an open problem (see [2], [3], [4], [5]). As discussed in [6], [7], the control bandwidth minimization problem is closely related to the expensive control case of the minimum quadratic Gaussian control problem shown in Fig. 1(iii), the objective being (see [6]) the stabilization of an unstable feedforward system over an AWGN channel, given by $N(0, \sigma_w^2)$, using a control signal u having the minimal average power. It is well-known that the optimal feedback for the shown system is a gain of $-\alpha + 1/\alpha$ so that the minimal average power of u required for stabilization is given by $P = (\alpha^2 - 1)\sigma_w^2$, and $\ln(\alpha) = \frac{1}{2}\ln\left(1 + \frac{P}{\sigma_w^2}\right)$. Motivating question for the paper is to examine if the system performance degrades markedly if the set-points on the regulated variables (transmit power, number of bits, aggregation quality, etc.) across nodes vary to counter changes in the operating conditions such as node failures, environmental disturbances, and other factors. If the sensory information is routed through the network using a tree topology (see Fig. 1(i)), the data aggregation policies can be synthesized using team-optimized resource allocation techniques, recently applied for power control in multi-cell CDMA networks by [8]. Following [8] and [9], we shall establish in this paper the largest possible class of linear time-invariant aggregation policies that are stable in the input-output sense, and examine whether the stability holds if the set-points across the network are changed simultaneously.

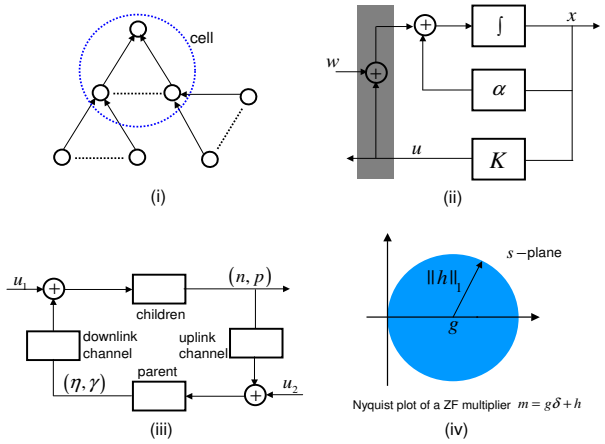

Fig. 1. (i): Data aggregation using a tree topology. (ii) Interpreting the performance degradations as the loss of stability, bounds on the minimal control bandwidth required can be established [6], [7]. (iii) Block diagram of the feedback system considered in this paper. (iv) [9] uses the Zames-Falb multipliers, the Nyquist plot of which lies in shown disc, to improve the performance of the system shown in (ii).

A key observation is that the transmit power used to communicate a message increases monotonically with the number of bits used so that it is possible to define a cost function $C_i(\vartheta_i)$ for an i-th node, where $\vartheta_i \doteq [p_i \;\; n_i]^T$, p_i being transmit power and n_i being the number of bits, as a game between a convex function of ϑ_i and a strictly concave function of the aggregation quality; the convex pricing function represents price incurred by the user in the communication while the concave utility function represents the incentive to increase ϑ_i to maximize some desired performance metric [10]. For example, consider the scenario shown in Fig. 1(i) where a parent receives data from its K children. Suppose the child i transmits $n_i(t)$ bits at the transmit power $p_i(t)$ at the time instant t, where $0 \leq p_i(\cdot) \leq p^*$, for some p^*. The corresponding signal received at the parent is $y_i(t) = h_i p_i(t)$ where h_i is the channel gain. The *signal-to-interference ratio* (SIR) for the child is $\gamma_i = Lh_i p_i / \left(\sum_{k \neq i} h_k p_k + \sigma^2\right)$, where L is the spreading gain, and σ^2 is the noise variance due to the background interference [11], [12], [13]. One instance of the system optimization problem is

$$\min_{p \doteq [p_1 \;\ldots\; p_K]} \sum_i C_i(p_i) \quad \text{s.t.} \quad \gamma_i \geq \bar{\gamma}_i \geq 0, p_i^* \geq p_i \geq 0 \quad \forall\, i$$

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TABLE I
NOTATION

Symbol	Meaning
$(f(x))_x^+$	$\doteq \begin{cases} f(x) & \text{if } x>0 \vert\vert (x=0 \text{ and } f(x)\geq 0); \\ 0 & \text{if } x=0 \text{ and } f(x)<0. \end{cases}$
$(x)^+$	$\doteq \begin{cases} x & \text{if } x\geq 0; \\ 0 & \text{else.} \end{cases}$

with the cost function $C_i(\cdot) \doteq P_i(p_i) - U_i(\gamma_i)$, where the *pricing function* $P_i(p_i)$ is twice continuously differentiable, non-decreasing, and strictly convex, and the *utility function* $U_i(\gamma_i) \doteq u_i \log(\gamma_i + L)$ where $u_i > 0$ is user-specific (see [9]). In this paper, we use the Zames-Falb multipliers (see [14], [15], [16]) to establish the largest possible class of linear time-invariant controllers to update the transmit power and the number of bits used in stable data aggregation policies. We also show that, if the set-points for the parent and the children vary simultaneously with time, this class of controllers reduces dramatically.

