Volterra Integral Approach to Impulsive Renewal Systems: Application to Networked Control

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Abstract

We analyze impulsive systems with independent and identically distributed intervals between transitions. Our approach involves the derivation of novel results for Volterra integral equations with positive kernel. We highlight several applications of these results, and show that when applied to the analysis of impulsive systems they allow us to (i) provide necessary and sufficient conditions for mean square stability, stochastic stability and mean exponential stability, which can be equivalently tested in terms of a matrix eigenvalue computation, an LMI feasibility problem, and a Nyquist criterion condition; (ii) assess performance of the impulsive system by computing a second moment Lyapunov exponent. The applicability of our results is illustrated in a benchmark problem considering networked control systems with stochastically spaced transmissions, for which we can guarantee stability for inter-sampling times roughly twice as large as in previous papers.

Index Terms

Impulsive systems; Stochastic Systems; Volterra Integral Equations; Positive Systems; Networked Control.

The work of D. Antunes was supported by the FCT PhD Student Scholarship, SFRH/BD/24632/2005.

This research is partially supported by the National Science Foundation under the grants CNS-0720842, ECCS-0725485 and ECCS-0835847

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Preprint submitted to IEEE Transactions on Automatic Control. Received: November 24, 2010 09:41:30 PST

I. Introduction

In this paper we provide analytical results for *impulsive renewal systems*, which are deeply rooted in a set of novel results for *Volterra integral equations with positive kernel*, and have applications to the stability analysis of *networked control systems*.

Impulsive renewal systems are described by a *vector field* that determines the evolution of the state between *transition times* at which the state undergoes a jump determined by a *reset map*. The intervals between transition times are assumed to be independent and identically distributed (i.i.d.) random variables. In the present paper we restrict ourselves to linear dynamics and reset maps. Impulsive renewal systems are a special class of the piecewise deterministic processes considered in [1].

Impulsive renewal systems with general non-linear dynamic and reset maps are considered in [2]. The nomenclature introduced therein to address such systems is motivated by the fact that the process that counts the number of transitions up to the current time is a renewal process [3]. The paper [2] considers a more general class of impulsive systems in which the system evolves according to a stochastic differential equation between jumps (as opposed to a deterministic vector field). For this class of impulsive renewal systems, *sufficient* conditions for a form of exponential stability are provided.

Early work on systems with i.i.d. distributed parameters can be found in [4], [5], [6]. The stability problems considered in these early references are typically cast into a discrete-time setting, even if the motivating stability problem is in a continuous-time setting, as it is the case for randomly sampled systems [6]. *Sufficient* conditions for continuous-time mean square stability are provided in [7] for impulsive renewal systems that arise in model-based networked control systems with stochastically spaced transmissions. The approach followed in [4], [5], [6], [7] involves a stability test in terms of a matrix eigenvalue computation. An alternative approach that can be found in the literature to investigate the stability of such systems resorts to stochastic Lyapunov functions, leading to stability conditions formulated in terms of the existence of a Lyapunov function satisfying appropriate inequalities [8], [9].

In the present paper, we propose a new approach to analyze impulsive renewal system based on a Volterra integral equation, describing the expected value of a quadratic function of the system. Our main contribution to the stability of impulsive renewal systems is stated in two main theorems

Preprint submitted to IEEE Transactions on Automatic Control. Received: November 24, 2010 09:41:30 PST

that are proved using novel results for Volterra integral equations. We believe that these stability results cannot be obtained through any of the previous approaches [2], [4], [5], [7], [8], [9]. The first main stability theorem establishes *necessary and sufficient* conditions for mean square stability, stochastic stability and mean exponential stability. This result provides a unified treatment for these three stability notions and reveals that these are not equivalent in general. This result obviates the conservativeness of the results in [2], [7] where only *sufficient* stability conditions are presented. We prove that the stability conditions can be cast in terms of a matrix eigenvalue computation, the feasibility of a set of LMIs, and also tested using the Nyquist criterion. The stability tests in terms of an eigenvalue or an LMI computation resemble tests available for Markov linear systems [10], but, to the best of our knowledge, a Nyquist test has not appeared previously for this type of systems. The second main theorem characterizes the second moment Lyapunov exponent, which provides the asymptotic rate of decrease/growth for the expected value of a quadratic function of the systems' state. We discuss how one can assert the performance of the system by computing this Lyapunov exponent.

Volterra integral equations arise in many problems in physics, engineering and biology. The excellent textbook [11] discusses an extensive number of results for these equations. The Volterra equation considered here has the special property that its kernel is a positive operator in a sense to be defined below. For this specific class of Volterra equations we show that stability can be determined through a matrix eigenvalue computation or through a cone programming problem, and provide a method to obtain the Lyapunov exponent of the Volterra equation. These two results are important contributions per se and can be used in problems unrelated to impulsive renewal systems. As an example, we show how they can be used to construct a simple stability condition for a class of LTI closed-loop systems with non-rational transfer functions. It is important to note that the notion of positiveness needed in this paper differs from the well-known notions of monotonicity (see [11, Ch.5]) and positiveness (see [11, Ch.16]).

We illustrate the application of our stability results for impulsive renewal systems in the networked control setup considered, e.g., in [2] and [12]. It is assumed that the time intervals between transmissions of sensor and actuation data are i.i.d., which is a reasonable assumption in networks using Carrier Sense Multiple Access (CSMA)-type protocols [12]. We consider an emulation-based design, i.e., we assume that a continuous-time stabilizing controller has been

designed, without regard to the network characteristics, and we are interested in analyzing the effects of the network on the closed-loop. For a benchmark problem considered in [2], [12], [13] we verify that the results in [2] were indeed conservative. In fact, with the results in the present paper one can guarantee stability for an average inter-sampling roughly twice as large as in [2].

The remainder of the paper is organized as follows. Section II defines impulsive renewal systems and introduces appropriate stability notions. The two main stability theorems are stated without a proof and discussed in Section III. Section IV derives the Volterra integral equation describing a second moment of the impulsive systems' state, and establishes general results for Volterra integral equations with positive kernel, leading to the proof of the results of Section III. The applicability of the results to networked control is addressed in Section V. Section VI contains final conclusions. Some technical results are proved in the Appendix.

Notation and Preliminaries: For a given matrix A, its transpose is denoted by $A^\intercal$, its hermitian by A^*, its trace by $\mathrm{tr}(A)$. We use $A > 0$ ($A \geq 0$) to denote that a real or complex symmetric matrix is positive definite (semi-definite). The identity and zero matrices are denoted by I and 0, respectively. We work only in finite dimensional spaces, which can be identified with $\mathbb{C}^n$, or $\mathbb{R}^n$, subsumed to be Hilbert spaces with the usual inner product $\langle x, y\rangle = y^*x$, Banach spaces with the usual norm $\|x\|^2 = \langle x, x\rangle$ and endowed with the usual topology inherited by the norm. We consider the usual vector identifications $\mathbb{C}^{n\times n} \cong \mathbb{C}^{n^2}$, $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ for matrices and this results in the following inner product $\langle A, B\rangle = tr(B^*A)$. The Kronecker product is denoted by $\otimes$ and $(A, B) := [A^\intercal\, B^\intercal]^\intercal$. The expected value is denoted by $\mathbb{E}[.]$. We consider scalar *real* measures μ over $\mathbb{R}_{\geq 0}$, and we omit the 'over $\mathbb{R}_{\geq 0}$', since these are the only scalar measures that we consider. We consider also matrix real measures Θ (each entry θ_{ij} is a scalar real measure), with the usual total variation norm $|\Theta|(E) := \sup \sum_{j=1}^{\infty} \|\Theta(E_j)\|$, where the supremum is taken over all countable partitions $\{E_j\}$ of a set E (cf. [11, Ch.3, Def.5.2]), and $\|\Theta(E_j)\|$ denotes the induced matrix norm by the usual vector norm. One can prove that $|\Theta|(E)$ is a positive measure (cf. [11, Ch.3,Th.5.3]). For a measurable vector function $b(s) \in \mathbb{R}^n$ and an interval $I \subseteq [0, \infty]$, the integral $\int_I \Theta(ds)b(s)$ is a vector function with components $\sum_{j=1}^{n} \int_I b_j(s)\theta_{ij}(ds)$. We say that $\int_I \Theta(ds)b(s)$ converges absolutely if $\int_I \|b(s)\||\Theta|(ds) < \infty$, and use the same nomenclature when we replace the real measure $\Theta(ds)$ by the positive Lebesgue measure ds.

II. System, stability and performance definitions

An impulsive renewal system is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad && t \neq t_k, \;\; t \geq 0, \;\; k \in \mathbb{Z}_{>0} \\ x(t_k) &= Jx(t_k^-), && t_0 = 0, \;\; x(t_0) = x_0, \end{aligned} \tag{1}$$

where the state x evolves in $\mathbb{R}^n$ and the notation $x(t_k^-)$ indicates the limit from the left of a function $x(t)$ at the transition time t_k. The intervals between consecutive transition times $\{h_k := t_{k+1} - t_k, k \geq 0\}$ are assumed to be i.i.d.. The matrices A and J are real. The value at time t of a sample path of (1) is given by $x(t) = T(t)x_0$, where $T(t) = e^{A(t-t_r)}Je^{Ah_{r-1}} \dots Je^{Ah_0}$, $r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\}$ is the transition matrix.

The probability measure of the random variables h_k is denoted by μ. The support of μ may be unbounded but we assume that $\mu\big((0,\infty)\big) = 1$, $\mu(\{\infty\}) = 0$. We also assume that $\mu(\{0\}) = 0$. The measure μ can be decomposed into a continuous and a discrete components as in $\mu = \mu_c + \mu_d$, with $\mu_c([0,s)) = \int_0^s f(r)dr$, for some density function $f(r) \geq 0$, and μ_d is a discrete measure that captures possible point masses $\{b_i > 0, i \geq 1\}$ such that $\mu(\{b_i\}) = w_i$. The integral with respect to the measure μ is defined as

$$\int_0^t W(s)\mu(ds) = \int_0^t W(s)a(s)ds + \sum_{i:b_i \in [0,t]} w_i W(b_i). \tag{2}$$

We consider the following three stability notions for (1).

Definition 1. The system (1) is said to be

(i) *Mean Square Stable (MSS)* if for every x_0,

$$\lim_{t \to +\infty} \mathbb{E}[x(t)^\intercal x(t)] = 0,$$

(ii) *Stochastic Stable (SS)* if for every x_0,

$$\int_0^{+\infty} \mathbb{E}[x(t)^\intercal x(t)]dt < \infty,$$

(iii) *Mean Exponentially Stable (MES)* if there exists constants $c > 0$ and $\alpha > 0$ such that for every x_0,

$$\mathbb{E}[x(t)^\intercal x(t)] \leq ce^{-\alpha t} x_0^\intercal x_0, \forall_{t \geq 0}.$$

□

The following definition of second order Lyapunov exponent is adapted from [9, Ch.2].

