A converse Lyapunov theorem for asymptotic stability in probability

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Abstract

A converse Lyapunov theorem is established for discrete-time stochastic systems with non-unique solutions. In order to prove the result, mild regularity conditions are imposed on the set-valued mapping that characterizes the update of the system state. It is shown that global asymptotic stability in probability implies the existence of a continuous Lyapunov function, smooth outside of the attractor, that decreases in expected value along solutions. Additional results on the robustness of global asymptotic stability in probability are used to construct a Lyapunov function with the desired regularity properties.

I. Introduction

Converse Lyapunov theorems address when asymptotic stability implies the existence of a Lyapunov function, usually with nice regularity properties. Converse Lyapunov theorems have a long history and have been used frequently in the nonlinear control literature to establish robustness of asymptotic stability, to demonstrate the existence of stabilizing control laws, and to motivate the notion of control Lyapunov functions, among other applications.

For non-stochastic systems, the most recent converse Lyapunov theorems address asymptotic stability for systems with non-unique solutions. This trend started in [1], which addressed locally Lipschitz differential inclusions with not necessarily convex values. In turn, [2] established a converse Lyapunov theorem for global asymptotic stability of the origin for differential inclusions satisfying mild regularity conditions, including differential inclusions associated with Krasovskii solutions and Filippov solutions for discontinuous differential equations. In [3], these results were extended to compact attractors and local asymptotic stability. The existence of smooth Lyapunov functions was established for discrete-time systems with non-unique solutions in [4], [5], and [6], and for hybrid systems with non-unique solutions in [7].

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For stochastic systems, converse Lyapunov theorems appear in [8] for a type of asymptotic stability in probability and in [9, Chapter V] for a type of stochastic exponential stability. Converse Lyapunov theorems for various stochastic recurrence properties are summarized in the comprehensive text [10]. Converse theorems related to almost sure stabilizability are in [11].

In this paper, we present a converse Lyapunov theorem for discrete-time stochastic systems with non-unique solutions. Under mild regularity conditions, we establish that global asymptotic stability in probability of a compact set implies the existence of a continuous Lyapunov function that is smooth outside of the attractor and that decreases in expected value along solutions. Unlike the non-stochastic case, global asymptotic stability in probability does not always guarantee the existence of a Lyapunov function that decreases exponentially in expected value. The phenomenon discussed in [12, pg. 214] is related to this limitation. The construction that we use has some connection to the Lyapunov function construction used for [13, Theorem 19.3-19.10] for continuous-time deterministic systems. Inspired by [2], [3], and [4], we establish the regularity properties of the Lyapunov function by first establishing that global asymptotic stability in probability is robust to sufficiently small, positive definite state perturbations. This idea has also been explored in a stochastic setting in [14].

The results of the paper are organized as follows: We cover notation and some basic definitions in Section II. In Section III, we describe the class of stochastic systems that we consider, including a short review of stochastic processes with non-unique solutions. Our main results are stated in Section IV, which includes our definition of asymptotic stability in probability and the converse theorem in Section IV-A and robustness results in Section IV-B. Section V describes some probability functions that are used to give equivalent characterizations of global asymptotic stability in probability. The Lyapunov function is constructed in Section VI from the functions in Section V. The proofs of robustness of asymptotic stability in probability are provided in Sections VIII and X, using some extra facts about probability functions that are summarized in Sections VII and IX. We draw some conclusions and indicate future work in Section XII.

II. Notation and basic definitions

$\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers; $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$ is the Euclidean distance to S. $\mathbb{B}$ (resp., $\mathbb{B}^\circ$) denotes the closed (resp., open) unit ball in $\mathbb{R}^n$. For a closed set $S \subset \mathbb{R}^n$ and $\varepsilon > 0$, $S + \varepsilon\mathbb{B}$

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(resp., $S+\varepsilon\mathbb{B}^\circ$) denotes the set $\{x\in\mathbb{R}^n : |x|_S\le\varepsilon\}$ (resp., $\{x\in\mathbb{R}^n : |x|_S<\varepsilon\}$). The function $\mathbb{I}_S:\mathbb{R}^n\to\{0,1\}$ satisfies $\mathbb{I}_S(x)=1$ for $x\in S$ and $\mathbb{I}_S(x)=0$ otherwise. A function $\phi:\mathbb{R}^n\to\mathbb{R}$ is *upper semicontinuous* if $\limsup_{i\to\infty}\phi(x_i)\le\phi(x)$ whenever $\lim_{i\to\infty}x_i=x$. The function $\mathbb{I}_S$ is upper semicontinuous for closed S. A set-valued mapping $M:\mathbb{R}^p\rightrightarrows\mathbb{R}^n$ is *outer semicontinuous* if, for each $(x_i,y_i)\to(x,y)\in\mathbb{R}^p\times\mathbb{R}^n$ satisfying $y_i\in M(x_i)$ for all $i\in\mathbb{Z}_{\ge0}$, $y\in M(x)$. A mapping M is *locally bounded* if, for each bounded set $K\subset\mathbb{R}^p$, $M(K):=\bigcup_{x\in K}M(x)$ is bounded. $\mathbf{B}(\mathbb{R}^m)$ denotes the Borel field, the subsets of $\mathbb{R}^m$ generated from open subsets of $\mathbb{R}^m$ through complements and finite and countable unions. A set $F\subset\mathbb{R}^m$ is *measurable* if $F\in\mathbf{B}(\mathbb{R}^m)$. A mapping $M:\mathbb{R}^p\rightrightarrows\mathbb{R}^n$ is *measurable* [15, Def. 14.1] if for each open set $\mathcal{O}\subset\mathbb{R}^n$ the set $M^{-1}(\mathcal{O}):=\{v\in\mathbb{R}^p : M(v)\cap\mathcal{O}\neq\varnothing\}$ is measurable. When the values of M are closed, measurability is equivalent to $M^{-1}(\mathcal{C})$ being measurable for each closed set $\mathcal{C}\subset\mathbb{R}^n$ [15, Thm. 14.3]. A function $\alpha:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ is *of class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0)=0$. It is *of class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\psi:\mathbb{Z}_{\ge0}\to\mathbb{R}_{\ge0}$ is *of class* $\mathcal{L}$ if it is nonincreasing and $\lim_{\ell\to\infty}\psi(\ell)=0$. A function $\gamma:\mathbb{R}_{\ge0}\times\mathbb{Z}_{\ge0}\to\mathbb{R}_{\ge0}$ is *of class* $\mathcal{KL}$ if $\gamma(\cdot,k)\in\mathcal{K}$ for each $k\in\mathbb{Z}_{\ge0}$ and $\gamma(s,\cdot)\in\mathcal{L}$ for each $s\in\mathbb{R}_{\ge0}$. Given a compact set $\mathcal{A}\subset\mathbb{R}^n$, $\rho:\mathbb{R}^n\to\mathbb{R}_{\ge0}$ is *of class* $\mathcal{PD}(\mathcal{A})$ if it is continuous and $\rho(x)=0\iff x\in\mathcal{A}$. A function $\rho:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ is *of class* $\mathcal{PD}$ if it is continuous and $\rho(s)=0\iff s=0$.

III. Difference inclusions with random inputs

Consider a set-valued mapping $G:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{R}^n$ and a stochastic discrete-time system with state $x\in\mathbb{R}^n$ and (random) input $v\in\mathbb{R}^m$ written formally as

$$x^+\in G(x,v)\ . \tag{1}$$

We impose the following regularity conditions throughout the paper.

Standing Assumption 1: The set-valued mapping G satisfies the following properties:

1) For each $v\in\mathbb{R}^m$ the mapping $x\mapsto G(x,v)$ is outer semicontinuous.
2) For each $x\in\mathbb{R}^n$ the mapping $v\mapsto G(x,v)$ is measurable.
3) The mapping $v\mapsto\text{graph}(G(\cdot,v)):=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : y\in G(x,v)\}$ is measurable.
4) The mapping G is locally bounded.

Without the first and last condition, even for non-stochastic systems, it may be impossible to construct a *continuous* Lyapunov function [4]. The second and third conditions guarantee that G

generates stochastic processes, as we show below. If G is a (single-valued) function that satisfies the first two conditions then it also satisfies the third condition; see [15, Example 14.15]. We allow $G(x,v)$ to be empty for some (x,v). Consequently, solutions may not be defined for all initial conditions, times, and inputs.