II. Notation and System Description

A. Notation and Mathematical Preliminaries

Notation is fairly standard in control theory (see [17], [18], [19]). We say that the *projection* $(f(x))_x^+$ is *active* if it is zero-valued, and *inactive* otherwise. We denote i-th row of a matrix A as $\text{row}_i(A)$. A feedback system $\dot{x} = f(x,u),\ y = h(x)$ is said to be *passive* if there exists a continuously differentiable positive semi-definite function, i.e. a *storage function*, $V(x)$ such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) \leq u^T h(x) \quad \forall x, u.$$

The system $y = h(t,u)$ is *passive* if $u^T y \geq 0\ \forall u$. A *multiplier* is a *convolution operator* such as, e.g., the Popov multipliers, the Zames-Falb multipliers, and the RC multipliers [20], [14], [15]. The terms *internal stability*, *input-output stability*, *incremental stability*, and *incremental positivity* are defined in [20], [14], [17], [18], [19].

Definition 1: [$\mathcal{N}_M$: monotone nonlinearity]
The class $\mathcal{N}_M$ of *monotone* nonlinearities consists of all memoryless mappings $N : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that:

1) N is the gradient of a convex real-valued function; and
2) there exists $C \in \mathbb{R}^+$ s.t. $\|N(x)\| \leq C\|x\| \quad \forall x \in \mathcal{L}_2$.

The class $\mathcal{N} \doteq \{N \in \mathcal{N}_M | N(0) = 0\}$ and the class $\mathcal{N}_{odd} \doteq \{N \in \mathcal{N} | N(x) = -N(-x)\ \forall x\}$. □

Definition 2: [repeated monotone nonlinearity]
Every element N of $\mathcal{N}^{RS}$, the class of *repeated single-input-single-output (SISO) monotone* nonlinearities, is of the form

$$N(\zeta) \doteq [\phi(\zeta_1)\ \phi(\zeta_2)\ \ldots\ \phi(\zeta_p)]^T \quad \forall \zeta \in \mathbb{R}^p$$

where $\phi \in \mathcal{N},\ \phi$ is SISO, and $\zeta = [\zeta_1\ \zeta_2\ \ldots\ \zeta_p]^T$. □

Definition 3: [$\mathcal{M}_{ZF}$: Zames-Falb multipliers]
$\mathcal{M}_{ZF}$ of Zames-Falb multipliers denotes the class of convolution operators, either continuous-time or discrete-time, such that the impulse response of an $M \in \mathcal{M}_{ZF}$ is of the form

$$m(\cdot) = g\ \delta(\cdot) + h(\cdot) \quad \text{with} \quad \|h\|_1 < g,$$

where $g, h(\cdot) \in \mathbb{R}$. □

Remark 1: A multiplier preserves positivity of a $\mathcal{N}_M$ nonlinearity if and only if it is in $\mathcal{M}_{ZF}$ [15], [21]. □

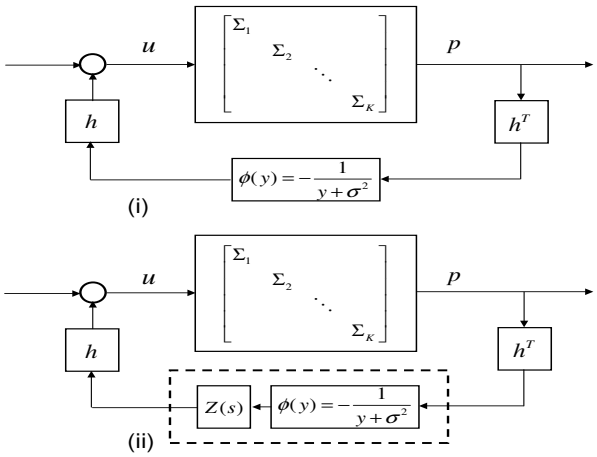

Fig. 2. (i): Block-diagram of the first-order gradient algorithm proposed in [9] using which a parent assigns transmit powers to its children. (ii): As shown in [9], an inclusion of the Zames-Falb multipliers in the price/utility updates does not alter the Nash equilibrium of the scheme depicted in (i), and allows a greater flexibility in controlling the transient response.