Definition 2. Suppose that $\mathbb{E}[x(t)^\intercal x(t)] \neq 0, \forall t_{\geq 0}$ and that for every x_0 the following limit exists

$$\lambda_L(x_0) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[x(t)^\intercal x(t)].$$

Then the second order Lyapunov exponent λ_L for the system (1) is defined as

$$\lambda_L := \sup_{x_0\in\mathbb{R}^n} \lambda_L(x_0).$$

Moreover, if $\exists_{b>0} : \mathbb{E}[x(t)^\intercal x(t)] = 0, \forall t > b$ then $\lambda_L := -\infty$. □

The Lyapunov exponent provides a measure of how quickly the probability of $\|x(t)\|$ being large decays with time. One can see this, e.g., through the Chebyshev's inequality

$$\text{Prob}[\|x(t)\| > \epsilon] \leq \frac{\mathbb{E}[x(t)^\intercal x(t)]}{\epsilon^2}.$$

A small second moment Lyapunov exponent corresponds to a fast exponential decrease of $\mathbb{E}[x(t)^\intercal x(t)]$ and consequently to a fast decrease of the probability that $\|x(t)\|$ is larger than some positive constant ϵ.

III. Stability and performance analysis of Impulsive Renewal Systems

In this section we state the two main theorems of the paper. The first one characterizes the stability of (1) and to state it we need to introduce the complex function

$$\hat{\Theta}(z) := \int_0^\infty (Je^{As})^\intercal \otimes (Je^{As})^\intercal e^{-zs} \mu(ds),$$

which can be partitioned as in (2), $\hat{\Theta}(z) = \hat{\Theta}_c(z) + \hat{\Theta}_d(z)$. The following two technical conditions on the function $\hat{\Theta}$ will be needed:

(T1) $\hat{\Theta}(-\epsilon)$ converges absolutely for some $\epsilon > 0$.

(T2) $\inf_{z\in\mathcal{C}(R,\epsilon)}\{|\det(I - \hat{\Theta}_d(z))|\} > 0$ for some $\epsilon > 0$, and $R > 0$, where $\mathcal{C}(R,\epsilon) := \{z : |z| > R, \Re[z] > -\epsilon\}$.

These conditions hold trivially when μ has bounded support and no discrete component ($\mu_d([0,t]) = 0, \forall_{t\geq 0}$), but they also hold for much more general classes of probability measures. Let $\sigma(M)$ be the spectral radius of a $m \times m$ matrix M, i.e.,

$$\sigma(M) := \max\{|\lambda| : Az = \lambda z, \text{ for some } z \in \mathbb{C}^m\}.$$

The following is our first main theorem. Each of the conditions appearing in its statement will be commented in the sequel.

Theorem 3. Suppose that (T1) and (T2) hold. Then, the following conditions are equivalent

(A) $\det(I - \hat{\Theta}(z)) \neq 0, \ \Re[z] \geq 0,$

(B) $\sigma(M) < 1,$ where

$$M := \int_0^\infty (Je^{As})^\intercal \otimes (Je^{As})^\intercal \mu(ds),$$

(C) $\exists_{P>0} : L(P) - P < 0,$ where

$$L(P) := \int_0^\infty (Je^{As})^\intercal P Je^{As} \mu(ds).$$

Moreover, (1) is

(i) MSS if and only if (A), (B) and (C) hold and

$$e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) \to 0 \text{ as } t \to \infty; \tag{3}$$

(ii) SS if and only if (A), (B) and (C) hold and

$$\int_0^\infty e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) dt < \infty; \tag{4}$$

(iii) MES if and only if (A), (B) and (C) hold and

$$e^{2\lambda_\Re(A)t} t^{2(m_\Re(A)-1)} r(t) \leq c e^{-\alpha_1 t} \text{ for some } c > 0, \ \alpha_1 > 0, \tag{5}$$

where $r(t) := \mu\big((t, \infty]\big)$ denotes the survivor function, $\lambda_\Re(A)$ denotes the real part of the eigenvalue of A with largest real part and $m_\Re(A)$ the dimension of the larger Jordan block associated with this eigenvalue. □

The second main result provides a method to compute the Lyapunov exponent λ_L for (1). For simplicity, we restrict ourselves to the case where μ has bounded support. We recall that if $\hat{\Theta}(a)$ converges absolutely for a real b then $\hat{\Theta}(b)$ converges absolutely for every $a > b$ (cf. [11, Ch. 3, Th.3.8]).

Theorem 4. Suppose that (T1) and (T2) hold, that μ has finite support, and let $b := \inf\{a : \hat{\Theta}(a) \text{ converges absolutely}\}$. Then, the spectral radius $\sigma(\hat{\Theta}(a))$ of $\hat{\Theta}(a)$ is a monotone non-

increasing function of a for $a > b$ and the Lyapunov exponent for (1) is given by

$$\lambda_L = \begin{cases} a \in \mathbb{R} : \sigma(\hat{\Theta}(a)) = 1, \text{ if such } a \text{ exists} \\ -\infty \text{ otherwise} \end{cases}$$

□

Note that, since $\sigma(\hat{\Theta}(a))$ is non-increasing, one can compute λ_L by performing a simple binary search on $\mathbb{R}$.

We comment next on each condition of Theorem 3.

Condition (A): When $\det(I-\hat{\Theta}(i\omega)) \neq 0$, $\forall\, \omega \in \mathbb{R}$, the Nyquist criterion can be used to check if (A) holds , i.e., the number of zeros of $\det(I - \hat{\Theta}(z))$ in the closed-right half complex plane counted according to their multiplicities equals the number of times that the curve $\det(I-\hat{\Theta}(i\omega))$ circles anticlockwise around the origin as ω goes from ∞ to $-\infty$.

Eigenvalue condition (B): Using the properties

$$(AB) \otimes (CD) = (A \otimes B)(C \otimes D), \tag{6}$$

and $(A \otimes B)^\intercal = A^\intercal \otimes B^\intercal$ (cf. [14]), and considering the Jordan normal form decomposition of $A = VDV^{-1}$ we obtain

$$M^\intercal = (JV) \otimes (JV) \int_0^\infty e^{Ds} \otimes e^{Ds} \mu(ds)(V^{-1} \otimes V^{-1}).$$

Thus M can typically be obtained by integrating exponentials with respect to the measure μ.

LMI condition (C): By choosing a basis C_i for the linear space of symmetric matrices, we can write $P = \sum_{i=1}^m c_i C_i$, $m \coloneqq \frac{n\times(n+1)}{2}$, and express (C) it terms of the LMIs:

$$\exists_{\{c_i, i=1,\ldots,m\}} : \sum_{i=1}^m c_i C_i > 0, \qquad \sum_{i=1}^m c_i (L(C_i) - C_i) < 0. \tag{7}$$

Let ν denote the operator that transforms a matrix into a column vector $\nu(A) = \nu([a_1 \ \ldots \ a_n]) = [a_1^\intercal \ \ldots \ a_n^\intercal]^\intercal$ and recalling that

$$\nu(ABC) = (C^\intercal \otimes A)\nu(B) \tag{8}$$

(cf. [14]), we conclude that

$$\nu(L(C_i)) = M\nu(C_i). \tag{9}$$

This shows that the integrals that appear in (7) can be obtained by computing the matrix M, which can be done efficiently, as noted in the discussion regarding the condition (B).

A generalized version of [10, Th.3.9], can be used to prove that (C) is the stability condition for the discrete-time stochastic process $v_k := x(t_k)$ obtained by sampling (1) at jump times, which is described by $v_{k+1} = Je^{Ah_k}v_k$. However, Theorems 3 and 4 show that the stability of (1) is not equivalent to the stability of the process v_k.

Conditions (i), (ii) and (iii): These conditions pertain to the inter-jump behavior of (1). To gain intuition on why (A) could not suffice for stability, and on why there may exist systems for which the stability notions differ, suppose that $A=a\in\mathbb{R}_{>0}, J=0$ in (1) (in which case (A), (T1) and (T2) hold trivially). For this system, $x(t) = 0$ if one or more transitions have occurred up to time t, and $x(t) = e^{at}x_0$ if no jump has occurred up to time t. This latter event is equivalent to the first interval between transitions being greater than t, and has probability Prob$[h_0>t]=r(t)$. Thus, $\mathbb{E}[x(t)^\intercal x(t)]=e^{2at}x_0 r(t)$ and (1) is:

(i) MSS but not SS nor MES if $r(t) = e^{-2at}\frac{1}{1+t}$,

(ii) MSS and SS but not MES if $r(t) = e^{-2at}\frac{1}{1+t^2}$.

Conditions (T1) and (T2): Note that (T1) holds if μ has bounded support or if A is Hurwitz, or, more generally, if for some $\lambda > \lambda_{\Re}(A)$, we have that $\int_0^\infty e^{2\lambda s}\mu(ds) < \infty$. Moreover, whenever (T1) holds, the following proposition provides a simple condition to verify if (T2) holds. The proof can be found in the appendix.

Proposition 5. Assuming that (T1) holds, the condition (T2) also holds provided that

$$\sigma(\hat{\Theta}_d(0)) = \sigma\Big(\sum_{i=1}^{\infty} w_i (Je^{Ab_i})^\intercal \otimes (Je^{Ab_i})^\intercal\Big) < 1.$$

IV. Volterra Integral Equations with Positive Kernel

In this section we consider a Volterra equation taking the form

$$y(t) = \int_0^t \Psi(ds)y(t-s) + g(t), \quad t \geq 0 \tag{10}$$

where $g : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^m$ and the *kernel* Ψ is a finite real measure on $\mathbb{R}_{\geq 0}$ (i.e., $\int_0^\infty \Psi(ds)$ converges absolutely). We assume that Ψ has no singular part with respect to the Lebesgue measure other than a possible set of point masses $\{c_i, i \geq 1\}$, which means that we can decompose the integral of a measurable function $a(t)$ with respect to Ψ into an absolutely continuous part and a discrete part, i.e.,

$$\int_0^t \Psi(ds)a(s) = \int_0^t K_c(s)a(s)ds + \sum_{i:c_i\in[0,t]} K_{di}a(c_i), \tag{11}$$

where $K_c(s)$ is a real matrix valued measurable function that only needs to be defined a.e., i.e., up to a set of zero Lebesgue measure, and $\{K_{di}, i \geq 1\}$ are real matrices (cf. [11, p.79]). We assume there are no point masses at zero, i.e., $c_i > 0$.

We will see shortly in Subsection IV-A that

$$\mathbb{E}[x(t)^\intercal x(t)], \tag{12}$$

can be obtained as the solution to a Volterra equation of the form (10). Moreover, we will see in Subsection IV-B that this Volterra equation has a positive kernel. We provide general results for Volterra equations with positive kernel in Subsection IV-C, prove the results of Section III in Subsection IV-D and discuss more applications of these results in Subsection IV-E.