Let $\mathcal{S}_{c,m}(x)$ denote the set of maximal (non-random) solutions to (1) starting at x that are causal, measurable functions of the inputs. That is, $\phi \in \mathcal{S}_{c,m}(x)$ if ϕ comprises a sequence of measurable functions $\phi_i : \mathrm{dom}\ \phi_i \subset (\mathbb{R}^m)^i \to \mathbb{R}^n$, $i \in \mathbb{Z}_{\geq 0}$, with $\phi_0 = x$ such that $\phi_{i+1}(v_0,\ldots,v_i) \in G(\phi_i(v_0,\ldots,v_{i-1}),v_i)$ for all $i \in \mathbb{Z}_{\geq 0}$ and all $(v_0,\ldots,v_i) \in \mathrm{dom}\ \phi_{i+1}$ with the property that $\mathrm{dom}\ \phi_{i+1} = \{(v_0,\ldots,v_{i-1},v_i) \in \mathrm{dom}\ \phi_i \times \mathbb{R}^m : G(\phi_i(v_0,\ldots,v_{i-1}),v_i) \neq \varnothing\}$. Under Standing Assumption 1, $\mathcal{S}_{c,m}(x)$ is non-empty for each $x \in \mathbb{R}^n$. See [16, Lemma 3].

Now we add a probability structure to (1). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. For $i \in \mathbb{Z}_{\geq 0}$, let $\mathbf{v}_i : \Omega \to \mathbb{R}^m$ be a sequence of independent, identically distributed (i.i.d.) random variables. Thus, $\mathbf{v}_i^{-1}(F) := \{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\} \in \mathcal{F}$ for each $F \in \mathbf{B}(\mathbb{R}^m)$. Let $\mathcal{F}_{\mathbf{v}_i} := \left\{\mathbf{v}_i^{-1}(F) : F \in \mathbf{B}(\mathbb{R}^m)\right\}$ be the σ-algebra generated by $\mathbf{v}_i$ and let $\mathcal{F}_{\mathbf{v},i} := \left\{\mathcal{F}_{\mathbf{v}_j} : j \in \{0,\ldots,i\}\right\}$ be the natural filtration of $\mathbf{v}$. From the i.i.d. property, each random variable has the same probability measure $\mu : \mathbf{B}(\mathbb{R}^m) \to [0,1]$ defined as $\mu(F) := \mathbb{P}\{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\}$ and $\mathbb{E}[f(\mathbf{v}_{i+1})|\mathcal{F}_{\mathbf{v},i}] = \int_{\mathbb{R}^m} f(v)\mu(dv)$ for each $i \in \mathbb{Z}_{\geq 0}$ and each measurable $f : \mathbb{R}^m \to \mathbb{R}$. A random process $\mathbf{x}$ is an *adapted solution candidate from* $x \in \mathbb{R}^n$ if it comprises a sequence of random variables $\mathbf{x}_i : \Omega \to \mathbb{R}^n$, $i \in \mathbb{Z}_{\geq 0}$, with $\mathbf{x}_0 = x$ for all $\omega \in \Omega$, and $\mathcal{F}_{i+1} := \mathcal{F}_{\mathbf{x}_{i+1}} \subset \mathcal{F}_{\mathbf{v},i}$ for each $i \in \mathbb{Z}_{\geq 0}$ so that $\mathbf{x}$ is adapted to the natural filtration of $\mathbf{v}$, imposing a causality structure on $\mathbf{x}$ relative to $\mathbf{v}$. An adapted solution candidate $\mathbf{x}$ from $x \in \mathbb{R}^n$, together with a random variable $\mathbf{T}_{\mathbf{x}} : \Omega \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, is an *adapted random solution* of (1) if $\mathbf{x}_{i+1}(\omega) \in G(\mathbf{x}_i(\omega),\mathbf{v}_i(\omega))$ for all $\omega \in \mathrm{dom}\ \mathbf{x}_{i+1} := \{\omega \in \Omega : i+1 < \mathbf{T}_{\mathbf{x}}(\omega)\}$ and $i \in \mathbb{Z}_{\geq 0}$. An adapted random solution $(\mathbf{x},\mathbf{T}_{\mathbf{x}})$ from $x \in \mathbb{R}^n$ is said to be *maximal* if it cannot be extended, i.e., there does not exist another adapted random solution $(\mathbf{y},\mathbf{T}_{\mathbf{y}})$ from x such that $\mathrm{dom}\ \mathbf{x}_i \subset \mathrm{dom}\ \mathbf{y}_i$ for all $i \in \mathbb{Z}_{\geq 0}$, $\mathbf{y}_i(\omega) = \mathbf{x}_i(\omega)$ for all $\omega \in \mathrm{dom}\ \mathbf{x}_i$ and all $i \in \mathbb{Z}_{\geq 0}$, and $\mathrm{dom}\ \mathbf{x}_i \neq \mathrm{dom}\ \mathbf{y}_i$ for some $i \in \mathbb{Z}_{\geq 0}$. We use $\mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ to denote the set of maximal adapted random solutions of (1) from $x \in \mathbb{R}^n$ and write $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$, suppressing the associated random variable $\mathbf{T}_{\mathbf{x}}$. The next result, established in Section XI-A, relates adapted random solutions $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ to causal, measurable non-random solutions $\phi \in \mathcal{S}_{c,m}(x)$ through the conditions

$$\left.\begin{array}{rcl} \mathrm{dom}\ \mathbf{x}_i & = & \{\omega \in \Omega : (\mathbf{v}_0(\omega),\ldots,\mathbf{v}_{i-1}(\omega)) \in \mathrm{dom}\ \phi_i\} \\ \mathbf{x}_i(\omega) & = & \phi_i(\mathbf{v}_0(\omega),\ldots,\mathbf{v}_{i-1}(\omega)) \quad \forall \omega \in \mathrm{dom}\ \mathbf{x}_i \end{array}\right\} \quad \forall i \in \mathbb{Z}_{\geq 0}\ . \tag{2}$$

Proposition 1: i) For each $\phi \in \mathcal{S}_{c,m}(x)$ there exists $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ satisfying (2).
ii) For each $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ there exists $\phi \in \mathcal{S}_{c,m}(x)$ satisfying (2).

Henceforth, to save on notation, subsets of Ω expressed in terms of the inverse image of the random variables $\mathbf{x}_i$ will be assumed to be relative to $\mathrm{dom}\ \mathbf{x}_i$. In particular, for $\mathcal{I} \subset \mathbb{Z}_{\geq 0}$,

$$\cap_{i\in\mathcal{I}}\mathbf{x}_i^{-1}(F) \ := \ \cap_{i\in\mathcal{I}}\left(\mathrm{dom}\ \mathbf{x}_i\right) \cap \left(\mathbf{x}_i^{-1}(F)\right) = \cap_{i\in\mathcal{I}}\left\{\omega \in \Omega : (\omega \in \mathrm{dom}\ \mathbf{x}_i) \wedge (\mathbf{x}_i(\omega) \in F)\right\}$$
$$\cup_{i\in\mathcal{I}}\mathbf{x}_i^{-1}(F) \ := \ \cup_{i\in\mathcal{I}}\left(\mathrm{dom}\ \mathbf{x}_i\right) \cap \left(\mathbf{x}_i^{-1}(F)\right) = \cup_{i\in\mathcal{I}}\left\{\omega \in \Omega : (\omega \in \mathrm{dom}\ \mathbf{x}_i) \wedge (\mathbf{x}_i(\omega) \in F)\right\}.$$

IV. Main results

A. Converse Lyapunov theorem

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *stable in probability for (1)* if for each $\varepsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that, for each $x \in \mathcal{A} + \delta\mathbb{B}$ and $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$, $\mathbb{P}\left(\cup_{i\in\mathbb{Z}_{\geq 0}}\mathbf{x}_i^{-1}\left(\mathbb{R}^n\backslash(\mathcal{A} + \varepsilon\mathbb{B}^\circ)\right)\right) \leq \rho$. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *almost surely globally asymptotically recurrent for (1)* if, for each $x \in \mathbb{R}^n$, each $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$, and each $\varepsilon > 0$, $\mathbb{P}\left(\cap_{i\in\mathbb{Z}_{\geq 0}}\mathbf{x}_i^{-1}\left(\mathbb{R}^n\backslash(\mathcal{A} + \varepsilon\mathbb{B}^\circ)\right)\right) = 0$. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *globally asymptotically stable in probability for (1)* if it is both stable in probability for (1) and almost surely globally asymptotically recurrent for (1). An upper semicontinuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *strict Lyapunov function relative to $\mathcal{A}$ for (1)* if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\varrho \in \mathcal{PD}(\mathcal{A})$ such that $\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$ and[1] $\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V(g)\mu(dv) \leq V(x) - \varrho(x)$ for all $x \in \mathbb{R}^n$. Our main result is stated next.

Theorem 1: If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is globally asymptotically stable in probability for (1) then there exists a smooth on $\mathbb{R}^n\backslash\mathcal{A}$, continuous, strict Lyapunov function relative to $\mathcal{A}$ for (1).