B. System Description

For simplicity, we consider a single cell network, i.e., a network in which, say, K children transmit the sensory data to a single parent. Control variables at a child node i are the transmit power p_i and the number of bits n_i. Performance metric at the parent is the SIR for the transmit power and $\eta \doteq [\eta_1\ \ldots\ \eta_K]^T$ for the number of bits received where we define $\eta_i \doteq \alpha n_i / \left(\sum_j n_j + \nu^2\right)$ for some $\alpha, \nu \in \mathbb{R}^+$. Control variables at the parent will be described shortly. We have introduced η to account for the share of the child i in the data processing overload at the parent. The terms α and ν^2 are user specific, and their effect is characterized in Lemma 2. Let $\gamma \doteq [\gamma_1\ \ldots \gamma_k]^T$. Define p, n, and η likewise. Let $\kappa \doteq [\gamma^T\ \eta^T]^T$, $\omega \doteq [p^T\ n^T]^T$. By $\kappa(\omega_0)$, we refer to the value of κ when $\omega = \omega_0$. As shown in [22], the non-cooperative game between the pricing and utility functions admits a unique Nash equilibrium, and a gradient law used to update the price at a child that ensures asymptotic stability of this equilibrium is given by

$$\Sigma_i : \quad \dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \frac{u_i h_i \lambda_i}{\sum_j h_j p_j + \sigma^2} \right)_{p_i}^+,$$

where $\gamma_i > 0$. Define $h \doteq [h_1\ \ldots\ h_K]^T$. Let $q \doteq \phi(y) \doteq -1/(y+\sigma^2),\ y \doteq h^T p,\ w \doteq -h \cdot q$. Let us refer to

this system, shown in Fig. 2(i), as $\mathcal{S}_1$. Let p^* be the Nash equilibrium with y^*, q^*, and w^* being the corresponding values of y, q, and w, respectively. Using [9], it follows that $(q-q^*)(y-y^*) \geq 0$. As shown in [20], [14], [21], inserting a suitable multiplier at the output of $\phi(\cdot)$ (see fig. 2(ii)) can reduce the conservatism in determining stability of $\mathcal{S}_1$. Let $\mathcal{Z}$ be a class of multipliers, and $z(\cdot) \in \mathcal{Z}$. Then, the system of interest, $\mathcal{S}_2$, is given by

$$\dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i \widetilde{w}_i\right)_{p_i}^{+}, \quad \widetilde{w} = -h \cdot z(\cdot) \cdot q. \quad (1)$$

III. Problem Formulation and Background Results

The problems of interest are as follows.

Problem 1: Obtain the largest class $\mathcal{Z}$ of multipliers that ensures the input-output stability of $\mathcal{S}_2$. □

Problem 2: Determine whether the incremental stability of $\mathcal{S}_2$ can be verified using the multipliers obtained as the solution to the Problem 1. □

Consider the system problem described in Section I:

$$\min_{\vartheta} \sum_{i=1}^{K} C_i(\vartheta)$$

s.t. $\gamma_i \geq \bar{\gamma}_i \geq 0, p^* \geq p_i \geq 0, \eta_i \geq \bar{\eta}_i \geq 0, n^* \geq n_i \geq 0 \ \forall i.$

As in [8], this problem can be recast as

$$\min_{\vartheta} \sum_{i=1}^{K} C_i(\vartheta) \text{ s.t. } \vartheta \in \Omega, \quad (2)$$

where $\Omega \doteq \{\vartheta | A\vartheta \geq b, p_i \leq p^*, n_i \leq n^*, p_i, n_i \in \mathbb{R}^+\}$,

$$\begin{aligned}
b &\doteq [\frac{\sigma^2\bar{\gamma}_1}{L} \ \dots \ \frac{\sigma^2\bar{\gamma}_K}{L} \ \nu^2\bar{\eta}_1 \ \dots \ \nu^2\bar{\eta}_K]^T, \\
A &\doteq \text{diag}\,(A_{11}, A_{22}), \\
A_{11} &\doteq \begin{bmatrix} h_1 & -\frac{h_2\bar{\gamma}_1}{L} & \dots & -\frac{h_K\bar{\gamma}_1}{L} \\ -\frac{h_1\bar{\gamma}_2}{L} & h_2 & \dots & -\frac{h_K\bar{\gamma}_1}{L} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{h_1\bar{\gamma}_K}{L} & -\frac{h_2\bar{\gamma}_K}{L} & \dots & h_K \end{bmatrix}, \\
A_{22} &\doteq \begin{bmatrix} 1 & -\bar{\eta}_1 & \dots & -\bar{\eta}_1 \\ -\bar{\eta}_2 & 1 & \dots & -\bar{\eta}_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\eta}_K & -\bar{\eta}_K & \dots & 1 \end{bmatrix}.
\end{aligned}$$

Let us define $\theta \doteq \max\{\sum_j \frac{\bar{\gamma}_j}{L+\bar{\gamma}_j}, \sum_j \frac{\bar{\eta}_j}{\alpha}\}$.