A. Volterra Equations for Impulsive Renewal Systems

Using (6) and (8), we can write (12) as

$$\mathbb{E}[x(t)^\intercal I x(t)] = \mathbb{E}[(x(t)^\intercal \otimes x(t)^\intercal)\nu(I)] = (x_0^\intercal \otimes x_0^\intercal)w(t)$$

where

$$w(t) := \mathbb{E}[(T(t)^\intercal \otimes T(t)^\intercal)\nu(I)], \tag{13}$$

and I is the identity matrix. Note, however, that all results in this section would hold if we were to replace the identity matrix I by a general positive definite matrix, which would allow us to consider a more general quadratic form of the state of (1). The next result shows that $w(t)$ satisfies a Volterra equation.

Proposition 6. The function $w(t)$ satisfies

$$w(t) = \int_0^t \Theta(ds)w(t-s) + h(t),\ t \geq 0 \tag{14}$$

where $h(t) := e^{A^\intercal t} \otimes e^{A^\intercal t}\nu(I)r(t)$ and

$$\Theta([0,t]) := \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal \mu(ds). \tag{15}$$

□

Proof: Condition (13) on the time of the first jump t_1,

$$w(t) = \int_0^\infty \mathbb{E}[T(t)^\intercal \otimes T(t)^\intercal \nu(I)|t_1 = s]\mu(ds). \tag{16}$$

The integrand has a different expression for $s > t$ and $s \leq t$,

$$\mathbb{E}[T(t)^\intercal \otimes T(t)^\intercal \nu(I)|t_1 = s] = \tag{17}$$
$$\begin{cases} (e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I), \text{ if } s > t \\ \mathbb{E}[(T_1(t-s)Je^{As})^\intercal \otimes (T_1(t-s)Je^{As})^\intercal \nu(I)], \text{ if } s \leq t \end{cases}$$

where $T_1(t-s)$ is the transition matrix of (1) from $s = t_1$ to t, which depends on $\{h_k : k \geq 1\}$. Due to the i.i.d. assumption on the intervals between transitions $\mathbb{E}[T_1(t)^\intercal \otimes T_1(t)^\intercal \nu(I)] = w(t)$. Thus, partitioning (16) using (17) we obtain (14):

$$\begin{aligned} w(t) =& \int_0^t \mathbb{E}[(T_1(t-s)Je^{As})^\intercal \otimes (T_1(t-s)Je^{As})^\intercal \nu(I)]\mu(ds) \\ &+ \int_t^\infty \mathbb{E}[(e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I)]\mu(ds) \\ =& \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal w(t-s)\mu(ds) + (e^{A^\intercal t} \otimes e^{A^\intercal t})\nu(I)r(t), \end{aligned}$$

where we used the fact that we can write the first term on the right hand side of (14) as

$$\int_0^t \Theta(ds)w(t-s) = \int_0^t (Je^{As})^\intercal \otimes (Je^{As})^\intercal w(t-s)\mu(ds),$$

which can be concluded by extending [15, Th.2.9] to real matrix-valued measures. ■

The kernel Θ has a special property which we term of being positive with respect to a cone, as defined below.

B. Cones and positive linear maps

Following [16, p. 6], a *cone* in $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex set such that if $x, y \in \mathcal{K}$ then $\alpha_1 x + \alpha_2 y \in \mathcal{K}$, for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and such that the set $-\mathcal{K} := \{-x : x \in \mathcal{K}\}$ intersects $\mathcal{K}$ only at the zero vector. A cone is said to be *solid* if the cone contains a ball of $\mathbb{R}^m$, which in finite-dimensional spaces is equivalent to being *reproducing*, i.e., any element $v \in \mathbb{R}^m$ can be written as $v = v_1 - v_2$ where $v_1, v_2 \in \mathcal{K}$ (cf. [16, p. 10]). A linear operator $A : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to be *positive* with respect to (w.r.t.) a cone $\mathcal{K}$ if it maps elements in $\mathcal{K}$ into elements in $\mathcal{K}$. For a complex vector $u = u_R + iu_I \in \mathbb{C}^m$, $u_R, u_I \in \mathbb{R}^m$, define $Au = Au_R + iAu_I$. Let $\sigma(A)$ be the spectral radius of A, i.e.,

$$\sigma(A) := \max\{|\lambda| : Au = \lambda u, \text{ for some } u \in \mathbb{C}^m\}. \tag{18}$$

If $A : \mathbb{R}^m \mapsto \mathbb{R}^m$ is positive w.r.t. the solid cone $\mathcal{K}$ then

$$\exists_{x \in \mathcal{K}} : Ax = \sigma(A)x \tag{19}$$

(cf. [16, Ch.8,9]). The following are examples of cones which will be useful in the sequel.

Examples 7.

1. The positive orthant $\mathbb{R}^m_{\geq 0}$ consisting of all vectors with non-negative components is a solid cone in $\mathbb{R}^m$;
2. The set $\mathcal{S}^k_{\geq 0}$ of $k \times k$ real positive semi-definite matrices is a solid (and therefore reproducing) cone when viewed as a subset of the linear space of symmetric $k \times k$ matrices with dimension $m := k(k+1)/2$, denoted by $\mathcal{S}^k(\mathbb{R})$. Note however, that it is not a solid cone if we view it as a subset of the linear space of $k \times k$ matrices with dimension $m := k^2$. To verify that this is the case, note that any symmetric matrix can be written and the difference between two positive semidefinite matrices (which confirms the reproducing property), but a non-symmetric matrix cannot be decomposed in this way.

□

We can extend the above definition of positivity to the kernel of Volterra equations by defining Ψ in (11) to be *positive w.r.t. the cone* $\mathcal{K}$ if $K_c(s)$ is positive w.r.t. to $\mathcal{K}$ for s a.e. in $[0, \infty)$ and K_{di} is positive for every $i \geq 1$. Since the kernel Ψ is not affected by the values of $K_c(s)$ on sets of measure zero, we can arbitrate hereafter that $K_c(s)$ is positive w.r.t. K for every value of $s \in [0, \infty)$.

We introduce the Laplace transform of Ψ,

$$\hat{\Psi}(z) := \int_0^\infty e^{-zs} \Psi(ds), \tag{20}$$

which can be partitioned into $\hat{\Psi}(z) = \hat{\Psi}_d(z) + \hat{\Psi}_c(z)$ where

$$\hat{\Psi}_c(z) := \int_0^\infty K_c(s) e^{-sz} ds, \;\; \hat{\Psi}_d(z) := \sum_{i \geq 1} K_{di} e^{-c_i z}. \tag{21}$$

We state next that for a real a, $\hat{\Psi}(a)$ is a positive operator. The proof can be found in the appendix.

Proposition 8. If Ψ is positive w.r.t. a cone $\mathcal{K}$ then, for a given real a such that $\hat{\Psi}(a)$ converges absolutely, $\hat{\Psi}(a)$ is a positive operator w.r.t. $\mathcal{K}$. □

C. Main results for Volterra Integral Equations

We establish in this subsection two theorems that characterize the stability of the Volterra equation (10) when Ψ is positive w.r.t. a solid cone.

We recall that the stability condition for a Volterra equation (see [11, p.195] for the definition) taking the form (10) is $\det(I-\hat{\Psi}(z)) \neq 0,\ \Re[z] \geq 0$. The first of the two theorems in the present section provides computationally efficient alternatives to this stability condition when the kernel of (10) is positive. We denote by $\text{int}(B)$ the interior of a set $B \in \mathbb{R}^m$.

Theorem 9. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$. Then, the following conditions are equivalent

$$\begin{aligned}
&\text{(A)}\ \ \det(I-\hat{\Psi}(z)) \neq 0,\ \ \Re[z] \geq 0,\\
&\text{(B)}\ \ \sigma(N) < 1,\ \text{where}\ N := \int_0^\infty \Psi(ds),\\
&\text{(C)}\ \ \exists_{x \in \text{int}(\mathcal{K})} : x - Nx \in \text{int}(\mathcal{K}) \quad \square
\end{aligned}$$

Note that (A) can be tested by using the Nyquist criteria, and (B) by computing the absolute value of the eigenvalues of N. The condition (C) is a cone optimization problem for which numerical efficient algorithms can be found in [17].

As a preliminary to prove Theorem 9 we state two lemmas both proved in the appendix. The first lemma establishes a monotone property for the spectral radius of the Laplace transform of Ψ. We recall that if $\hat{\Psi}(a)$ converges absolutely for some real a then $\hat{\Psi}(z)$ converges absolutely in the set $\{z \in \mathbb{C} : \Re[z] \geq a\}$(cf. [11, Ch.3,Th.8.2]).

Lemma 10. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$, and that $\hat{\Psi}(a)$ converges absolutely for some $a \in \mathbb{R}$. Then for any $z : \Re[z] \geq a$, the following holds $\sigma(\hat{\Psi}(z)) \leq \sigma(\hat{\Psi}(a))$. $\square$

The second Lemma characterizes the geometric placement of the zeros of $\det(I - \hat{\Psi}(z))$.

Lemma 11. Suppose that Ψ is positive w.r.t. a solid cone $\mathcal{K}$, and that $\hat{\Psi}(a)$ converges absolutely for some $a \in \mathbb{R}$. Then, if $\sigma(\hat{\Psi}(a)) \geq 1$, there exists $a_1 \geq a$ with the following properties:

(i) a_1 is the unique real number such that $\sigma(\hat{\Psi}(a_1)) = 1$,

(ii) $\det(I - \hat{\Psi}(a_1)) = 0$,

(iii) if $z : \det(I - \hat{\Psi}(z)) = 0$ then $\Re[z] \leq a_1$. $\square$

Proof: (of Theorem 9)

(B)⇒(A) Note that $N = \hat{\Psi}(0)$. From Lemma 10, $\sigma(\hat{\Psi}(z)) \leq \sigma(\hat{\Psi}(0)) = \sigma(N)$ for every $z : \Re[z] \geq 0$. This implies that if (B) holds, all the eigenvalues of $\hat{\Psi}(z)$ have a modulus strictly less than one, and therefore $\det(I - \hat{\Psi}(z))$ cannot be zero in $\Re[z] \geq 0$, which implies (A).

(A)⇒(B) If $\sigma(N) = \sigma(\hat{\Psi}(0)) \geq 1$, from Lemma 11 we have that there exists some $a_1 \geq 0$ such that $\det(I - \hat{\Psi}(a_1)) = 0$, and therefore condition (A) does not hold at $z = a_1$.