Theorem 1 is the converse of [16, Corollary 1], for example, which establishes that Lyapunov conditions imply global asymptotic stability in probability for (1).[2]

B. Robust global asymptotic stability in probability: the effect of inflations

To prove Theorem 1, we use two results of independent interest on the preservation of global asymptotic stability in probability under inflations of the mapping G. The first inflation is

$$x^+ \in G_\nu(x, v) := G(x, v) \cup M_\nu(x) \tag{3}$$

[1] If ϕ is upper semicontinuous and $G(x, v) \neq \varnothing$ then there exists $g^* \in G(x, v)$ such that $\sup_{g\in G(x,v)} \phi(g) = \phi(g^*)$. Hence, we use $\max_{g\in G(x,v)} V(g)$ for $\sup_{g\in G(x,v)} V(g)$. We also use $\max_{g\in G(x,v)} \phi(g) = 0$ when $G(x, v) = \varnothing$ and $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

[2] The definition of global asymptotic stability in probability in [16] appears to be stronger than the definition here but is, in fact, equivalent due to Theorem 1 and the results in [16]. It is similar to the characterization in [9, Theorem 4.3].

where $M_\nu(x) = \varnothing$ for $x \in \mathcal{A} + 2\mathbb{B}^\circ$ and $M_\nu(x) = \mathcal{A} + \nu(|x|_\mathcal{A})\mathbb{B}$ otherwise, where $\nu \in \mathcal{K}_\infty$.

Proposition 2: For each $\nu \in \mathcal{K}_\infty$, the set-valued mapping G_ν defined in (3) satisfies the conditions of Standing Assumption 1.

The previous result is proved in Section XI-B while the next result is proved in Section VIII.

Theorem 2: If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is globally asymptotically stable in probability for (1) then there exists $\nu \in \mathcal{K}_\infty$ such that $\mathcal{A}$ is globally asymptotically stable in probability for (3).

When we apply the Lyapunov construction to (3) we get a Lyapunov function for (1) since $G(x,v) \subset G_\nu(x,v)$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^m$. Moreover, the introduction of the function $\nu \in \mathcal{K}_\infty$ guarantees that the Lyapunov function V satisfies $\alpha_1(|x|_\mathcal{A}) \leq V(x)$ for some $\alpha_1 \in \mathcal{K}_\infty$.

The second inflation is relative to (3) and has the form

$$x^+ \in G_{\rho,\nu}(x,v) := \{w \in \mathbb{R}^n : w \in g + \rho(g)\mathbb{B}\ ,\ g \in G_\nu(x + \rho(x)\mathbb{B}, v)\} \qquad \rho \in \mathcal{PD}(\mathcal{A}). \quad (4)$$

Proposition 3: For each $\nu \in \mathcal{K}_\infty$ and each continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the set-valued mapping $G_{\rho,\nu}$ defined in (4) satisfies the conditions of Standing Assumption 1.

The previous result is proved in Section XI-C while the next result is proved in Section X.

Theorem 3: If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is globally asymptotically stable in probability for (3) then there exists $\rho \in \mathcal{PD}(\mathcal{A})$ such that $\mathcal{A}$ is globally asymptotically stable in probability for (4).

Thus global asymptotic stability in probability is robust to small, state-dependent perturbations. When we apply our Lyapunov construction to (4) we get a Lyapunov function V for (1) that satisfies $\alpha_1(|x|_\mathcal{A}) \leq V(x)$, since $G(x,v) \subset G_\nu(x,v) \subset G_{\rho,\nu}(x,v)$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^m$. Moreover, the introduction of the function ρ guarantees that the Lyapunov function we build can be made smooth on $\mathbb{R}^n \backslash \mathcal{A}$ while retaining the property that it is a Lyapunov function for (1).

V. Stability in terms of probability functions

A. Weak viability

The probability that a closed set $S \subset \mathbb{R}^n$ is weakly viable is quantified through the functions $m_{S,\cap}(j+1,\xi) := \int_{\mathbb{R}^m} \max_{g \in G(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(j,g)\mu(dv)$ for each $(j,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ where $m_{S,\cap}(0,\xi) := 1$. Due to [17, Lemma 3], the functions $\xi \mapsto m_{S,\cap}(j,\xi) \in [0,1]$ are well-defined, upper semicontinuous, and $m_{S,\cap}(j+1,\xi) \leq m_{S,\cap}(j,\xi)$ for each $(j,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$. For the relation to weak viability, consider the next result, proved in Section XI-D.

Proposition 4: Let $S \subset \mathbb{R}^n$ be closed. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}_{\geq 1}$ there exists $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ such that $m_{S,\cap}(k,x) = \mathbb{P}\left(\cap_{i=1}^k \mathbf{x}_i^{-1}(S)\right) = \sup_{\mathbf{z} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)} \mathbb{P}\left(\cap_{i=1}^k \mathbf{z}_i^{-1}(S)\right)$.

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If $S_1 \subset S_2$ then $m_{S_1,\cap}(j,\xi) \leq m_{S_2,\cap}(j,\xi)$ for all $(j,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$. Moreover, $\widehat{m}_{S,\cap}(\xi) := \lim_{j\to\infty} m_{S,\cap}(j,\xi)$ is well defined for each $\xi \in \mathbb{R}^n$. The next result is proved in Section XI-E.

Proposition 5: Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. The following statements are equivalent:

1) $\mathcal{A}$ is almost surely globally asymptotically recurrent for (1).
2) For each $\varepsilon > 0$ there exists $\rho \in [0,1)$ such that $\widehat{m}_{\mathbb{R}^n\setminus(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cap}(\xi) \leq \rho$ for all $\xi \in \mathbb{R}^n$.
3) For each ε and $\xi \in \mathbb{R}^n$, $\widehat{m}_{\mathbb{R}^n\setminus(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cap}(\xi) = 0$.
4) For each compact set $K \subset \mathbb{R}^n$, $\rho > 0$, and $\varepsilon > 0$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that $m_{\mathbb{R}^n\setminus(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cap}(\ell,\xi) \leq \rho$ for all $\xi \in K$.

B. Weak reachability

The probability that a closed set $S \subset \mathbb{R}^n$ is weakly reachable is quantified through the functions $m_{S,\cup}(j+1,\xi) := \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \max\{\mathbb{I}_S(g), m_{S,\cup}(j,g)\}\,\mu(dv)$ for each $(j,\xi) \in \mathbb{Z}_{\geq 0}\times\mathbb{R}^n$ where $m_{S,\cup}(0,\xi) := 0$ for each $\xi \in \mathbb{R}^n$. Due to [17, Lemma 2], the functions $\xi \mapsto m_{S,\cup}(j,\xi) \in [0,1]$ are well-defined, upper semicontinuous, and $m_{S,\cup}(j,\xi) \leq m_{S,\cup}(j+1,\xi)$ for each $(j,\xi) \in \mathbb{Z}_{\geq 0}\times\mathbb{R}^n$. For the relation to weak reachability, consider the next result, proved in Section XI-F.

Proposition 6: Let $S \subset \mathbb{R}^n$ be closed. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}_{\geq 1}$ there exists $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ such that $m_{S,\cup}(k,x) = \mathbb{P}\left(\cup_{i=1}^k \mathbf{x}_i^{-1}(S)\right) = \sup_{\mathbf{z}\in\mathcal{S}_{\mathbf{a},\mathbf{r}}(x)} \mathbb{P}\left(\cup_{i=1}^k \mathbf{z}_i^{-1}(S)\right)$.

Evidently, if $S_1 \subset S_2$ then $m_{S_1,\cup}(j,\xi) \leq m_{S_2,\cup}(j,\xi)$ for all $(j,\xi) \in \mathbb{Z}_{\geq 0}\times\mathbb{R}^n$. Also, $\widehat{m}_{S,\cup}(\xi) := \max\{\mathbb{I}_S(\xi), \lim_{j\to\infty} m_{S,\cup}(j,\xi)\}$ is well defined for each $\xi \in \mathbb{R}^n$. Using Proposition 5 and the proof of [17, Prop. 5], under the conditions of Theorem 1, for each $\varepsilon > 0$, with $S := \mathbb{R}^n\setminus(\mathcal{A} + \varepsilon\mathbb{B}^\circ)$, $\xi \mapsto \widehat{m}_{S,\cup}(\xi)$ is upper semicontinuous and $\int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \widehat{m}_{S,\cup}(g)\mu(dv) \leq \widehat{m}_{S,\cup}(\xi)$ for all $\xi \in \mathbb{R}^n$. The next result follows from the stability in probability definition and Proposition 6.

Proposition 7: The compact set $\mathcal{A} \subset \mathbb{R}^n$ is stable in probability for (1) if and only if for each $\varepsilon > 0$ there exists $\alpha_\varepsilon \in \mathcal{K}$ such that $\widehat{m}_{\mathbb{R}^n\setminus(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cup}(\xi) \leq \alpha_\varepsilon(|\xi|_{\mathcal{A}})$ for all $\xi \in \mathbb{R}^n$.