Lemma 1: If Ω is nonempty, the optimization problem described in (2) has a unique global minimum. □

Proof: The arguments used in [8] to prove [8, Lemma 3.1] give the required proof. QED.

Lemma 2: If $\theta < 1$ and if (p^*, n^*) are chosen sufficiently large, Ω is nonempty and every ϑ satisfying $A\vartheta \geq b$ satisfies $\vartheta \geq 0$. Furthermore, if $\theta > 1$, Ω is empty. □

Proof: The arguments used in [8] to prove [8, Lemma 3.2] give the required proof as follows. Define

$$\widetilde{A} \doteq \text{diag}\left(\widetilde{A}_{11}, \widetilde{A}_{22}\right), \quad x \doteq [h_1 p_1 \ \dots \ h_K p_K \ n_1 \ \dots \ n_K]^T,$$

$$\begin{aligned}
\widetilde{A}_{11} &\doteq \text{diag}\left(1 + \frac{\bar{\gamma}_i}{L}\right) - \begin{bmatrix} \frac{\bar{\gamma}_1}{L} & \frac{\bar{\gamma}_1}{L} & \dots & \frac{\bar{\gamma}_1}{L} \\ \frac{\bar{\gamma}_2}{L} & \frac{\bar{\gamma}_2}{L} & \dots & \frac{\bar{\gamma}_2}{L} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{\gamma}_K}{L} & \frac{\bar{\gamma}_K}{L} & \dots & \frac{\bar{\gamma}_K}{L} \end{bmatrix}, \\
\widetilde{A}_{22} &\doteq I - \begin{bmatrix} \frac{\bar{\eta}_1}{\alpha} & \frac{\bar{\eta}_1}{\alpha} & \dots & \frac{\bar{\eta}_1}{\alpha} \\ \frac{\bar{\eta}_2}{\alpha} & \frac{\bar{\eta}_2}{\alpha} & \dots & \frac{\bar{\eta}_2}{\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{\eta}_K}{\alpha} & \frac{\bar{\eta}_K}{\alpha} & \dots & \frac{\bar{\eta}_K}{\alpha} \end{bmatrix}, \quad \widehat{A}_{11} \doteq I - T_{11}, \\
T_{11} &\doteq \begin{bmatrix} \frac{\bar{\gamma}_1}{L+\bar{\gamma}_1} & \frac{\bar{\gamma}_1}{L+\bar{\gamma}_1} & \dots & \frac{\bar{\gamma}_1}{L+\bar{\gamma}_1} \\ \frac{\bar{\gamma}_2}{L+\bar{\gamma}_2} & \frac{\bar{\gamma}_2}{L+\bar{\gamma}_2} & \dots & \frac{\bar{\gamma}_2}{L+\bar{\gamma}_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{\gamma}_K}{L+\bar{\gamma}_K} & \frac{\bar{\gamma}_K}{L+\bar{\gamma}_K} & \dots & \frac{\bar{\gamma}_K}{L+\bar{\gamma}_K} \end{bmatrix}. \\
\widehat{b} &\doteq [\frac{b_1 L}{L+\bar{\gamma}_1} \ \dots \ \frac{b_K L}{L+\bar{\gamma}_K} \ b_{K+1} \ \dots \ b_{2K}],
\end{aligned}$$

Now, $A\vartheta \geq b$ is equivalent to $\widetilde{A}x \geq b$ which, in turn, is equivalent to $\widehat{A}x \geq \widehat{b}$ where $\widehat{A} \doteq \text{diag}(\widehat{A}_{11}, \widetilde{A}_{22})$. Now, T_{11} has $K-1$ eigenvalues at 0 and one eigenvalue at $\sum_j \bar{\gamma}_j/(L+\bar{\gamma}_j)$. Hence, $\widehat{A}$ has $2K-2$ eigenvalues at 1, one eigenvalue at $1 - \sum_j \bar{\gamma}_j/(L+\bar{\gamma}_j)$, and one eigenvalue at $1 - \sum_j \bar{\eta}_j/\alpha$. Hence, if $\theta < 1$, $\widehat{A}$ is a nonsingular M-matrix whence it follows that $\widehat{A}x \geq \widehat{b} \geq 0$ implies $x \geq 0$, i.e., $\vartheta \geq 0$. To prove that Ω is empty if $\theta > 1$, suppose, on the contrary, there exists $x \geq 0$ such that $\widehat{A}x \geq \widehat{b}$. Since each entry of $\widehat{b}$ is strictly positive, it follows that $\widehat{A}$ is a nonsingular M-matrix (see [23]) which implies that $\theta < 1$, i.e., a contradiction. QED. □