(B)⇒(C) If $\sigma(N) < 1$ the system of equations

$$Nx - x = -v$$

has a unique solution given by $x = \sum_{k=0}^{+\infty} N^k(v)$, which satisfies $x \in \mathcal{K}$ if $v \in \mathcal{K}$ due to the fact that N is a positive operator (since $N = \hat{\Psi}(0)$ and due to Proposition 8). If $v \in \text{int}(\mathcal{K})$ then since $x = v + w$, where $w = \sum_{k=1}^{+\infty} N^k(y) \in \mathcal{K}$ and $v \in \text{int}(\mathcal{K})$, we have that $x \in \text{int}(\mathcal{K})$. In fact, since $v \in \text{int}(\mathcal{K})$ there exists an ϵ such that $B_\epsilon := \{z : \|z - v\| < \epsilon\} \subseteq \mathcal{K}$. Thus the set $\{w + z, z \in B_\epsilon\}$ is a ball of radius ϵ centered at x which belongs to $\mathcal{K}$ since w and z belong to $\mathcal{K}$, and thus $x \in \text{int}(\mathcal{K})$.

(C)⇒(B) If (C) holds, then $x - Nx = v$ for some $v \in \text{int}(\mathcal{K})$, from which we conclude that, for each $k \geq 1$, $x - \sum_{j=0}^{k-1} N^j v = N^k x \in \mathcal{K}$. This implies that $\| \sum_{j=0}^{k} N^j v\| \leq c\|x\|$ since in finite-dimensional every cone is normal (cf [16, Ch.4]). Letting $k \to \infty$ we conclude that $\sum_{j=0}^{\infty} N^j v$ converges which implies that it converges absolutely, a general fact in Banach spaces (as it is the case since $\mathcal{K} \in \mathbb{R}^m$), and this implies that $N^k v \to 0$ as $k \to \infty$. Given any $w \in \mathcal{K}$, there exists $\alpha > 0$ such that $v - \alpha w \in \mathcal{K}$ because $v \in \text{int}(\mathcal{K})$. Then $N^k v - \alpha N^k w = N^k(v - \alpha w) \in \mathcal{K}$, and hence $N^k w \to 0$ as $k \to \infty$ because $N^k v \to 0$ as $k \to \infty$ and ($\|\alpha N^k w\| \leq c\|N^k v\|$). Finally, since $\mathcal{K}$ is reproducing, for every $u \in \mathbb{R}^m$, there exists $u_1, u_2 \in \mathcal{K}$ such that $u = u_1 - u_2$, and therefore $N^k u = N^k u_1 - N^k u_2 \to 0$ as $k \to \infty$, which implies that $\sigma(N) < 1$. ■

The *Lyapunov exponent* of the Volterra equation (10) is defined as follows.

Definition 12. Suppose that the solution to (10) satisfies $\|y(t)\| \neq 0, \forall t_{\geq 0}$. Then the second order Lyapunov exponent for (10) is defined as

$$\lambda_V := \lim_{t \to \infty} \frac{1}{t} \log \|y(t)\| \tag{22}$$

when such limit exists. Moreover, if $\exists_{b>0} \colon \|y(t)\| = 0, \forall t > b$ then $\lambda_V := -\infty$. □

The next Theorem provides a method to determine the Lyapunov exponent λ_V for (10). Recall that a singularity of a complex function is said to be removable if the function is bounded in a neighborhood of z_i.

Theorem 13. Suppose that (V1), (V2) and (V3) hold, $\hat{\Psi}(a)$ converges absolutely for some real a and $\sigma(\hat{\Psi}(a)) \geq 1$. Then there is a unique

$$\lambda > a : \sigma(\hat{\Psi}(\lambda)) = 1 \tag{23}$$

which is a singularity of $[I - \hat{\Psi}(z)]^{-1}\hat{g}(z)$, and, if it is not removable, the Lyapunov exponent λ_V is given by $\lambda_V = \lambda$. Moreover, if there does not exists a real a such that $\sigma(\hat{\Psi}(a)) \geq 1$, then $\lambda_V = -\infty$. □

Note that $\sigma(\hat{\Psi}(a))$ is a non-increasing function of a (cf. Lemma 10), and therefore (23) can be easily computed by performing a binary search.

To prove the Theorem 13 we need to derive two results. The first states how a special perturbation of $g(t)$ and Ψ affects the solution of (10). We introduce the following Laplace transforms

$$\hat{y}(z) := \int_0^\infty w(s)e^{-zs}ds, \hat{g}(z) := \int_0^\infty g(s)e^{-zs}ds. \tag{24}$$

The following proposition can be obtained by direct substitution.

Proposition 14. Let $g_\delta(t) := g(t)e^{\delta t}$ and Ψ_δ be a measure such that,

$$\int_0^t \Psi_\delta(ds)a(s) = \int_0^t K_c(s)e^{\delta s}a(s)ds + \sum_{i:c_i\in[0,t]} K_{di}e^{\delta c_i}a(c_i), \tag{25}$$

for a measurable function $a(t)$, where K_c and K_{di} are specified by (11). Then the solution to

$$y_\delta(t) = \int_0^t \Psi_\delta(ds)y_\delta(t-s) + g_\delta(t), t \geq 0 \tag{26}$$

satisfies $y_\delta(t) = y(t)e^{\delta t}$. Moreover

$$\hat{y}_\delta(z) = \hat{y}(z-\delta),\ \hat{g}_\delta(z) = \hat{g}(z-\delta),\ \hat{\Psi}_\delta(z) = \hat{\Psi}(z-\delta), \tag{27}$$

where $\hat{y}_\delta(z)$, $\hat{g}_\delta(z)$, and $\hat{\Psi}_\delta(z)$ are the Laplace transforms of w_δ and g_δ and Ψ_δ defined as in (20) and (24). □

The second is an instability result that takes into account the critical case, i.e, it allows the characteristic equation ($\det(I - \hat{\Psi}(z)) = 0$) to have zeros on the imaginary axis. It is important

to emphasize that this result avoids the hard-to-test assumptions typically used to handle the critical case (e.g. [11, Ch.7,Th.3.7]). Consider the following assumptions.

(V1) $\hat{\Psi}(-\epsilon)$ converges absolutely for some $\epsilon > 0$.

(V2) $\inf_{z\in\mathcal{C}(R,\epsilon)}\{|\det(I-\hat{\Psi}_d(z))|\}>0$ for some $\epsilon > 0$, and $R > 0$, where $\mathcal{C}(R,\epsilon) := \{z : |z| > R, \Re[z] > -\epsilon\}$.

(V3) g is bounded and $g(t) = 0, t \in [b,\infty)$ for some $b > 0$.

The same arguments utilized in the proof of Proposition 5 can be used to prove that $\sigma(\hat{\Psi}_d(0)) < 1$ implies (V2).

Lemma 15. Consider a general Volterra equation taking the form (10) and suppose that (V1), (V2), (V3) hold. Then

(i) there exists at most a finite number of $z_i \in \mathbb{C}$ such that $\Re[z_i] \geq 0$ and $\det(I - \hat{\Psi}(z_i)) = 0$.

(ii) The solution to (10) is given by

$$y(t) = w_s(t) + \sum_{i=1}^{n_z}\sum_{j=0}^{m_i-1} s_{i,j}t^j e^{z_i t}, \quad t \geq 0, \tag{28}$$

and its Laplace transform is given by $\hat{y}(z) = [I - \hat{\Psi}(z)]^{-1}\hat{g}(z)$, where $w_s(t)$ is a bounded function that converges to zero and $s_{i,j}$ are such that in a neighborhood of z_i,

$$\hat{y}(z) = u_i(z) + \sum_{j=0}^{m_i-1} s_{i,j}\frac{j!}{(z-z_i)^{j+1}}, \tag{29}$$

where $u_i(z)$ are analytic vector functions. □

The proof of Lemma 15 is given in the appendix. Note that the second term on the right hand side of (29) is the principal part of $\hat{y}(z)$ at z_i and $s_{i,j}$ can be uniquely determined by the characterization (29) (see [15, Th.10.21]).

Proof: (of Theorem 13)

If there exists a real number a such that $\sigma(\hat{\Psi}(a)) \geq 1$, then there exists a unique $\lambda \in \mathbb{R}$ such that $\sigma(\hat{\Psi}(\lambda)) = 1$, which satisfies $\det(I-\hat{\Psi}(\lambda)) = 0$ (cf. Lemma 11), and therefore λ is a singularity of $[I-\hat{\Psi}(z)]^{-1}\hat{g}(z)$. Consider a perturbation to the Volterra equation (10) as in Proposition (14) with $\delta = -\lambda$. Since λ is the zero of $\det(I - \hat{\Psi}(z))$ with largest real part (cf. Lemma 10), we have that $z_{1\delta} = 0$ is the zero of $\det(I-\hat{\Psi}_\delta(z)) = \det(I-\hat{\Psi}(z-\delta))$ with largest real part, where we used (27). Also, due to (27), the principal parts of $\hat{y}(z)$ about z_i and of $\hat{y}_\delta(z)$ about $z_i + \delta$

have the same coefficients $s_{i,j}$. Thus, from Lemma 15, the solution to the Volterra equation (26) becomes $y_\delta(t) = v_\delta(t) + u_\delta(t) + w_\delta(t)$ where $v_\delta(t) = \sum_{j=0}^{m_i-1} s_{1,j}t^j$ corresponds to the zero of $\det(I - \hat{\Phi}(z))$ labeled $z_{1\delta} = 0$, u_δ is a sum of exponent and polynomial functions such that $u_\delta(t)e^{-ct} \to 0,$ as $t \to \infty$ for every $c > 0$ and $w_\delta(t)$ is a bounded function that converges to zero. Note that u_δ may not tend to zero since there may exist zeros of $\det(I - \hat{\Phi}_\delta(z))$ on the imaginary axis other than $z_{1\delta}$. Since the singularity $z_1 = \lambda$ is not removable, $\exists_j : s_{1,j} \neq 0$. Therefore $y(t) = y_\delta(t)e^{-\delta t}$ is such that

$$\lambda_V = \lim_{t\to\infty} \frac{\log(\|y(t)\|)}{t} = \lim_{t\to\infty} \frac{\log(\|y_\delta(t)\|)}{t} - \delta = 0 - \delta = \lambda.$$

If there does not exists a real number a such that $\sigma(\hat{\Psi}(a)) \geq 1$, then using Lemma 10, we have that $\sigma(\hat{\Psi}(z)) < 1$ for every complex z and therefore $\det(I - \hat{\Phi}(z)) \neq 0$ for all z. Using the arguments above we can choose δ arbitrarily large, and prove that $y(t) = y_\delta(t)e^{-\delta t}$ where $y_\delta(t)$ is a bounded function that converges to zero. This implies that $\Lambda_V = -\infty$. Note that, this latter case encompasses the case where, e.g., $\Psi([0, t]) = 0, \forall_{t\geq 0}$ and $y(t) = g(t)$ is such that $y(t) = 0, t > b$ for some $b > 0$. ■