C. Post-transient weak reachability

For each $\xi \in \mathbb{R}^n$, define $\vec{m}_{S,\cup}(0,\xi) := \widehat{m}_{S,\cup}(\xi)$ and, for $j \in \mathbb{Z}_{\geq 0}$, define $\vec{m}_{S,\cup}(j+1,\xi) := \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \vec{m}_{S,\cup}(j,g)\mu(dv)$. As established in [17, Lemma 5], under the conditions of Theorem 1, for $\varepsilon > 0$ and $S := \mathbb{R}^n\setminus(\mathcal{A} + \varepsilon\mathbb{B}^\circ)$, the functions $\xi \mapsto \vec{m}_{S,\cup}(j,\xi)$ are well defined, upper semicontinuous, and satisfy $0 \leq \vec{m}_{S,\cup}(j+1,\xi) \leq \vec{m}_{S,\cup}(j,\xi) \leq 1$ for all $(j,\xi) \in \mathbb{Z}_{\geq 0}\times\mathbb{R}^n$. While it will not be proved because it is not used in this paper, it can be shown that if the compact set $\mathcal{A}$ is globally asymptotically stable in probability for (1) then, for each $\varepsilon > 0$, $j \in \mathbb{Z}_{\geq 0}$ and

$\xi \in \mathbb{R}^n$, we have $\vec{m}_{S,\cup}(j,\xi) = \sup_{\mathbf{z}\in\mathcal{S}_{\mathbf{a},\mathbf{r}}(\xi)} \mathbb{P}\left(\cup_{i\in\mathbb{Z}_{\geq j}} \mathbf{z}_i^{-1}(S)\right)$ when $S := \mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ)$. The following result is proved in Section XI-G.

Proposition 8: The compact set $\mathcal{A} \subset \mathbb{R}^n$ is globally asymptotically stable in probability for (1) if and only if for each $\varepsilon > 0$ there exists $\gamma_\varepsilon \in \mathcal{KL}$ such that $\vec{m}_{\mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cup}(k,\xi) \leq \gamma_\varepsilon(|\xi|_\mathcal{A}, k)$ for all $(k,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$.

VI. Proof of Theorem 1

All probabilities are generated from (4) with $\nu \in \mathcal{K}_\infty$ and $\rho \in \mathcal{PD}(\mathcal{A})$ chosen according to Theorems 2 and 3 so that the compact set $\mathcal{A}$ is globally asymptotically stable in probability for (4). Fix $\varepsilon > 0$, define $S_\varepsilon := \mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ)$, let $\kappa_\varepsilon \in \mathcal{K}_\infty$, and define $W_\varepsilon : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$W_\varepsilon(\xi) := \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \vec{m}_{S_\varepsilon,\cup}(i,\xi) \ . \tag{5}$$

Due to Proposition 8 applied to (4), $W_\varepsilon(\xi) = 0$ for all $\xi \in \mathcal{A}$.

Proposition 9: There exists $\kappa_\varepsilon \in \mathcal{K}_\infty$ such that W_ε in (5) is well defined, locally bounded, upper semicontinuous, arbitrarily small near $\mathcal{A}$, and $W_\varepsilon(\xi) \geq \kappa_\varepsilon(1)$ for all $\xi \in S_\varepsilon$.

Proof: Define $\widetilde{m}_{S_\varepsilon}(k,\xi) := \vec{m}_{S_\varepsilon,\cup}(k-1,\xi) - \vec{m}_{S_\varepsilon,\cup}(k,\xi) \geq 0$ for each $(k,\xi) \in \mathbb{Z}_{\geq 1}\times\mathbb{R}^n$. Then $W_\varepsilon(\xi) = \sum_{i=1}^\infty \kappa_\varepsilon(i)\widetilde{m}_{S_\varepsilon}(i,\xi)$. Using Proposition 8, there exists $\gamma_\varepsilon \in \mathcal{KL}$ such that $\vec{m}_{S_\varepsilon,\cup}(\ell,\xi) \leq \gamma_\varepsilon\left(|\xi|_\mathcal{A}, \ell\right)$ for all $(\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$. Hence, for all $(\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$,

$$\textstyle\sum_{i=\ell+1}^\infty \widetilde{m}_{S_\varepsilon}(i,\xi) = \sum_{i=\ell+1}^\infty \left(\vec{m}_{S_\varepsilon,\cup}(i-1,\xi) - \vec{m}_{S_\varepsilon,\cup}(i,\xi)\right) = \vec{m}_{S_\varepsilon,\cup}(\ell,\xi) \leq \gamma_\varepsilon\left(|\xi|_\mathcal{A}, \ell\right) \ .$$

Let $i \mapsto \rho_i$ be a strictly increasing, unbounded mapping satisfying $\rho_0 = 0$ and $\gamma_\varepsilon(2^i, \rho_i) \leq 2^{-i}$ for all $i \in \mathbb{Z}_{\geq 1}$. Let $\widetilde{\rho}_\varepsilon \in \mathcal{K}_\infty$ satisfy $\widetilde{\rho}_\varepsilon(i) = \rho_i$ for each $i \in \mathbb{Z}_{\geq 0}$. Define $\kappa_\varepsilon(s) := \widetilde{\rho}_\varepsilon^{-1}(s)$. Then

$$\begin{aligned}\sum_{i=1}^\infty \kappa_\varepsilon(i)\widetilde{m}_{S_\varepsilon}(i,\xi) &= \sum_{j=1}^\infty \left(\sum_{i=\rho_{j-1}+1}^{\rho_j} \kappa_\varepsilon(i)\widetilde{m}_{S_\varepsilon}(i,\xi)\right) \leq \sum_{j=1}^\infty \left(\sum_{i=\rho_{j-1}+1}^{\rho_j} \kappa_\varepsilon(\rho_j)\widetilde{m}_{S_\varepsilon}(i,\xi)\right) \\ &= \sum_{j=1}^\infty \left(j \sum_{i=\rho_{j-1}+1}^{\rho_j} \widetilde{m}_{S_\varepsilon}(i,\xi)\right) \leq \sum_{j=1}^\infty \left(j \sum_{i=\rho_{j-1}+1}^{\infty} \widetilde{m}_{S_\varepsilon}(i,\xi)\right) \leq \sum_{j=1}^\infty j\gamma_\varepsilon(|\xi|_\mathcal{A}, \rho_{j-1}) \ .\end{aligned}$$

Given $\xi \in \mathbb{R}^n$, let $i \in \mathbb{Z}_{\geq 1}$ satisfy $|\xi|_\mathcal{A} \leq 2^{i-1}$ and observe that

$$\begin{aligned}\sum_{j=1}^\infty j\gamma_\varepsilon\left(|\xi|_\mathcal{A}, \rho_{j-1}\right) &= \sum_{j=1}^{i-1} j\gamma_\varepsilon\left(|\xi|_\mathcal{A}, \rho_{j-1}\right) + \sum_{j=i}^\infty j\gamma_\varepsilon\left(|\xi|_\mathcal{A}, \rho_{j-1}\right) \\ &\leq \sum_{j=1}^{i-1} j\gamma_\varepsilon(2^{i-1}, 0) + \sum_{j=i}^\infty j\gamma_\varepsilon(2^{j-1}, \rho_{j-1}) \leq \frac{i(i-1)}{2}\gamma_\varepsilon(2^{i-1}, 0) + \sum_{j=i}^\infty j2^{-(j-1)}.\end{aligned}$$

Since $\sum_{j=1}^{\infty} j2^{-(j-1)} < \infty$ it follows that $\xi \mapsto W_\varepsilon(\xi)$ is well defined and locally bounded.

Let $\lambda > 0$. Pick $i^* \in \mathbb{Z}_{\geq 0}$ so that $\sum_{j=i^*}^{\infty} j2^{-(j-1)} \leq 0.5\lambda$. Next, pick $\delta \in (0,1]$ such that $\gamma_\varepsilon(\delta, 0) \leq \lambda/(i^*)^2$. Then, following the calculations above, for $|\xi|_{\mathcal{A}} \leq \delta$ we have

$$W_\varepsilon(\xi) \leq \sum_{j=1}^{i^*-1} j\gamma_\varepsilon\left(|\xi|_{\mathcal{A}}, 0\right) + \sum_{j=i^*}^{\infty} j2^{-(j-1)} \leq \tfrac{(i^*)^2}{2}\gamma_\varepsilon(\delta, 0) + \tfrac{\lambda}{2} \leq \tfrac{\lambda}{2} + \tfrac{\lambda}{2} = \lambda \ .$$

Therefore, W_ε is arbitrarily small near $\mathcal{A}$.