We now show how the above results can be used to synthesize the decentralized data aggregation algorithms. Assuming ϑ^* is large enough, since the solution ϑ^o to the system problem is componentwise positive, it suffices to consider the Lagrangian

$$L(\vartheta, \lambda) \doteq \sum_i C_i(\vartheta_i) + \lambda^T(A\vartheta - b). \quad (3)$$

Let $q \doteq A^T\lambda, r \doteq \text{diag}(p_i)q$. Then, (3) is recast as

$$L(\vartheta, \lambda) \doteq \sum_i (C_i(\vartheta_i) - r_i) + \lambda^T b, \quad (4)$$

with ϑ^o satisfying $\frac{dC(\vartheta)}{d\vartheta}|_{\vartheta=\vartheta^o} - q = 0$. Define the convex user and network problems as

$$\text{user } i\text{: } \min_{r_i} C_i(r_i/q_i) - r_i, \ r_i \geq 0 \quad (5)$$

$$\text{network: } \min_{\vartheta} \sum_i -r_i \log(\vartheta_i), \quad \vartheta \in \Omega. \quad (6)$$

As shown in [8], a primal update algorithm is as follows:

$$\dot{\vartheta} = K\left(\frac{dC(\vartheta)}{d\vartheta} + q_i\right), q(\vartheta) = A^T\rho(A\vartheta), \quad (7)$$

where $K \doteq \text{diag}(k_i)$ (k_i user specific), $\rho_i(\omega) \doteq f(\bar{\kappa}_i - \kappa_i(\omega))$, $\rho(A\vartheta) \doteq [\rho_1(\text{row}_1(A\vartheta)) \dots \ \rho_K(\text{row}_K(A\vartheta))]^T$; the

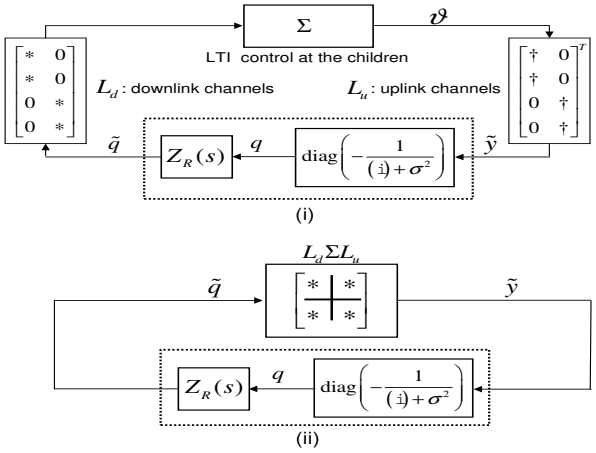

Fig. 3. The above interpretation of the system described in Fig. 2(ii) allows one to implement the Zames-Falb multipliers for repeated monotone nonlinearities. In principle, this facilitates the greatest flexibility possible (*via convolution operators*) in governing the price/utility fluctuations in the dynamical systems on interest. Note that the computational load is pushed into the network (i.e., at the parent) and away from the leaf nodes.

user specific function f is monotone continuous and zero-valued if the argument is negative-valued. A block diagram representation of an instance of this system is Fig. 2(ii) where we have designed the feedback nonlinearities to be repetitive (in general, those need not be repetitive). We want to examine how the choice of k_i (and other user specific functions) affects the closed-loop stability, and whether the transient and the steady-state response can be improved by introducing multipliers at the output of the parent (see $\mathcal{S}_2$ of (1)).

IV. Main Results

We first note down the complete class of positivity preserving multipliers for repeated monotone nonlinearities.