D. Proofs of Theorem 3 and Theorem 4

As a preliminary, we rewrite (14) in two forms. Applying ν^{-1} to both sides of (14), and using (8), we obtain

$$W(t) = K(W)(t) + H(t), \quad t \geq 0 \tag{30}$$

where $W(t) := \nu^{-1}(w(t))$, $H(t) := e^{A^\intercal t} X e^{At} r(t)$ and

$$K(W)(t) := \int_0^t (Je^{As})^\intercal W(t-s) Je^{As} \mu(ds).$$

Let $K^j(W)(t)$ denote the composition operator obtained by applying j times K. The unique solution to (30) is given by

$$W(t) = \sum_{j=1}^{\infty} K^j(H)(t) + H(t), \quad t \geq 0, \tag{31}$$

(cf. [11, Th.1.7,Ch.4]). Note that $H(t)$ and $K(H)(t)$ are symmetric and therefore $W(t)$ is symmetric. Let $\{C_i,\, 1 \leq i \leq m\}$, with $m := \frac{n(n+1)}{2}$, be an orthogonal basis for the space of symmetric matrices $\mathcal{S}^n(\mathbb{R})$ and let $\Pi(x) : \mathbb{R}^m \mapsto \mathcal{S}^n(\mathbb{R})$ be the invertible map defined by

$$\Pi(x) = \sum_{i=1}^{m} x_i C_i.$$

Then (30) (or equivalently (14)) can be written in terms of $u(t) := \Pi^{-1}(W(t))$ as

$$u(t) = \int_0^t \Phi(ds)u(t-s) + e(t) \tag{32}$$

where $e(t) = \Pi^{-1}(E(t))$ and $\Phi(ds)$ is such that

$$\int_0^t \Phi(ds)a(s) = \int_0^t F_c(s)a(s)ds + \sum_{i:c_i\in[0,t]} F_{di}a(c_i)$$

where $a(t)$ is a measurable function and

$$\begin{aligned} F_c(s) &= \Pi^{-1}\nu^{-1}(Je^{As})^\intercal \otimes (Je^{As})^\intercal f(s)\nu\Pi, \\ F_{di} &= \Pi^{-1}\nu^{-1}(Je^{Ac_i})^\intercal \otimes (Je^{Ac_i})^\intercal w_i\nu\Pi. \end{aligned} \tag{33}$$

The kernel Φ is positive w.r.t. the cone in $\mathbb{R}^m$ corresponding to $S^n_{\geq 0}(\mathbb{R})$, i.e., $\mathcal{K}^S := \{v \in \mathbb{R}^m : \sum_{i=1}^m v_i C_i > 0\}$. In fact, if $x \in \mathcal{K}^S$, i.e., $X = \Pi(x) \geq 0$ then using (33) and (9)

$$F_c(s)x = \Pi^{-1}(Y) \in \mathcal{K}^S, \text{ where } Y = (Je^{As})^\intercal X Je^{As} f(s).$$

Similarly one can prove that all the F_{di} are positive operators.

Let $\hat{\Phi}(z) := \int_0^\infty e^{-zs}\Phi(ds)$. Since Φ is a positive kernel w.r.t. the solid cone $\mathcal{K}^S$ (cf. Example 7.2)), from Theorem 9 we obtain that the following are equivalent:

(A') $\det(I - \hat{\Phi}(z)) \neq 0, \ \ \Re[z] \geq 0,$

(B') $\sigma(\hat{\Phi}(0)) < 1,$

(C') $\exists_{x\in \text{int}(\mathcal{K}^S)} : x - \hat{\Phi}(0)x \in \text{int}(\mathcal{K}^S).$

We shall also need the following facts: Using (33), we have

$$\hat{\Phi}(z) = \Pi^{-1}\nu^{-1}\hat{\Theta}(z)\nu\Pi. \tag{34}$$

Since Φ is positive w.r.t. a solid cone we have that $\hat{\Phi}(a)$ for real a is a positive operator (cf. Proposition 8), and therefore from (19),

$$\exists_{x\in\mathcal{K}^S} : \hat{\Phi}(a)x = \sigma(\hat{\Phi}(a))x \tag{35}$$

which, using (34), can be written as

$$\exists_{X\geq 0}\hat{\Theta}(a)\nu(X) = \sigma(\hat{\Phi}(a))\nu(X). \tag{36}$$

Proof: (of Theorem 3)

Since, as discussed above, (A')-(C') are equivalent, to prove that (A)-(C) are equivalent we can show that

$$\text{(C)} \Leftrightarrow \text{(C')} \text{ , (C) } \Rightarrow \text{(B), (B)} \Rightarrow \text{ (A), (A)} \Rightarrow \text{(A').}$$

(C)$\Leftrightarrow$(C') Using (34) with $z = 0$ we write (C') as

$$\exists_{P>0} : \nu(P) - M\nu(P) \in \{\nu(Q) : Q > 0\}. \tag{37}$$

Applying ν^{-1} to (37), and using (9), we conclude that (37) is equivalent to (C).

(C) $\Rightarrow$(B) Let $\mathcal{S}^n_{\geq 0}(\mathbb{C})$ be the set of $n \times n$ complex positive semi-definite matrices and consider the system

$$u_{k+1} = M^{\intercal} u_k, \quad U_{k+1} = L'(U_k), \quad \nu(U_k) = u_k, \quad U_0 \in \mathcal{S}^n_{\geq 0}(\mathbb{C}), \tag{38}$$

where $L'(U_k) := \nu^{-1} \circ M^{\intercal} \circ \nu(U_k)$ is given by

$$L'(U_k) = \int_0^T Je^{As} U_k (Je^{As})^{\intercal} F(ds),$$

and for any $Y, Z \in S^n_+(\mathbb{C})$ satisfies

$$\text{tr}(L(Z)^* Y) = \text{tr}(Z^* L'(Y)). \tag{39}$$

Note that $U_k \geq 0$ for a given k implies that $U_{k+1} = L'(U_k) \geq 0$, and therefore, by induction, we conclude that $U_k \geq 0$ for all k. We show that (C) implies that this system is stable by considering a Lyapunov function $V(u_k) = \text{tr}(P\nu^{-1}(u_k))$ for (38), where P satisfies (C), that is $P > 0$ and $L(P) - P < 0$. In fact, this function V is radially unbounded and positive definite for $X_k \geq 0$, and verifies $V(0) = 0$. Using (C) and (39) we have that for every $U_k \in \mathcal{S}^n_{\geq 0}(\mathbb{C}) - \{0\}$,

$$\begin{aligned} V(u_{k+1}) - V(u_k) &= \text{tr}(P\nu^{-1}(M^{\intercal}\nu(U_k))) - \text{tr}(PU_k) \\ &= \text{tr}(PL'(U_k)) - \text{tr}(PU_k) = \text{tr}((L(P) - P)U_k) = -\text{tr}(ZU_k) < 0, \end{aligned}$$

where $Z := -(L(P) - P) > 0$. Therefore (38) is stable for any $U_0 \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$, i.e., $(M^{\intercal})^k U_0 \to 0$. Since any complex matrix Z can be written as $Z = Z_1 - Z_2 + i(Z_3 - Z_4)$, where $Z_i \in \mathcal{S}^n_{\geq 0}(\mathbb{C}), 1 \leq i \leq 4$ (cf. [18, Rem. 2]) this implies that $(M^{\intercal})^k Z \to 0$, which implies that $\sigma(M) = \sigma(M^{\intercal}) < 1$.

(B)⇒(A) For $z : \Re[z] \geq 0$ and $X, Y \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$, we have

$$\begin{aligned}
&|\langle Y, \nu^{-1}(\hat{\Theta}^k(z)\nu(X))\rangle| \\
&= |\int_0^T \dots \int_0^T \mathrm{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X J e^{Ah_k} \dots \\
&\qquad \dots Je^{Ah_k} e^{-zh_1} \dots e^{-zh_k} \mu(dh_1) \dots \mu(dh_k)| \\
&\leq \int_0^T \dots \int_0^T |\mathrm{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X (Je^{Ah_1}) \dots \\
&\qquad \dots Je^{Ah_k}| \mu(dh_1) \dots \mu(dh_k) \\
&= \langle Y, \nu^{-1}(M^k \nu(X))\rangle
\end{aligned} \tag{40}$$

where the last equality holds because $\mathrm{tr}(Y^*(Je^{Ah_k})^\intercal \dots (Je^{Ah_1})^\intercal X J e^{Ah_1} \dots Je^{Ah_k})$ is a non-negative function since it can be written as the inner product of two (complex) positive semi-definite matrices. If (B) holds then $\sigma(M) < 1$, $M^k \nu(X)$ converges to zero as k tends to infinity and by the inequality (40) this implies that $|\langle Y, \nu^{-1}(\hat{\Theta}^k(z)\nu(X))\rangle|$ converges to zero for any matrices $Y, X \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$. Given $Z^1, Z^2 \in \mathbb{C}^{n\times n}$ we can write $Z^j = Z^j_1 - Z^j_2 + i(Z^j_3 - Z^j_4)$, where for $1 \leq l \leq 4$ and $1 \leq j \leq 2$, $Z^j_l \in \mathcal{S}^n_{\geq 0}(\mathbb{C})$ (cf. [18]). Thus,

$$|\langle Z^1, \nu^{-1}(\hat{\Theta}^k(z)\nu(Z^2))\rangle| = \sum_{1\leq l\leq 4, 1\leq j\leq 2} c_{lj} |\langle Z^j_l, \nu^{-1}(\hat{\Theta}^k(z)\nu(Z^j_l))\rangle|$$

also converges to zero as k tends to infinity, where c_{lj} are constants. This implies $\sigma(\hat{\Theta}(z)) < 1$ in $\Re[z] \geq 0$ and thus $\det(I - \hat{\Theta}(z))$ cannot be zero in $\Re[z] \geq 0$, which is (A).

(A) ⇒ (A') If (A') does not hold then there must exist a $z_1 : \Re[z_1] \geq 0$ such that $\det(I - \hat{\Phi}(z_1)) = 0$ and this implies that

$$\exists_{x\in\mathbb{C}^m} : \hat{\Phi}(z)x = x. \tag{41}$$

Using (34) we conclude that (41) can be rewritten as $\exists_{X=Y+iZ:Y,Z\in\mathcal{S}^n(\mathbb{R})} : \hat{\Theta}(z_1)\nu(X) = \nu(X)$ and this implies that $\det(I - \hat{\Theta}(z_1)) = 0$, which means that (A) does not hold.

Sufficiency of (A) and (3), (4), (5) *for MSS, SS and MES*

We start by noticing that (3) is equivalent to $h(t)$ (or $H(t)$) converges to zero, and (4) is equivalent to $\int_0^T h(t)dt$ (or $\int_0^T H(t)dt$) converging absolutely. Moreover, since $\mathbb{E}[x(t)^\intercal x(t)] = x_0 \otimes x_0 w(t) = x_0^\intercal W(t) x_0$, where $W(t)$, given by (31), is non-zero and positive semi-definite, MSS is equivalent to $w(t) \to 0$ (or $W(t) \to 0$), SS is equivalent to $\int_0^\infty w(t)dt$ (or $\int_0^\infty W(t)dt$) converges absolutely.