By the definition of $\vec{m}_{S_\varepsilon,\cup}$, we have $\vec{m}_{S_\varepsilon,\cup}(0,\xi) = \widehat{m}_{S_\varepsilon,\cup}(\xi) = 1$ for all $\xi \in S_\varepsilon$. Hence, from the definition of W_ε in (5), $W_\varepsilon(\xi) \geq (\kappa_\varepsilon(1) - \kappa_\varepsilon(0))\vec{m}_{S_\varepsilon,\cup}(0,\xi) = \kappa_\varepsilon(1)$ for all $\xi \in S_\varepsilon$.

For upper semicontinuity, the calculations above give that for each $x \in \mathbb{R}^n$, $\delta > 0$ and $\lambda > 0$ there exists $i^* \in \mathbb{Z}_{\geq 0}$ such that $W_\varepsilon(z) \leq \sum_{i=0}^{i^*-1} \left[\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right] \vec{m}_{S_\varepsilon,\cup}(i,z) + \lambda$ for all $z \in x + \delta\mathbb{B}$. The function $z \mapsto \vec{m}_{S_\varepsilon,\cup}(i,z)$ is upper semicontinuous for each $i \in \mathbb{Z}_{\geq 0}$. Also, since $\kappa \in \mathcal{K}_\infty$, the values $\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)$ are positive. Let the sequence x_j converge to x. Then

$$\begin{aligned} \limsup_{j\to\infty} W_\varepsilon(x_j) &\leq \limsup_{j\to\infty} \left(\sum_{i=0}^{i^*-1} \left[\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right] \vec{m}_{S_\varepsilon,\cup}(i,x_j)\right) + \lambda \\ &\leq \sum_{i=0}^{i^*-1} \left[\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right] \vec{m}_{S_\varepsilon,\cup}(i,x) + \lambda \leq W_\varepsilon(x) + \lambda \ . \end{aligned}$$

Since $\lambda > 0$ is arbitrary, W_ε is upper semicontinuous. ■

Proposition 10: There exists $\varrho_\varepsilon \in \mathcal{PD}$ such that $\int_{\mathbb{R}^m} \max_{g\in G_{\rho,\nu}(x,v)} W_\varepsilon(g)\mu(dv) \leq W_\varepsilon(x) - \varrho_\varepsilon(\max\{0, |x|_{\mathcal{A}} - \varepsilon\})$ for all $x \in \mathbb{R}^n$.

Proof: Let $\widehat{\ell}_\varepsilon \in \mathcal{L}$ satisfy $\gamma_\varepsilon(s, \widehat{\ell}_\varepsilon(s)) \leq 0.5$ for all $s \geq 0$. Let $\varrho_\varepsilon \in \mathcal{PD}$ satisfy

$$\varrho_\varepsilon(\max\{0, s-\varepsilon\}) \leq 0.5 \min_{i\in\left\{0,\ldots,\lceil\widehat{\ell}_\varepsilon(s)\rceil-1\right\}} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \qquad \forall s \geq 0 \ . \tag{6}$$

Since $\vec{m}_{S_\varepsilon,\cup}(0,\xi) = \widehat{m}_{S_\varepsilon,\cup}(\xi) = 1$ for $\xi \in S_\varepsilon$, it follows that

$$\sum_{i=0}^{\ell-1} \left(\vec{m}_{S_\varepsilon,\cup}(i,\xi) - \vec{m}_{S_\varepsilon,\cup}(i+1,\xi)\right) = 1 - \vec{m}_{S_\varepsilon,\cup}(\ell,\xi) \geq 1 - \gamma_\varepsilon(|\xi|_{\mathcal{A}}, \ell) \qquad \forall \xi \in S_\varepsilon. \tag{7}$$

Using the definition of W_ε in (5),

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g\in G_{\rho,\nu}(\xi,v)} W_\varepsilon(g)\mu(dv) &= \int_{\mathbb{R}^m} \max_{g\in G_{\rho,\nu}(\xi,v)} \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \vec{m}_{S_\varepsilon,\cup}(i,g)\mu(dv) \\ &\leq \int_{\mathbb{R}^m} \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \max_{g\in G_{\rho,\nu}(\xi,v)} \vec{m}_{S_\varepsilon,\cup}(i,g)\mu(dv) \\ &= \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \int_{\mathbb{R}^m} \max_{g\in G_{\rho,\nu}(\xi,v)} \vec{m}_{S_\varepsilon,\cup}(i,g)\mu(dv) \\ &= \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \vec{m}_{S_\varepsilon,\cup}(i+1,\xi) \\ &= W_\varepsilon(\xi) - \sum_{i=0}^{\infty} \left(\kappa_\varepsilon(i+1) - \kappa_\varepsilon(i)\right) \left(\vec{m}_{S_\varepsilon,\cup}(i,\xi) - \vec{m}_{S_\varepsilon,\cup}(i+1,\xi)\right) . \end{aligned} \tag{8}$$

Then, combining (6), (7), and (8), we get

$$\begin{aligned}\int_{\mathbb{R}^m}\max_{g\in G_{\rho,\nu}(\xi,v)} W_\varepsilon(g)\mu(dv) - W_\varepsilon(\xi) &\le -\sum_{i=0}^{\lceil\widehat{\ell}_\varepsilon(|\xi|_{\mathcal{A}})\rceil-1}\left(\kappa_\varepsilon(i+1)-\kappa_\varepsilon(i)\right)\left(\widetilde{m}_{S_\varepsilon,\cup}(i,\xi)-\widetilde{m}_{S_\varepsilon,\cup}(i+1,\xi)\right)\\ &\le -\min_{i\in\left\{0,\ldots,\lceil\widehat{\ell}_\varepsilon(|\xi|_{\mathcal{A}})\rceil-1\right\}}\left(\kappa(i+1)-\kappa(i)\right)\left(1-\gamma_\varepsilon\left(|\xi|_{\mathcal{A}},\widehat{\ell}_\varepsilon\left(|\xi|_{\mathcal{A}}\right)\right)\right)\\ &\le -0.5\min_{i\in\left\{0,\ldots,\lceil\widehat{\ell}_\varepsilon(|\xi|_{\mathcal{A}})\rceil-1\right\}}\left(\kappa(i+1)-\kappa(i)\right)\le -\varrho_\varepsilon\left(\max\left\{0,|\xi|_{\mathcal{A}}-\varepsilon\right\}\right).\end{aligned}$$

This bound proves the proposition. ■

Since W_ε is locally bounded for each $\varepsilon>0$, there exists a nondecreasing sequence of positive numbers $\overline{W}_i$ such that, for each $i\in\mathbb{Z}_{\ge0}$, $W_{2^{-i}}(x)\le\overline{W}_i$ for all $x\in\mathcal{A}+2^i\mathbb{B}$. Let the strictly decreasing sequence of positive numbers α_i, $i\in\mathbb{Z}_{\ge0}$, satisfy $\sum_{i=0}^\infty\alpha_i\overline{W}_i<\infty$. Define

$$V(x):=\sum_{i=0}^\infty\alpha_i W_{2^{-i}}(x)\ . \tag{9}$$

Since, for each $i\in\mathbb{Z}_{\ge0}$, $W_{2^{-i}}(x)=0$ for all $x\in\mathcal{A}$ it follows that $V(x)=0$ for all $x\in\mathcal{A}$.

Proposition 11: The function V in (9) is well defined, locally bounded, upper semicontinuous, bounded away from zero on closed subsets of $\mathbb{R}^n\backslash\mathcal{A}$, and arbitrarily small near $\mathcal{A}$.

Proof: Let $x\in\mathbb{R}^n$. Let $i^*\in\mathbb{Z}_{\ge0}$ be such that $|x|_{\mathcal{A}}\le2^{i^*}$. Then

$$V(x)=\sum_{i=0}^{i^*-1}\alpha_i W_{2^{-i}}(x)+\sum_{i=i^*}^\infty\alpha_i W_{2^{-i}}(x)\le\sum_{i=0}^{i^*-1}\alpha_i W_{2^{-i}}(x)+\sum_{i=i^*}^\infty\alpha_i\overline{W}_i\ .$$

Since $\sum_{i=0}^\infty\alpha_i\overline{W}_i<\infty$, $V(x)$ is well defined and locally bounded.

Let x_j be a sequence converging to x. Let $\lambda>0$ and let i^* be such that $|x|_{\mathcal{A}}<2^{i^*}$ and $\sum_{i=i^*}^\infty\alpha_i\overline{W}_i\le\lambda$. Then, since $W_{2^{-i}}$ is upper semicontinuous for each $i\in\mathbb{Z}_{\ge0}$,

$$\limsup_{j\to\infty}V(x_j)\le\limsup_{j\to\infty}\left(\sum_{i=0}^{i^*-1}\alpha_i W_{2^{-i}}(x_j)\right)+\sum_{i=i^*}^\infty\alpha_i\overline{W}_i\le\sum_{i=0}^{i^*-1}\alpha_i W_{2^{-i}}(x)+\lambda\le V(x)+\lambda\ .$$

Since $\lambda>0$ is arbitrary, the upper semicontinuity of V follows.