Definition 4: [$\mathcal{M}^{RS}$ [16]]
$\mathcal{M}^{RS}_{odd}$ denotes the class of MIMO convolution operators, either continuous or discrete, such that the impulse response of an $M \in \mathcal{M}^{RS}_{odd}$ is of the form $m = g\ \delta - h$ where $g, h(\cdot) \in \mathbb{R}^{p\times p}$ satisfy

$$g_{ii} \geq \sum_{i=1, i\neq j}^{n} |g_{ij}| + \sum_{i=1}^{n} \|h_{ij}\|_1 \quad \forall i = 1, 2, \ldots, 2K \tag{8}$$

$$g_{ii} \geq \sum_{j=1, j\neq i}^{n} |g_{ij}| + \sum_{j=1}^{n} \|h_{ij}\|_1 \quad \forall i = 1, 2, \ldots, 2K. \tag{9}$$

Further stipulating

$$g_{ij} \leq 0\ \forall i \neq j, \quad h_{ij}(\cdot) \geq 0\ \forall i, j, \tag{10}$$

the subclass $\mathcal{M}^{RS}$ is obtained. □

Basic algebraic operations yield Fig. 3 as the block diagram representation of $\mathcal{S}_2$. If the children are homogenous in the sense that they all use the same parameter values λ_i and u_i, the feedback nonlinearity is a repeated memoryless scalar nonlinearity, the graph of which lies in the first-and-third quadrant. Then, solution to Problem 1 is as follows.

Theorem 1: Consider $\mathcal{S}_2$ with $z \in \mathcal{M}^{RS}$. The feedforward system mapping $(\tilde{w} - \tilde{w}^*)$ into $(\tilde{y} - \tilde{y}^*)$ is passive. Further, the system is input-output stable at the Nash equilibrium ϑ^*. □

Proof: The proof follows through a slight modification of the arguments used in [9] to prove [9, Proposition 1]. We present it all the same to point out some singular points pertaining to the number of bits used in the feedback. Choose the storage function to be

$$V(\tilde{\vartheta} - \tilde{\vartheta}^*) = \frac{1}{2} \sum_i \frac{1}{\mu_i \lambda_i} \left(\tilde{\vartheta}_i - \tilde{\vartheta}_i^*\right)^2. \tag{11}$$

Its derivative along the solutions of (8) is given by

$$\begin{aligned} \dot{V} &= \sum_i \frac{1}{u_i \lambda_i} \left(\tilde{\vartheta}_i - \tilde{\vartheta}_i^*\right) \left(-\lambda_i \frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i} + u_i \lambda_i \tilde{w}_i\right)^+_{\tilde{\vartheta}_i} \\ &\leq \sum_i \frac{1}{u_i \lambda_i} \left(\tilde{\vartheta}_i - \tilde{\vartheta}_i^*\right) \left(-\lambda_i \frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i} + u_i \lambda_i \tilde{w}_i\right), \end{aligned}$$

for the reasons explained in [9]. Whence, an active projection implies

$$\begin{aligned} 0 &= \sum_i \frac{1}{u_i \lambda_i} \left(\tilde{\vartheta}_i - \tilde{\vartheta}_i^*\right) \left(-\lambda_i \frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i} + u_i \lambda_i \tilde{w}_i\right) \\ &\leq \sum_i \frac{1}{u_i \lambda_i} \underbrace{\left(-\tilde{\vartheta}_i^*\right)}_{\leq 0} \underbrace{\left(-\lambda_i \frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i} + u_i \lambda_i \tilde{w}_i\right)}_{\text{Term 1}}. \end{aligned}$$

Term 1 may be verified to be non-negative. After adding and subtracting $u_i \lambda_i \tilde{w}_i^*$ to Term 1, it may be verified that

$$\begin{aligned} \dot{V} \leq\ & \underbrace{\sum_i \frac{1}{u_i} \left(\tilde{\vartheta}_i - \tilde{\vartheta}_i^*\right) \left(-\frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i} + \frac{dC_i(\tilde{\vartheta}_i)}{d\tilde{\vartheta}_i}|_{\tilde{\vartheta}_i = \tilde{\vartheta}_i^*}\right)}_{\text{Term 2}} \\ & + \left(\tilde{\vartheta} - \tilde{\vartheta}^*\right)^T (\tilde{w} - \tilde{w}^*). \end{aligned}$$

As Term 2 is strictly negative since $C_i(\cdot)$ is strictly convex,

$$\dot{V} < \left(\tilde{\vartheta} - \tilde{\vartheta}^*\right)^T \text{diag}(h_i)\, (\tilde{q} - \tilde{q}^*) \quad \forall \tilde{\vartheta} \neq \tilde{\vartheta}^*,$$

whence the strict passivity of the feed-forward system. Passivity of the feedback system mapping $(\tilde{\vartheta} - \tilde{\vartheta}^*)$ into $(\tilde{q} - \tilde{q}^*)$ follows using [16, Theorem 1].