Sufficiency of conditions (A) and (3) for MSS and of conditions (A) and (4) for SS follow then directly from [11, Ch.4,Th.4.9], which states that for a Volterra equation taking the form (14) if $\inf_{\Re[z]\geq 0} |\det(I-\hat{\Theta}(z))| > 0$ then, $w(t)$ is bounded and converges to zero if $h(t)$ is bounded and converges to zero, $\int_0^\infty w(t)dt$ converges absolutely if $\int_0^\infty h(t)dt$ converges absolutely. It remains to prove that $\inf |\det(I-\hat{\Theta}(z))| > 0$ for z in $\Re[z] \geq 0$ is equivalent to $\det(I-\hat{\Theta}(z)) \neq 0$ in the same complex region. Necessity is trivial. To prove sufficiency note that, $\det(I-\hat{\Theta}(z)) \neq 0$ in $\Re[z] \geq 0$ is equivalent to $\sigma(\hat{\Theta}(0)) < 1$, due to the equivalence between (A) and (B) in the present theorem. If $\sigma(\hat{\Theta}(0)) < 1$, then by Lemma 10, $\sigma(\hat{\Theta}(z)) < 1$ in the closed right half plane and therefore $\inf(|\det(I-\hat{\Theta}(z))|) > 0$ in $\Re[z] \geq 0$. To prove sufficiency of the conditions (A) and (5) for MES, we use a perturbation as in Proposition 14. For $\delta < \min(\epsilon, \alpha_1)$ where ϵ is such that (T1) holds and α_1 is such that (5) holds, we obtain that $h_\delta(t) = h(t)e^{\delta t}$, and Θ_δ, satisfy the conditions (A) and (3) of the present theorem for MSS and therefore $w_\delta(t)$ converges to zero, whereas $w(t) = w_\delta(t)e^{-\delta t}$ converges to zero exponentially fast.

Necessity of (A) and (3), (4), (5) *for MSS, SS and MES*

From (31), $W(t) \geq H(t)$. Hence, $W(t) \to 0$ (MSS), $W(t)$ is integrable (SS), and $W(t)$ converges to zero exponentially fast (MES), imply that the respective condition must hold for $H(t)$ (or equivalently for $h(t)$) and therefore that (3), (4), and (5) must hold, respectively. Necessity of (A) for MSS, SS and MES, follows from Lemma 15, which requires (V1) and (V2), implied by (T1) and (T2), and also (V3), which is assumed for now to hold. If (A) does not hold, from Lemma 11 there exists a real $a > 0$ such that $\det(I-\hat{\Theta}(a)) = 0$. Thus, from Lemma 15, $w(t)$ converges to infinity exponentially fast, since the zero $z_i = a$ is not a removable singularity ($\exists_j : s_{i,j} \geq 0$, cf. Proof of Theorem 4). If (V3) does not hold, we can use a similar reasoning to prove that (A) is necessary for (MSS), (SS), (MES). Let $W_2(t)$ be the solution to (30) when $H(t)$ is replaced by $H_2(t) := H(t)$ if $t > b, H(t) := 0$ otherwise for some $b > 0$. Then, $W_2(t) \leq W(t)$ since $W(t) - W_2(t) = K(W - W_2)(t) + H(t) - H_2(t)$, whose solution is positive semi-definite (cf. (31)). Thus, since (A) is necessary for $W_2(t)$ to be MSS, SS and MES, it is also for $W(t)$. ■

Proof: (of Theorem 4) Note that the conditions (T1) and (T2) imply (V1) and (V2) and the fact that μ has finite support implies (V3). Also, recall that $\mathbb{E}[x(t)^\intercal x(t)] = (x_0^\intercal \otimes x_0^\intercal)w(t) = x_0^\intercal W(t) x_0$. Since $W(t)$ is positive semidefinite, the limit $\lambda_L(x_0) = \lim_{t\to\infty} \frac{1}{t}\log(\mathbb{E}[x(t)^\intercal x(t)])$ exists for every x_0 if and only if the limit $\lambda_V = \lim_{t\to\infty} \frac{1}{t}\log(\|w(t)\|)$ exists, and $\lambda_V =$

$\sup_{x_0} \lambda_L(x_0)$. Note also that $\mathbb{E}[x(t)^\intercal x(t)] = 0, \forall_{x_0}, t > b > 0$ if and only if $w(t) = 0, t > b > 0$ in which case both λ_L and λ_V equal $-\infty$ according to our definitions. Thus the present theorem follows directly from the Theorem 13 applied to the Volterra equation (14) provided that we can show that the real zero a of $\det(I - \hat{\Theta}(a))$ such that $\sigma(\hat{\Theta}(a)) = 1$, is not a removable singularity of $R(a)\hat{h}(a)$ where $R(a) := [I - \hat{\Theta}(a)]^{-1}$.

To this effect, we apply ν^{-1} on both sides of (36) and obtain

$$\exists_{X \geq 0} L_a(X) = \sigma(\hat{\Phi}(a))X. \tag{42}$$

where

$$L_a(X) := \int_0^\infty (Je^{As})^\intercal X (Je^{As}) e^{-as} \mu(ds).$$

Similarly to the equivalence between (A') and (A) established in the proof of Theorem 4, we can prove that if $\sigma(\hat{\Phi}(a)) < 1$ then $\sigma(\hat{\Theta}(a)) < 1$. It is also clear that $\sigma(\hat{\Phi}(a)) \leq \sigma(\hat{\Theta}(a)) = 1$. This implies that $\sigma(\hat{\Phi}(a)) = 1$. Let $\hat{H}(b) := \int_0^\infty e^{A^\intercal t} e^{At} r(t) e^{-bt} dt$ and note that there exists α such that $\alpha X \leq \hat{H}(b)$. For $b > a$, $\sigma(\hat{\Theta}(b)) < 1$ due to the characterization of a of Lemma 10 and due to the monotonicity property of Lemma 11. Thus, we can expand $R(b)$ as a von Neumann series and conclude using (8) that

$$R(b)\hat{h}(b) = \sum_{k=0}^{\infty} \hat{\Theta}(b)^k \hat{h}(b) = \sum_{k=0}^{\infty} \nu[L_b^k(\hat{H}(b))] \geq \sum_{k=0}^{\infty} \alpha \nu(L_b^k(X)), \tag{43}$$

where L_b^k denotes the composition of applying L_b, k times. Note that from (42) and from the fact that $\sigma(\hat{\Phi}(a)) = 1$, we have that $\lim_{a \to b} L_b^k(X) = L_a^k(X) = X$. This implies, taking the limit as $b \to a$ in (43) , that $\lim_{b \to a, b > a} R(b)\hat{h}(z)$ cannot be finite and therefore a is not a removable singularity.

■

E. Additional applications of the Theorems 9 and 13

We can apply the results provided in Subsection IV-C to other problems where Volterra equations arise and have the special property of having a positive kernel, e.g., in physics, engineering, biology (cf. [11, Ch.1]). As an example, consider an LTI system $v(t) = (H * e)(t)$ where $e(t) \in \mathbb{R}^m$ is the input, $v(t) \in \mathbb{R}^m$ the output, $H(t) = [h_{i,j}]_{m \times m}$ is the impulse response, and $*$ denotes convolution. All signals are defined for $t \geq 0$. The Laplace transform of H is

given by $\hat{H}(z) := \int_0^\infty H(t)e^{-zt}dt$, and the closed loop is defined by $e(t) = v(t) + r(t)$ for a reference signal $r(t)$, which leads to the following Volterra equation

$$v(t) = \int_0^t H(s)v(t-s)ds + \int_0^t H(s)r(t-s)ds. \tag{44}$$

The transfer function H is not rational, e.g., when the impulsive response $H(t)$ exhibits delays. Following [11, Ch.1], we say that the closed-loop is stable if $v(t)$ belongs to L^p (i.e., $\int_0^t |v(t)|^p dt < \infty$, if $p < \infty$ and $v(t)$ is bounded a.e. if $p = \infty$) whenever $r(t)$ belongs to L^p. Assume that $\int_0^\infty H(t)dt$ converges absolutely which, in the case where $\hat{H}(z)$ is rational, is equivalent to saying that the open-loop is stable. The stability of the closed-loop can be tested by the condition

$$\det(I - \hat{H}(z)) \neq 0, \forall_{z \in \Re[z] \geq 0}, \tag{45}$$

One can conclude this from the stability condition for the Volterra equation (44) or from [11, Th.4.9,Ch.4]. However, when $H(t)$, $\forall t$ is a positive operator w.r.t. a given cone $\mathcal{K} \in \mathbb{R}^m$ (e.g., $\mathbb{R}^m_{\geq 0}$), we can use the following very simple algebraic test for stability.

Corollary 16. If $h_{ij}(t) \geq 0$ for every $t \geq 0$ and $1 \leq i, j \leq m$, then the closed-loop described (44) is stable if and only if

$$\sigma(\hat{H}(0)) < 1. \tag{46}$$

□

Proof: Since $H(t)$ is positive w.r.t the solid cone $\mathbb{R}^m_{\geq 0}$, Theorem 3 assures that (46) and (45) are equivalent. ■

V. Emulation in Networked Control

We now consider a networked control system for which a set of nodes consisting on sensors, actuators, and a controller, are connected through a communication network possibly shared by other users. The plant and controller are described by:

$$\text{Plant:} \qquad \dot{x}_P = A_P x_P + B_P \hat{u}, \quad y = C_P x_P \tag{47}$$

$$\text{Controller:} \quad \dot{x}_C = A_C x_C + B_C \hat{y}, \quad u = C_C x_C + D_C \hat{y}. \tag{48}$$

We assume that the controller has been designed to stabilize the closed-loop, when the process and the controller are directly connected, i.e., when $\hat{u}(t) = u(t)$, $\hat{y}(t) = y(t)$, and we are

interested in analyzing the effect of the network characteristics on the stability and performance of the closed loop. This parallels the emulation approach in digital control. The components $y_i \in \mathbb{R}, 1 \leq i \leq n_y$ and $u_j \in \mathbb{R}, 1 \leq j \leq n_u$ of y and u, respectively, are associated with the n_y sensors and n_u actuators of the networked control system. The components of these vectors are sampled and transmitted through the network generally at different instants of time. The data generated by a given sensor is received by the controller shortly after it is sampled and transmitted by the sensor, and likewise a given actuator receives its data shortly after it is sent by the controller. We denote the times at which (at least) one of the nodes transmits by $\{t_k, k \in \mathbb{N}\}$. The transmitted data may arrive corrupted at the receiver and we denote such an event by a packet drop. We are interested in scenarios for which the following assumptions hold:

(i) The time intervals $\{t_{k+1} - t_k\}$ are i.i.d.;

(ii) The transmission delays are negligible when compared to the time constants of the system dynamics;

(iii) Corresponding to each transmission there is a probability p_{drop} of a packet drop. Packet drops at different transmission times are independent.