We show that V is bounded away from zero on each closed set $S\subset\mathbb{R}^n\backslash\mathcal{A}$. Let $\ell\in\mathbb{Z}_{\ge0}$ be sufficiently large so that $S\subset\mathbb{R}^n\backslash(\mathcal{A}+2^{-\ell}\mathbb{B}^\circ)=:S_{2^{-\ell}}$. Using Proposition 9, $W_{2^{-\ell}}(\xi)\ge\kappa_{2^{-\ell}}(1)$ for all $\xi\in S_{2^{-\ell}}$. Thus, $V(\xi)\ge\alpha_\ell\kappa_{2^{-\ell}}(1)$ for all $\xi\in S$.

Finally we show that V is arbitrarily small near $\mathcal{A}$. Let $\lambda>0$ be arbitrary. Pick $i^*\in\mathbb{Z}_{\ge0}$ so that $\sum_{i=i^*+1}^\infty\alpha_i\overline{W}_i\le0.5\lambda$. Pick $\widetilde{\lambda}>0$ such that $\sum_{i=0}^{i^*}\alpha_i\widetilde{\lambda}\le0.5\lambda$. According to Proposition 9, there exists $\delta\in(0,1]$ such that $|x|_{\mathcal{A}}\le\delta$ implies $W_{2^{-i}}(x)\le\widetilde{\lambda}$ for all $i\in\{0,\ldots,i^*\}$. Then

$$V(x)\le\textstyle\sum_{i=0}^{i^*}\alpha_i\widetilde{\lambda}+\sum_{i=i^*+1}^\infty\alpha_i\overline{W}_i\le0.5\lambda+0.5\lambda=\lambda\qquad\forall x\in\mathcal{A}+\delta\mathbb{B}\ .$$

It follows that V is arbitrarily small near $\mathcal{A}$. ■

Proposition 12: There exists $\varrho \in \mathcal{PD}$ such that $\int_{\mathbb{R}^m} \max_{g \in G_{\rho,\nu}(x,v)} V(g)\mu(dv) \leq V(x) - \varrho(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.

Proof: Define $\varrho(s) := \sum_{i=0}^{\infty} \alpha_i \varrho_{2^{-i}}(\max\{0, s - 2^{-i}\})$. This function is well defined for the same reason that V is well defined. Indeed, from the bound in Proposition 10 and the fact that $W_{2^{-i}}(g) \geq 0$ for all $g \in \mathbb{R}^n$, it follows that $\varrho_{2^{-i}}(\max\{0, |x|_{\mathcal{A}} - 2^{-i}\}) \leq W_{2^{-i}}(x)$ for all $x \in \mathbb{R}^n$. Thus $\varrho_{2^{-i}}(\max\{0, s - 2^{-i}\}) \leq \overline{W}_i$ for $s \leq 2^i$. From here, the proof that ϱ is well defined and continuous follows the same lines as the proof that V is well defined and upper semicontinuous. Also, $\varrho \in \mathcal{PD}$ since each $\varrho_{2^{-i}} \in \mathcal{PD}$. Using Proposition 10 and the definition of V in (9),

$$\begin{aligned}\int_{\mathbb{R}^m} \max_{g \in G_{\rho,\nu}(x,v)} V(g)\mu(dv) &\leq \int_{\mathbb{R}^m} \sum_{i=0}^{\infty} \alpha_i \max_{g \in G_{\rho,\nu}(x,v)} W_{2^{-i}}(g)\mu(dv) = \sum_{i=0}^{\infty} \alpha_i \int_{\mathbb{R}^m} \max_{g \in G_{\rho,\nu}(x,v)} W_{2^{-i}}(g)\mu(dv) \\ &\leq \sum_{i=0}^{\infty} \alpha_i \left(W_{2^{-i}}(x) - \varrho_{2^{-i}}(\max\left\{0, |x|_{\mathcal{A}} - 2^{-i}\right\})\right) = V(x) - \varrho(|x|_{\mathcal{A}}),\end{aligned}$$

which establishes the result. ■

We use the following result to get the desired lower bound on V.

Proposition 13: For the system (4), for each $\nu \in \mathcal{K}_\infty$, $\rho \in \mathcal{PD}(\mathcal{A})$, and $k \in \mathbb{Z}_{\geq 0}$ there exists $R > 0$ such that $m_{\mathbb{R}^n \setminus (\mathcal{A} + \mathbb{B}^\circ), \cap}(k, \xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus (\mathcal{A} + R\mathbb{B}^\circ)$.

Proof: Define $S := \mathbb{R}^n \setminus (\mathcal{A} + 2\mathbb{B}^\circ)$. From the definition of G_ν in (3) and $G_{\rho,\nu}$ in (4) we have that $\mathcal{A} + \nu(|x|_{\mathcal{A}})\mathbb{B} \subset G_{\rho,\nu}(x, v)$ for all $x \in S$. We claim that, for the system (4),

$$m_{S,\cap}(k, \xi) \geq \mathbb{I}_{[2,\infty)}\left(\min_{i \in \{1,\ldots,k\}} \nu^i(|\xi|_{\mathcal{A}})\right) \qquad \forall (k, \xi) \in \mathbb{Z}_{\geq 0} \times S \ . \tag{10}$$

This bound holds by definition for $k = 0$. Suppose it holds for some $k \in \mathbb{Z}_{\geq 0}$. Then, for $\xi \in S$,

$$\begin{aligned} m_{S,\cap}(k+1, \xi) &= \int_{\mathbb{R}^m} \max_{g \in G_{\rho,\nu}(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(k, g)\mu(dv) \\ &\geq \int_{\mathbb{R}^m} \max_{g \in \mathcal{A} + \nu(|\xi|_{\mathcal{A}})\mathbb{B}} \mathbb{I}_S(g) \mathbb{I}_{[2,\infty)}\left(\min_{i \in \{1,\ldots,k\}} \nu^i(|g|_{\mathcal{A}})\right) \mu(dv) \\ &= \max_{g \in \mathcal{A} + \nu(|\xi|_{\mathcal{A}})\mathbb{B}} \mathbb{I}_S(g) \mathbb{I}_{[2,\infty)}\left(\min_{i \in \{1,\ldots,k\}} \nu^i(|g|_{\mathcal{A}})\right) \\ &= \mathbb{I}_{[2,\infty)}(\nu(|\xi|_{\mathcal{A}})) \mathbb{I}_{[2,\infty)}\left(\min_{i \in \{1,\ldots,k\}} \nu^{i+1}(|\xi|_{\mathcal{A}})\right) = \mathbb{I}_{[2,\infty)}\left(\min_{i \in \{1,\ldots,k+1\}} \nu^i(|\xi|_{\mathcal{A}})\right). \end{aligned}$$

By induction, (10) holds for all $k \in \mathbb{Z}_{\geq 0}$. Now let $k \in \mathbb{Z}_{\geq 0}$ be given. Pick $R > 2$ such that $\min_{i \in \{1,\ldots,k\}} \nu^i(R) \geq 2$, which can be achieved since $\nu^i \in \mathcal{K}_\infty$ for each $i \in \{1, \ldots, k\}$. With this choice, it follows from (10) that $m_{S,\cap}(k, \xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus (\mathcal{A} + R\mathbb{B}^\circ)$. The result of the proposition now follows from $S \subset \mathbb{R}^n \setminus (\mathcal{A} + \mathbb{B}^\circ)$. ■

Corollary 1: When constructed from the system (4) with $\nu \in \mathcal{K}_\infty$ and $\rho \in \mathcal{PD}(\mathcal{A})$ such that $\mathcal{A}$ is globally asymptotically stable in probability, the function W_1 in (5) is radially unbounded.

Proof: Define $S_1 : \mathbb{R}^n \backslash (\mathcal{A} + \mathbb{B}^\circ)$. Using Proposition 13, given $i^* \in \mathbb{Z}_{\geq 0}$, let $R > 0$ be such that $\widetilde{m}_{S_1,\cup}(i^* - 1, x) = 1$ for all $x \in \mathbb{R}^n \backslash (\mathcal{A} + R\mathbb{B}^\circ)$. Then, for $x \in \mathbb{R}^n \backslash (\mathcal{A} + R\mathbb{B}^\circ)$,

$$W_1(x) = \sum_{i=0}^{\infty} \kappa_1(i) \widetilde{m}_{S_1}(i, x) \geq \kappa_1(i^*) \sum_{i=i^*}^{\infty} \widetilde{m}_{S_1}(i, x) = \kappa_1(i^*) \widetilde{m}_{S_1,\cup}(i^* - 1, x) = \kappa_1(i^*) \ .$$

Since $\kappa_1 \in \mathcal{K}_\infty$ and i^* is arbitrary, the corollary follows. ■

Corollary 2: When constructed from the system (4) with $\nu \in \mathcal{K}_\infty$ and $\rho \in \mathcal{PD}(\mathcal{A})$ such that $\mathcal{A}$ is globally asymptotically stable in probability, the function V in (9) is radially unbounded.