Validity in the discrete-time case can be verified as follows. Note that continuous time signals x_c, y_c can be sampled with sampling interval ϵ to produce discrete time signals in ℓ_2, say $x_{d,\epsilon}, y_{d,\epsilon}$ such that $x_{d,\epsilon}(k) = \sqrt{\epsilon}\ x_c(k\epsilon),\ y_{d,\epsilon}(k) = \sqrt{\epsilon}\ y_c(k\epsilon)$. Given a continuous time linear operator $M_c : \mathcal{L}_2 \to \mathcal{L}_2$,

$$z_c(t) \doteq \int_{-\infty}^{\infty} \overline{m}_c(t, \tau) y(\tau)\, d\tau\ \forall t,$$

may be discretized as $z_{d,\epsilon} \doteq M_{d,\epsilon} y_{d,\epsilon}$ i.e. as

$$z_{d,\epsilon}(k) \doteq \sum_{l=-\infty}^{\infty} \overline{m}_{k,l} y_{d,\epsilon}(l) \quad \forall k$$

where, for all k and l,

$$\overline{m}_{kl} \doteq \frac{1}{\epsilon} \int_{t\in((k-1)\epsilon,k\epsilon]} \int_{\tau\in((l-1)\epsilon,l\epsilon]} \overline{m}_c(t,\tau)\ d\tau\ dt.$$

With this discretization, the continuous time inner-product $\langle x_c, M_c y_c\rangle$ is then recoverable as the limit

$$\langle x_c, M_c y_c\rangle = \lim_{\epsilon\to 0} \langle x_{d,\epsilon}, M_{d,\epsilon} y_{d,\epsilon}\rangle .$$

The discrete-time case proof follows by setting a finite value to ϵ. The input-output stability of $\mathcal{S}_2$ then follows using [14, Theorems 1-2]. □

Remark 2: Theorem 1 gives the complete class of linear time-invariant controllers that ensure the closed-loop stability. The controller can be implemented at the parent, i.e., *inside* the network, rather that at the leaf nodes thereby pushing the processing overload inside the network. □

Remark 3: We have assumed constant channel gains. In principle, the *integral quadratic constraints* for time-varying gains (see [19]) can be coupled with $\mathcal{M}^{RS}$ to derive the update algorithms that ensure closed-loop stability when the channels have time-varying gains as is the case with Rayleigh channels and the AWGN channels. A far simpler, though more conservative, solution can be obtained by noting that the gain variation Δh_i in the channel gain h_i leads to an *area loss* of $\Delta h_i/\sigma^2$ in the nonlinearity mapping $\Delta\tilde{\vartheta}$ into $\Delta\tilde{q}$. Then, it suffices to add a DC gain shift of $\Delta h_i/\sigma^2$ to the multiplier $\mathcal{M}^{RS}$. As the background noise σ^2 appears in the denominator of the area loss term, it seems to mitigate the effect of the channel gain variations. □

We next examine whether the closed-loop system refrains from exhibiting undesirable jump phenomena if the primal update of (8) employing an $M \in \mathcal{M}^{RS}$ is exercised while the set-points $\bar{\kappa}$ and $\bar{\omega}$ are being changed arbitrarily, i.e., whether the system employing an $M \in \mathcal{M}^{RS}$ is *incrementally stable*. Clearly, the incremental stability is assured if M preserves the incremental positivity of the monotone nonlinearity ϕ [24]. However, the following result shows that such is not the case in general.

*Lemma 3 (*rephrased [24, Theorem 1]): The incremental positivity of a monotone nonlinearity N is preserved by all Zames-Falb multipliers iff N is linear. A Zames-Falb multiplier preserves incremental positivity of all monotone nonlinearities iff it is a constant gain. □

Fortunately, a close examination of the proofs in [24] shows that the closed-loop stability is assured provided the setpoint at the parent is kept constant, i.e., $\bar{\kappa}$ is kept constant, while the setpoints, namely $\bar{\omega}$ at all children vary arbitrarily (and vice versa). Thus, we have the following result.

Theorem 2: Consider $\mathcal{S}_2$ with $z \in \mathcal{M}^{RS}$. The system is incrementally stable at the Nash equilibrium ϑ^* if and only if $z(\cdot)$ is a constant gain. The system is incrementally stable at the Nash equilibrium ϑ^* under a dynamic $z(\cdot)$ provided either $\bar{\omega}$ or $\bar{\kappa}$ is held constant. □

V. Simulation

We implemented a 21 node network topology, including a single parent, in MATLAB to see how the simulation results