Assumption (ii) is common. We assume (iii) for simplicity, but extending the results in this paper along the lines of [19] we could consider a more general set-up where the packet drops in the channel are modeled by a Markov chain. This is more appropriate when errors are likely to occur in bursts. Assumption (i) holds, at least approximately, for scenarios in which nodes attempt to do periodic transmissions of data, but these regular transmissions are perturbed by the medium access protocol. For example, nodes using CSMA for medium access, may be forced to back-off for a random amount of time until the network becomes available. In [12], it is argued that for variants of CSMA protocols such as slotted p-persistent and pure ALOHA, the intervals between consecutive node transmissions are indeed i.i.d. random variables. This argument is especially compelling if one does not need to restrict the distribution to exponential, which was necessary to apply the results in [12], but not for the results in the present paper.

Between sampling times, $\hat{u}$ and $\hat{y}$ are held constant, i.e.,

$$\hat{u}(t) = \hat{u}(t_k), \;\; \hat{y}(t) = \hat{y}(t_k), t \in [t_k, t_{k+1}), \;\; k \in \mathbb{Z}_{\geq 0}.$$

Let

$$e(t) := (e_u(t), e_y(t)) = (\hat{u}(t) - u(t), \hat{y}(t) - y(t)),$$

and $\omega_k := 1$ if a packet drop occurred at transmission time t_k and $\omega_k := 2$ otherwise. Denote by $\text{diag}([A_1 \dots A_n])$ a block diagonal matrix with blocks A_i and let

$$\Omega_k^{\omega_k} := \text{diag}([\pi_{1k}^{\omega_k} \dots \pi_{n_u k}^{\omega_k}]), \;\; \Gamma_k^{\omega_k} := \text{diag}([\nu_{1k}^{\omega_k} \dots \nu_{n_y k}^{\omega_k}])$$

where $\pi_{jk}^{\omega_k} := 0$ if u_j is transmitted at t_k and the packet is not dropped, and $\pi_{jk}^{\omega_k} := 1$ otherwise; and likewise $\nu_{ik}^{\omega_k} := 0$ if y_i is transmitted at t_k and the packet is not dropped and $\nu_{ik}^{\omega_k} := 1$ otherwise. The fact that in general only a subset of nodes transmits at a given sampling time t_k is captured by

$$e(t_k) = \Lambda_k^{\omega_k} e(t_k^-), \;\; \Lambda_k^{\omega_k} := \text{diag}([\Omega_k^{\omega_k} \, \Gamma_k^{\omega_k}]). \tag{49}$$

Defining $x := (x_A, e)$, where $x_A := (x_P, x_c)$, the following impulsive renewal system describes (47), (48), (49),

$$\begin{aligned} \begin{bmatrix} \dot{x}_A(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} I \\ A_e \end{bmatrix} \begin{bmatrix} A_{xx} & A_{xe} \end{bmatrix} \begin{bmatrix} x_A(t) \\ e(t) \end{bmatrix} \\ \begin{bmatrix} x_A(t_k) \\ e(t_k) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \Lambda_k^{\omega_k} \end{bmatrix} \begin{bmatrix} x_A(t_k^-) \\ e(t_k^-) \end{bmatrix}, \end{aligned} \tag{50}$$

where

$$A_{xx} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix},$$

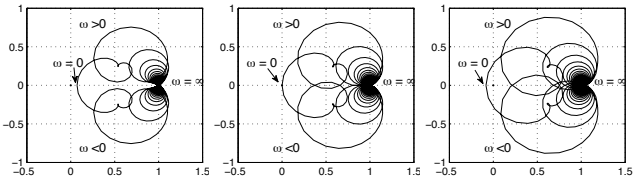
$$A_e = \begin{bmatrix} 0 & -C_C \\ -C_P & 0 \end{bmatrix}, \quad A_{xe} = \begin{bmatrix} B_P & B_P D_C \\ 0 & B_C \end{bmatrix}.$$

Suppose that the intervals between transmission $t_{k+1} - t_k$ are uniformly distributed on $[0, \tau]$. Then, we wish to: (i) determine for which values of the maximum sampling time τ the system (50) is stable (MSS sense); (ii) assess the performance of the networked control system by computing the second-moment Lyapunov exponent of (50).

A. Inverted Pendulum

Suppose that the plant (47) is described by

$$A_P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \;\; B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \;\; C_P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$



(a) $\tau = 1.35 (MSS)$ (b) $\tau = 1.391$ (c) $\tau = 1.43$(NOT MSS)

Fig. 1. Nyquist plot illustration of stability for the impulsive renewal pendulum system.

which by properly scaling the state and input can be viewed as a linearized model of a damp-free inverted pendulum. The controller is synthesized assuming $\hat{u}(t) = u(t),\ \hat{y}(t) = y(t)$ and it is chosen to be an LQR controller given by $u(t) = K_{\text{opt}}\hat{y}(t)$, $K_{\text{opt}} = \begin{bmatrix} -1.618 & -2.058 \end{bmatrix}$ which is the solution to the minimization of $\int_0^\infty [x_P(t)^\intercal x_P(t) + u(t)^2] dt$. In this example, we assume that there are no packet losses, and that only the control law is sent through the network. The equations for $(x_P(t), e_u)$ take the form (50) with $A_{xx} = A_P + B_P K_{\text{opt}}$, $A_{ex} = B_P K_{\text{opt}}$, $A_e = -K_{\text{opt}}$, $\Lambda_k^{\omega_k} = 0$ for all k. Thus, (50) takes the simplified form (1). We can test for which values of τ the system is MSS (or equivalently SS and MES since τ is finite, cf. Theorem 3), using the algebraic condition (B) or the LMI condition (C) of the Theorem 3. Performing a binary search we conclude that the maximum value of τ for which the system is MSS is $\tau_c = 1.391$. Using the condition (A) of the Theorem 3 and the Nyquist Theorem, we can illustrate this graphically in Fig. 1 by drawing the Nyquist plot of $\det(I - \hat{\Theta}(z))$. In Figure 1 we see that for a value $\tau = 1.35$ less than τ_c the number of anti-clockwise encirclements of the curve $\det(I - \hat{\Theta}(j\omega))$, when ω varies from ∞ to $-\infty$ is zero, while for a value $\tau = 1.43$ greater than τ_c, the number of encirclements is one, which by the Nyquist Theorem implies that the stability condition (A) does not hold.

To assess the closed-loop performance we consider

$$\mathbb{E}[x_P(t)^\intercal x_P(t) + \hat{u}(t)^2 dt], \tag{51}$$

which is the function whose integral is minimized when designing the controller in the absence of a network. Notice that, since $\hat{u}(t) = e(t) + K_{\text{opt}} x_P(t)$, we can re-write (51) as $\mathbb{E}[[x_P(t)^\intercal \ \ e(t)] U^\intercal U [x_P(t)^\intercal \ \ e(t)]^\intercal dt]]$, with $U = \begin{bmatrix} I & 0 \\ K_{\text{opt}} & 1 \end{bmatrix}$. Using the Theorem 4 we can compute the Lyapunov exponent of this function for different values of τ. The results are

TABLE I

VARIATION OF THE LYAPUNOV EXPONENT OF (51) WITH THE SUPPORT τ OF FOR THE INTER-SAMPLING DISTRIBUTION

τ	0.1	0.3	0.6	0.9	1.2
Lyap. exponent	-2.152	-2.111	-1.904	-1.836	-0.709

summarized in Table I, quantifying the performance degradation as the distribution of times between consecutive transmissions assigns high probability to slow sampling.

B. Batch Reactor

This example considers a linearized model of an open loop unstable batch reactor, described by (47), where

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \; C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

As in [2], we assume that only the outputs are sent through the network, using a round-robin protocol, i.e., $\Gamma_k^{\omega_k} = \Omega_k^{\omega_k}$, $\forall_k$ where: $\Gamma_k^{\omega_k} = I_2$, if $\omega_k = 1$; $\Gamma_k^{\omega_k} = \text{diag}([1 \;\; 0])$, if $\omega_k = 2$ and k is even; $\Gamma_k^{\omega_k} = \text{diag}([0 \;\; 1])$, if $\omega_k = 2$, and k is odd. The system is controller by a PI controller, described by (48), where

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}, D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

To compare our results with the ones in [2] we consider uniformly and exponentially ($\mu([0, x)) = 1 - \exp(-\lambda_{\text{exp}} x)$) distributed time intervals h_k. The results are summarized in Table II, and were obtained using an extension of Theorem 3 given in [19].

VI. CONCLUSIONS AND FUTURE WORK

We presented stability and performance results for impulsive renewal systems. Our approach is based on a Volterra equation that describes the time evolution of the expected value of a

TABLE II

STABILITY CONDITIONS FOR THE BATCH REACTOR EXAMPLE

	Nec. & Suf. Cond.		Results taken from [2]	
	no drops	$p = 0.5$	no drops	$p = 0.5$
Maximum support τ of Uniform Distribution	0.112	0.0385	0.0517	0.0199
Max. expected value $1/\lambda_{\text{exp}}$ of Exponential Distribution	0.0417	0.0188	0.0217	0.00924

quadratic function of the systems' state. A direction for future work consists of extending the results of this paper to compute any moment of the state of the system.

ACKNOWLEDGMENTS

The first author is grateful to Alexandre Mesquita for stimulating discussions.

APPENDIX

Proof: (of Proposition 5) The condition (T1) assures that $\hat{\Theta}_d(y)$ converges absolutely for $y > -\epsilon$ and in this region, $\sigma(\hat{\Theta}_d(y))$ is continuous (cf. [11, Ch. 3, Th.3.8]). Therefore, $\sigma(\hat{\Theta}_d(0)) < 1$ implies that $\sigma(\hat{\Theta}_d(-\epsilon_1)) < 1$ for some $\epsilon_1 \in (0, \epsilon)$, and from Lemma 10 we have $\sigma(\hat{\Theta}_d(z)) < 1$ in $\Re[z] \geq -\epsilon_1$, which encompasses $\mathcal{C}(\epsilon_1, R)$, $R > 0$. This implies $\inf |\det(I - \hat{\Theta}_d(z))| > 0$ in $\mathcal{C}(\epsilon_1, R)$. ■