Proof: Using (9), $V(x) \geq \alpha_0 W_1(x)$ for all $x \in \mathbb{R}^n$. Corollary 1 now gives the result. ■

The next result follows from the combination of Corollary 2 and Proposition 11.

Proposition 14: There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$ for all $x \in \mathbb{R}^n$.

Proof: For $s > 0$, define $\widetilde{\alpha}_1(s) := \inf_{x \in \mathbb{R}^n \backslash (\mathcal{A} + s\mathbb{B}^\circ)} V(x)$ and $\widetilde{\alpha}_2(s) := \sup_{x \in \mathcal{A} + s\mathbb{B}} V(x)$. Also, set $\widetilde{\alpha}_1(0) = \widetilde{\alpha}_2(0) = 0$. According to Proposition 11, these functions are well defined. By construction, and since $V(\xi) = 0$ for all $\xi \in \mathcal{A}$, they satisfy $\widetilde{\alpha}_1(|\xi|_\mathcal{A}) \leq V(\xi) \leq \widetilde{\alpha}_2(|\xi|_\mathcal{A})$ for all $\xi \in \mathbb{R}^n$. By construction, they are nondecreasing functions of s. Also, due to Proposition 11 and Corollary 2, they approach zero as s approaches zero, they are positive for $s > 0$, and they grow unbounded as $s \to \infty$. That is to say, that have all of the properties of $\mathcal{K}_\infty$ functions except for continuity and strict monotonicity. It is then straightforward to construct $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(s) \leq \widetilde{\alpha}_1(s)$ and $\widetilde{\alpha}_2(s) \leq \alpha_2(s)$ for all $s \geq 0$. These functions establish the proposition. ■

Now we smooth V to get the result of Theorem 1. Let the upper semicontinuous V and the continuous function ϱ satisfy the conditions of Propositions 12 and 14. Following [4], define $V_s(x) := \int_{\mathbb{R}^n} V(x + \sigma(x)\xi)\psi(\xi)d\xi$ and $\varrho_s(x) := \int_{\mathbb{R}^n} \varrho(|x + \sigma(x)\xi|_\mathcal{A})\psi(\xi)d\xi$ for all $x \in \mathbb{R}^n$, where $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is smooth with support on $\mathbb{B}$, and $\sigma \in \mathcal{PD}(\mathcal{A})$ is smooth on $\mathbb{R}^n \backslash \mathcal{A}$. In particular, from the properties of V, ϱ, and σ, we have $V_s(x) = \varrho_s(x) = 0$ for all $x \in \mathcal{A}$. As in [4, (33)-(34)], if we pick σ to satisfy $\sigma(x) \leq 0.5|x|_\mathcal{A}$ for all $x \in \mathbb{R}^n$, then $\varrho_s \in \mathcal{PD}(\mathcal{A})$ and V_s is smooth on $\mathbb{R}^n \backslash \mathcal{A}$, continuous on $\mathbb{R}^n$, and $\alpha_1(0.5|x|_\mathcal{A}) \leq V_s(x) \leq \alpha_2(1.5|x|_\mathcal{A})$ for all $x \in \mathbb{R}^n$. We also choose σ sufficiently small so that

$$\sigma(x) \leq 0.5\rho(x) \leq \rho(x + \sigma(x)\xi) \qquad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B} \ . \tag{11}$$

It follows from (11) that

$$x \in \{x + \sigma(x)\xi\} + \rho(x + \sigma(x)\xi)\mathbb{B} \qquad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B} \ . \tag{12}$$

Then, since $G(x,v) \subset G_\nu(x,v)$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$G_{\rho,\nu}(x+\sigma(x)\xi, v) = \{w : w = \{g\} + \rho(g)\mathbb{B}, g \in G_\nu(x+\sigma(x)\xi + \rho(x+\sigma(x)\xi)\mathbb{B}, v)\} \ ,$$

(11) and (12) imply that if $g \in G(x,v)$, $\tilde{g} = g + \sigma(g)\xi$, $\xi \in \mathbb{B}$ then $\tilde{g} \in G_{\rho,\nu}(x+\sigma(x)\xi, v)$. Consequently, with σ satisfying the constraint (11), for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V_s(g)\mu(dv) &= \int_{\mathbb{R}^m} \max_{g\in G(x,v)} \left(\int_{\mathbb{R}^n} V(g+\sigma(g)\xi)\psi(\xi)d\xi \right) \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \max_{g\in G(x,v)} V(g+\sigma(g)\xi)\psi(\xi)d\xi \right) \mu(dv) \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} \max_{g\in G(x,v)} V(g+\sigma(g)\xi)\mu(dv) \right) \psi(\xi)d\xi \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} \max_{\tilde{g}\in G_{\rho,\nu}(x+\sigma(x)\xi,v)} V(\tilde{g})\mu(dv) \right) \psi(\xi)d\xi \\
&\leq \int_{\mathbb{R}^n} \left(V(x+\sigma(x)\xi) - \varrho(|x+\sigma(x)\xi|_{\mathcal{A}}) \right) \psi(\xi)d\xi = V_s(x) - \varrho_s(x) \ .
\end{aligned}$$

It follows that V_s satisfies the conditions of Theorem 1.

VII. Simple bounds pertaining to reachability and viability probabilities

We collect bounds used to prove Theorems 2 and 3. Bounds that depend more heavily on the Standing Assumption 1 and that are needed for Theorem 3 are deferred to Section IX.

The following result is a special case of [17, Lemma 4].

Lemma 1: Let $S_1, S_2 \subset \mathbb{R}^n$ be closed. Then $m_{S_1\cup S_2,\cap}(j,\xi) \leq m_{S_1,\cap}(j,\xi) + m_{S_2,\cup}(j,\xi)$ for all $(j,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$.

The following result is contained in [17, Lemma 7].

Lemma 2: Let $S, S_1 \subset \mathbb{R}^n$ be closed. For each $(k,\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$,

$$m_{S,\cup}(\ell,\xi) \leq m_{S,\cup}(k,\xi) + m_{S_1,\cap}(k,\xi) + \sup_{\zeta\in\mathbb{R}^n\setminus S_1} m_{S,\cup}(\ell,\zeta) \ .$$

Lemma 3: Let $S, S_1 \subset \mathbb{R}^n$ be closed. For each $(k,\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$,

$$m_{S,\cap}(k+\ell,\xi) \leq m_{S_1,\cap}(k,\xi) + \sup_{\zeta\in\mathbb{R}^n\setminus S_1} m_{S,\cap}(\ell,\zeta) \ . \tag{13}$$

Proof: The bound (13) holds for $k = 0$ and all $(\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ since $m_{S_1,\cap}(0,\xi) = 1$ for all ξ and $m_{S,\cap}(k+\ell,\xi) \leq 1$ for all k, ℓ and ξ. Now suppose that (13) holds for some $k \in \mathbb{Z}_{\geq 0}$

and all $(\ell, \xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$. Then, using $m_{S,\cap}(k+\ell, g) \leq m_{S,\cap}(\ell, g)$ for all $g \in \mathbb{R}^n$, we get that

$$\begin{aligned} m_{S,\cap}(k+\ell, g) &= \mathbb{I}_{\mathbb{R}^n \setminus S_1}(g) m_{S,\cap}(k+\ell, g) + \mathbb{I}_{S_1}(g) m_{S,\cap}(k+\ell, g) \\ &\leq \mathbb{I}_{\mathbb{R}^n \setminus S_1}(g) m_{S,\cap}(k+\ell, g) + \mathbb{I}_{S_1}(g) \left[m_{S_1,\cap}(k, g) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \right] \\ &\leq \mathbb{I}_{\mathbb{R}^n \setminus S_1}(g) m_{S,\cap}(\ell, g) + \mathbb{I}_{S_1}(g) \left[m_{S_1,\cap}(k, g) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \right] \\ \leq \mathbb{I}_{\mathbb{R}^n \setminus S_1}(g) \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) &+ \mathbb{I}_{S_1}(g) \left[m_{S_1,\cap}(k, g) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \right] \\ = \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k, g) &+ \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \ . \end{aligned}$$

Therefore,

$$\begin{aligned} m_{S,\cap}(k+\ell+1, \xi) &= \int_{\mathbb{R}^m} \max_{g \in G(\xi, v)} \mathbb{I}_S(g) m_{S,\cap}(k+\ell, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in G(\xi, v)} \left(\mathbb{I}_{S_1}(g) m_{S_1,\cap}(k, g) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \right) \mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g \in G(\xi, v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k, g) \mu(dv) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \\ &= m_{S_1,\cap}(k+1, \xi) + \sup_{\zeta \in \mathbb{R}^n \setminus S_1} m_{S,\cap}(\ell, \zeta) \ . \end{aligned}$$

The result now follows by induction. ■

The next result establishes that the value $m_{S,\cup}(k, \xi)$ becomes small for fixed k, uniformly over ξ in a compact set, when $S = \mathbb{R}^n \setminus R\mathbb{B}^\circ$ and $R \to \infty$. It exploits local boundedness of G.