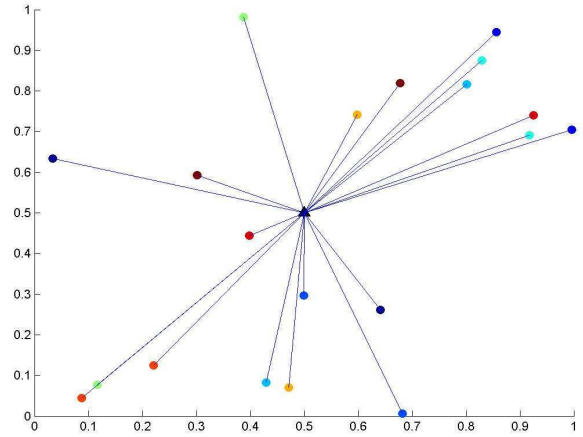

Fig. 4. Simulation set-up: the network topology showing 20 children transmitting data to a single parent

in [8] change after a $\mathcal{M}^{RS}$ multiplier $z(s) = 1/(s+1)$ is included in $\mathcal{S}_2$. The nodes are distributed randomly as shown in Fig. 4; d_i denotes the distance between the node i and the parent node. As in [8], we let the channel gain $h_i = 0.01d_i^2, L = 64, \sigma^2 = 0.1, k_i = 0.02, C_i(p_i) = 0.005h_ip_i^2 - 0.0001\log(p_i)$. The target SIR value was set to be 2 for ten nodes and 3 for the remaining ten nodes. Analytically, the closed-loop is stable. Fig. 5 and Fig. 6 show that the desired performance specification are met satisfactorily.

VI. Conclusion

We have extended [8] and [9] to derive the largest possible class of linear time-invariant controllers, $\mathcal{M}^{RS}$, to synthesize information dissemination policies in resource constrained sensor networks that ensure the input-output closed-loop stability. Our main contribution is the use of the Zames-Falb multipliers to preserve positivity of repeating monotone nonlinearities, encountered in the noncooperative game theoretic optimization. It turns out that although such controllers guarantee the input-output stability of the closed-loop system, the incremental stability is not assured. Therefore, if the setpoints are varied arbitrarily with time across the network, the class of stabilizing controllers reduces dramatically. However, if the setpoints at only the parent (or, alternatively, only the children) in the tree topology are varied with time, $\mathcal{M}^{RS}$ does ensure the closed-loop stability. This suggests that, sometimes, asynchrony should be introduced while varying setpoints across complicated large-scale sensor networks. Neither [9] nor [25] examine the robustness in the face of plant parameter uncertainty. A combination of these techniques together with a safe and reliable switching technique, such as [26], might lead to designs that are robust.

VII. Acknowledgment

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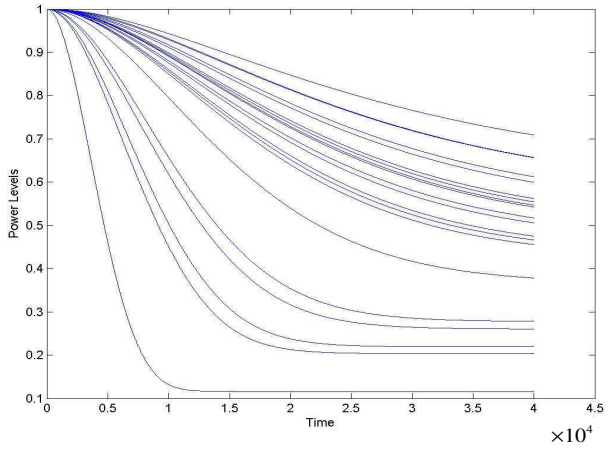

Fig. 5. Evolution of the transmit power of the children when $z(s) = 1/(s+1)$ is used as a $\mathcal{M}^{RS}$ multiplier in $\mathcal{S}_2$.

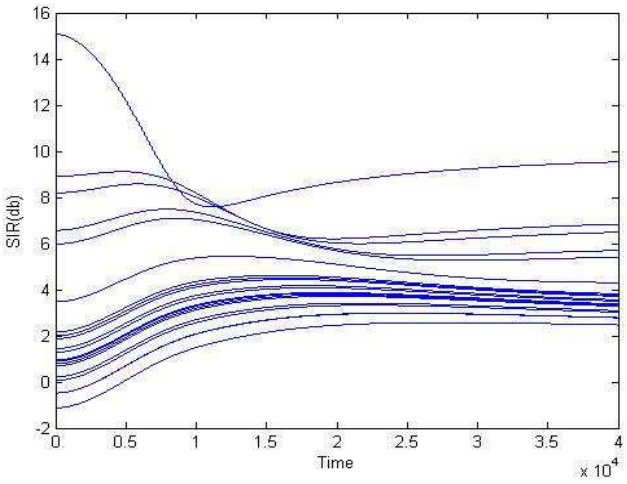

Fig. 6. Evolution of signal-to-interference ratio γ_i for the children when $z(s) = 1/(s+1)$ is used as a $\mathcal{M}^{RS}$ multiplier in $\mathcal{S}_2$.

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