Proof: (of Proposition (8)) From the fact that

$$\hat{\Psi}(a) = \int_0^\infty K_c(s)e^{-as}ds + \sum w_i K_{di} e^{-ac_i}, \tag{52}$$

it suffices to prove that $\int_0^\infty K_c(s)e^{-as}ds$ is a positive operator since this implies that (52) is a sum of positive operators and therefore a positive operator. It is clear that the fact that $\hat{\Psi}(a)$ converges absolutely implies that $\int_0^\infty K_c(s)e^{-as}ds$ converges absolutely. Since the space of continuous function with bounded support on $\mathbb{R}_{\geq 0}$, denoted by C_c, is dense in the space of measurable function whose Lebesgue integral on $\mathbb{R}_{\geq 0}$ is absolutely convergent, denoted by L_1, (cf. [15, Th.3.14]), there exists a sequence $K_n(s) \in C_c$ such that $|\int_0^\infty (K_n(s) - K_c(s))e^{-as}ds| < \frac{\epsilon_n}{2}$, where $\epsilon_n \to 0$. Since $K_n(s)$ is Riemann integrable, we can find a matrix step function $K_n^R(s)$, and h_n-spaced points s_i such that $\int_0^\infty K_n^R(s)e^{-as}ds = \sum_{i=1}^{N_n} h_n K_n^R(s_i)e^{-as_i}$. The result of this

integral is a positive operator since it is the sum of positive operators. Moreover for each n we can choose N_n sufficiently large such that $|\int_0^\infty (K_n^R(s) - K_n(s))e^{-as}ds| < \frac{\epsilon_n}{2}$. Thus,

$$|\int_0^\infty K_n^R(s)e^{-as}ds - \int_0^\infty K(s)e^{-as}ds| < \epsilon_n \to 0, \text{ as } n \to \infty$$

and therefore $\int_0^\infty K(s)e^{-as}ds$ is a positive operator since it is the limit of positive operators, and the space of positive operators w.r.t a given cone is closed (cf. [16, p.22]). ■

Proof: (of the Lemma 10) By Proposition 8, $\hat{\Psi}(a)$ is positive w.r.t. $\mathcal{K}$, and by induction so is $\hat{\Psi}(a)^k$. Let v belong to the dual cone $\mathcal{K}^*$ of $\mathcal{K}$, i.e., the set $v \in \mathbb{R}^m$ such that $\langle x, v\rangle \geq 0, \forall_{x\in\mathcal{K}}$. Such v different from zero always exists [16]. For every $v \in \mathcal{K}^*$ and for real a, $\langle \hat{\Psi}(a)^k x, v\rangle$ is a positive number by definition of $\mathcal{K}^*$. For $z : \Re[z] \geq a$ and for every $v \in \mathcal{K}^*$, $w \in \mathcal{K}$ and $k \geq 1$, one can show that

$$|\langle \hat{\Psi}(z)^k w, v\rangle| \leq \langle \hat{\Psi}(a)^k w, v\rangle. \tag{53}$$

For example for $k = 2$, let

$$a(r,s) := \langle K_c(r)K_c(s)w, v\rangle, \;\; b_j(s) := \langle K_c(s)K_{dj}w, v\rangle$$

$$d_i(r) =: \langle K_{di}K_c(r)w, v\rangle, \quad h_{ij} := \langle K_{di}K_{dj}w, v\rangle$$

which are positive functions due to the fact that the kernel Ψ is positive and by definition of $\mathcal{K}^*$. Then

$$\begin{aligned}
&|\langle \hat{\Psi}(z)^2 w, v\rangle| = \\
&|\int_0^\infty\int_0^\infty a(r,s)e^{-zr}e^{-zs}drds + \int_0^\infty \sum_{j=1}^\infty b_j(s)e^{-zc_j}e^{-zs}ds + \\
&\int_0^\infty \sum_{i=1}^\infty d_i(r)e^{-zc_i}e^{-zr}dr + \sum_{i=1}^\infty\sum_{j=1}^\infty h_{ij}e^{-zc_j}e^{-zc_i}| \\
&\leq \int_0^\infty\int_0^\infty |a(r,s)|e^{-ar}e^{-as}drds + \int_0^\infty \sum_{j=1}^\infty |b_j(s)|e^{-ac_j}e^{-as}ds + \\
&\int_0^\infty \sum_{i=1}^\infty |d_i(r)|e^{-ac_i}e^{-ar}dr + \sum_{j=1}^\infty\sum_{i=1}^\infty |h_{ij}|e^{-ac_i}e^{-ac_j} \\
&= \langle \hat{\Psi}(a)^2 w, v\rangle
\end{aligned}$$

where we used the fact that $a(r,s)$, $b_j(s)$, $d_i(r)$, h_{ji} are positive functions.

If $\sigma(\hat{\Psi}(a)) < 1$, then $\hat{\Psi}(a)^k$ converges to zero as k tends to infinity, and so does $\langle \hat{\Psi}(a)^k w, v\rangle$ for every $v \in \mathcal{K}^*, w \in \mathcal{K}$. Since $\mathcal{K}$ is solid, and therefore reproducing, so is $\mathcal{K}^*$(cf. [16, Th. 4.5]

and use the fact that a cone in $(\mathbb{R}^m, \|.\|)$ is normal [16, p. 37]). Thus, for any $u, y \in \mathbb{R}^m$ we can write $u = u_1 - u_2$ and $y = y_1 - y_2$, for $u_i \in \mathcal{K}$, $y_i \in \mathcal{K}^*$, $i \in \{1, 2\}$ and obtain that

$$|\langle \hat{\Psi}(z)^k u, y\rangle| \leq \sum_{1\leq i,j\leq 2} |\langle \hat{\Psi}(z)^k u_i, y_i\rangle|. \tag{54}$$

Each of the terms on the right hand side of (54) tend to zero as $k \to \infty$ due to (53) and the fact that $\sigma(\hat{\Psi}(a)) < 1$. This implies that $\langle \hat{\Psi}(z)^k u, y\rangle \to 0$ for every $u, y \in \mathbb{R}^m$ and therefore that $\hat{\Psi}(z)^k \to 0$ as $k \to \infty$, which in turn implies that $\sigma(\hat{\Psi}(z)) < 1$.

To conclude the proof we note that for $\alpha > 0$, if $\sigma(\hat{\Psi}(a)) < \alpha$ then $\sigma(\frac{1}{\alpha}\hat{\Psi}(a)) < 1$. Using the above arguments this implies that $\sigma(\frac{1}{\alpha}\hat{\Psi}(z)) < 1$, for $z : \Re[z] \geq a$, or equivalently that $\sigma(\hat{\Psi}(z)) < \alpha$. Since α is arbitrary, $\sigma(\hat{\Psi}(z)) \leq \sigma(\hat{\Psi}(a))$. ■

Proof: (of Lemma (11)) To prove (i), note first that, $\sigma(\hat{\Psi}(y))$ is continuous along $y \geq a$ and converges to zero as $y \to +\infty$. By the intermediate value theorem, if $\sigma(\hat{\Psi}(a)) \geq 1$ then $\exists_{a_1 \geq a}$: $\sigma(\hat{\Psi}(a_1)) = 1$. To see that a_1 is the unique real number that satisfies $\sigma(\hat{\Psi}(a_1)) = 1$, suppose that $\sigma(\hat{\Psi}(a_2)) = 1$ for another real number a_2. Due to Lemma 10, we must have $\sigma(\hat{\Psi}(y)) = 1$ in $y \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}]$. Since, by the property (19), $\sigma(\hat{\Psi}(a_1)) = 1$ is an eigenvalue, this implies that $\det(I - \hat{\Psi}(y)) = 0$, in $y \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}]$. But $\det(I - \hat{\Psi}(z)) = 0$ is analytic and the zeros of analytic functions (other than the identically zero function) are isolated (cf. [15, Th.10.18]), which leads to a contradiction.Since $\hat{\Psi}(a)$ is a positive operator due to Proposition 8, by the property (19) of positive operators w.r.t. solid cones, $\sigma(\hat{\Psi}(a_1)) = 1$ is an eigenvalue and this implies that $\det(I - \hat{\Psi}(a_1)) = 0$ (part (ii)). Finally, we argue by contradiction to prove (iii). If there exists w satisfying $\Re[w] > a_1$ and $\det(I - \hat{\Psi}(w)) = 0$ then $\hat{\Psi}(w)$ has an eigenvalue at 1 and therefore $\sigma(\hat{\Psi}(w)) \geq 1$. From Lemma 10, $\sigma(\hat{\Psi}(w)) \leq \sigma(\hat{\Psi}(\Re[w]))$ which implies that $\sigma(\hat{\Psi}(\Re[w])) \geq 1$. Now, $\sigma(\hat{\Psi}(\Re[w]))$ cannot be equal to one since a_1 is the unique real number that satisfies this property and $\Re[w] > a_1$. This leads to a contradiction since, again by Lemma 10, $\sigma(\hat{\Psi}(\Re[w])) \leq \sigma(\hat{\Psi}(a_1)) = 1$. ■

Proof: (of the Lemma 15)

Theorems [11, Ch.7,Th.2.4,2.5] and the argument in [11, p.192] would allow us to conclude

that (i) and (ii) hold if:

$$\inf_{z:\Re[z]=0} |\det(I-\hat{\Psi}(z))| > 0, \tag{55}$$

$$\lim_{|z|\to\infty,\Re[z]\geq 0} \inf |\det(I-\hat{\Psi}(z))| > 0, \tag{56}$$

$$g(t) \text{ is bounded and converges to zero.} \tag{57}$$

Note that (55) is restrictive since $\det(I-\hat{\Psi}(z))$ would not be allowed to have zeros or the imaginary axis (critical case). It is clear that (V3) implies (57). Note also that (V2) implies (56) and, since we assume (V2), the condition (55) can be simply stated as the characteristic equation $\det(I-\hat{\Psi}(z))=0$ has no zeros on the imaginary axis (non-critical case).

The proof of (i) under assumptions (V1), (V2) is a direct consequence of the arguments in [11, p.192] and [11, p.195] are is therefore omitted. To prove (ii) under assumptions (V1), (V2) and (V3) we argue as follows. Let $g_\delta(t)$ and Ψ_δ be as in Proposition 14. Due to (27) the zeros of $\det(I-\hat{\Psi}_\delta(z))$ are given by $z_i+\delta$, where z_i are the zeros of $\det(I-\hat{\Psi}(z))$. If $\det(I-\hat{\Psi}(z))$ has zeros on the imaginary axis (critical case), i.e., (55) does not hold, we can always choose (due to part (i) of the present theorem) a $\delta: 0<\delta<\epsilon$, where ϵ is the constant in (V1), and such that the characteristic equation of the Volterra equation (26) does not have zeros on the imaginary axis. Thus, the Volterra equation (26) satisfies the conditions (55), (56), (57) required by Theorems [11, Ch.7,Th.2.4,2.5] to assure (i) and (ii). Thus from Theorems [11, Ch.7,Th.2.4,2.5] one can conclude that

$$y_\delta(t) = y_{s\delta}(t) + \sum_{i=1}^{n_z}\sum_{j=0}^{m_i-1} s_{i,j} t^j e^{(z_i+\delta)t} + \sum_{i=1}^{n_w}\sum_{j=0}^{n_i-1} q_{i,j} t^j e^{v_i t}, \tag{58}$$

where v_i are the zeros of $\det(I-\hat{\Psi}_\delta(z))$ in $0<\Re[w_i]<\delta$ and we used the fact, due to (27), that the principal parts of $\hat{y}(z)$ about z_i and of $\hat{y}_\delta(z)$ about $z_i+\delta$ have the same coefficients $s_{i,j}$. Thus (28) follows from (58) due to $y(t)=y_\delta(t)e^{-\delta t}$.

■

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