Lemma 4: Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. For each $k \in \mathbb{Z}_{\geq 0}$, $\varepsilon > 0$, and $r > 0$ there exists $R > 0$ such that, with $S = \mathbb{R}^n \setminus (\mathcal{A} + R\mathbb{B}^\circ)$, $m_{S,\cup}(k, \xi) \leq \varepsilon$ for all $\xi \in \mathcal{A} + r\mathbb{B}$.

Proof: Let $F \in \mathbf{B}(\mathbb{R}^m)$ be a compact set such that $\mu(\mathbb{R}^m \setminus F) \leq \varepsilon/k$. Let $R_0 = r$ and, for $i \in \{1, \ldots, k\}$, let $R_i > R_{i-1}$ satisfy $G(\mathcal{A} + R_{i-1}\mathbb{B}, F) \subset \mathcal{A} + R_i\mathbb{B}$. Define $R := R_k + 1$ and $S := \mathbb{R}^n \setminus (\mathcal{A} + R\mathbb{B}^\circ)$. By definition, $m_{S,\cup}(0, x) = 0$ for all $x \in \mathcal{A} + R_k\mathbb{B}$. Suppose that, for some $\ell \in \{0, \ldots, k-1\}$, $m_{S,\cup}(\ell, x) \leq \ell\varepsilon/k$ for all $x \in \mathcal{A} + R_{k-\ell}\mathbb{B}$. Then, for all $x \in \mathcal{A} + R_{k-(\ell+1)}\mathbb{B}$,

$$\begin{aligned} m_{S,\cup}(\ell+1, x) &= \int_F \max_{g \in G(x, v)} \max \left\{ \mathbb{I}_S(g), m_{S,\cup}(\ell, g) \right\} \mu(dv) \\ &\qquad + \int_{\mathbb{R}^m \setminus F} \max_{g \in G(x, v)} \max \left\{ \mathbb{I}_S(g), m_{S,\cup}(\ell, g) \right\} \mu(dv) \\ &\leq \max_{g \in \mathcal{A} + R_{k-\ell}\mathbb{B}} \max \left\{ \mathbb{I}_S(g), m_{S,\cup}(\ell, g) \right\} + \varepsilon/k \\ &= \max_{g \in \mathcal{A} + R_{k-\ell}\mathbb{B}} m_{S,\cup}(\ell, g) + \varepsilon/k \leq \ell\varepsilon/k + \varepsilon/k = (\ell+1)\varepsilon/k \ . \end{aligned}$$

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By induction, we conclude that $m_{S,\cup}(k,x) \leq k\varepsilon/k = \varepsilon$ for all $x \in \mathcal{A} + R_{k-k}\mathbb{B} = \mathcal{A} + r\mathbb{B}$. ■

The last two results of this section give a relationship between probabilities, denoted with a subscripts "a" and "b" respectively, for $x^+ \in G_a(x,v)$ and $x^+ \in G_b(x,v)$ when there is an appropriate containment relation between G_a and G_b.

Lemma 5: Let $S \subset S_1 \subset \mathbb{R}^n$ be closed and suppose there exists $F \in \mathbf{B}(\mathbb{R}^m)$ with $\mu(F) = 0$ such that $G_a(\xi,v) \cap S \subset G_b(\xi,v)$ for all $(\xi,v) \in S_1 \times (\mathbb{R}^m \backslash F)$. Then $m_{a,S,\cap}(k,\xi) \leq m_{b,S,\cap}(k,\xi)$ for all $(k,\xi) \in \mathbb{Z}_{\geq 0} \times S_1$.

Proof: The asserted bound holds by definition for $k = 0$ and all $\xi \in S_1$. Now suppose the bound holds for some $k \in \mathbb{Z}_{\geq 0}$ and all $\xi \in S_1$. Let $\xi \in S_1$ and note that

$$\begin{aligned} m_{a,S,\cap}(k+1,\xi) &= \int_{\mathbb{R}^m} \max_{g \in G_a(\xi,v)} \mathbb{I}_S(g) m_{a,S,\cap}(k,g)\mu(dv) \\ &= \int_{\mathbb{R}^m \backslash F} \max_{g \in G_a(\xi,v) \cap S} \mathbb{I}_S(g) m_{a,S,\cap}(k,g)\mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in G_b(\xi,v)} \mathbb{I}_S(g) m_{b,S,\cap}(k,g)\mu(dv) = m_{b,S,\cap}(k+1,\xi) \ . \end{aligned}$$

The lemma now follows by induction. ■

Lemma 6: Let $S, S_1 \subset \mathbb{R}^n$ be closed and satisfy $\mathbb{R}^n \backslash S \subset S_1$, and suppose there exists $F \in \mathbf{B}(\mathbb{R}^m)$ with $\mu(F) = 0$ such that, for each $(\xi,v) \in S_1 \times (\mathbb{R}^m \backslash F)$, either $G_a(\xi,v) \subset G_b(\xi,v)$ or $G_b(\xi,v) \cap S \neq \varnothing$. Then $m_{a,S,\cup}(k,\xi) \leq m_{b,S,\cup}(k,\xi)$ for all $(k,\xi) \in \mathbb{Z}_{\geq 0} \times S_1$.

Proof: The asserted bound holds by definition for $k = 0$ and all $\xi \in S_1$. Now suppose the bound holds for some $k \in \mathbb{Z}_{\geq 0}$ and all $\xi \in S_1$. Let $(\xi,v) \in S_1 \times (\mathbb{R}^m \backslash F)$ and note that

$$\begin{aligned} \max_{g \in G_a(\xi,v)} \max\left\{\mathbb{I}_S(g), m_{a,S,\cup}(k,g)\right\} &= \max_{g \in G_a(\xi,v)} \max\left\{\mathbb{I}_S(g), \mathbb{I}_{\overline{\mathbb{R}^n \backslash S}}(g) m_{a,S,\cup}(k,g)\right\} \\ &\leq \max_{g \in G_a(\xi,v)} \max\left\{\mathbb{I}_S(g), \mathbb{I}_{S_1}(g) m_{a,S,\cup}(k,g)\right\} \\ &\leq \max_{g \in G_a(\xi,v)} \max\left\{\mathbb{I}_S(g), m_{b,S,\cup}(k,g)\right\} \\ &\leq \max_{g \in G_b(\xi,v)} \max\left\{\mathbb{I}_S(g), m_{b,S,\cup}(k,g)\right\} \end{aligned}$$

where the last inequality comes from $G_a(\xi,v) \subset G_b(\xi,v)$ or $G_b(\xi,v) \cap S \neq \varnothing$, the latter condition implying that the last quantity is equal to one. The lemma now follows by induction after integrating both sides of the preceding inequality and using $\mu(F) = 0$. ■

VIII. Proof of Theorem 2

We use the subscript ν in probabilities for (3). For $r > 0$, define $S_r := \mathbb{R}^n \backslash (\mathcal{A} + r\mathbb{B}^\circ)$. Since $G_\nu(x,v) = G(x,v)$ for all $x \in \mathcal{A} + 2\mathbb{B}^\circ$ and $\mathcal{A}$ is stable in probability for (1), due to Lemma 6

there exists $\delta > 0$ and, for each $r > 0$, there exists $\alpha_r \in \mathcal{K}_\infty$ such that, for each $\nu \in \mathcal{K}_\infty$,

$$\widehat{m}_{\nu,S_r,\cup}(\xi) \leq \alpha_r(|\xi|_{\mathcal{A}}) \qquad \forall \xi \in \mathcal{A} + \delta\mathbb{B} \ . \tag{14}$$

In particular, the choice of $\nu \in \mathcal{K}_\infty$ does not affect the characterization of stability in probability of $\mathcal{A}$ for (3). Therefore, according to Proposition 5, it is enough to show that for each $r > 0$ there exists $\varrho_r \in [0, 1)$ such that $\widehat{m}_{\nu,S_r,\cap}(\xi) \leq \varrho_r$ for all $\xi \in \mathbb{R}^n$.

Since $G_\nu(x, v) = G(x, v)$ for all $x \in \mathcal{A} + 2\mathbb{B}^\circ$ and since $\mathcal{A}$ is globally asymptotically stable in probability for (1), Proposition 5 implies that for each $r > 0$ there exists $\ell_0 \in \mathbb{Z}_{\geq 0}$ such that

$$m_{\nu,(\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ),\cap}(\ell_0, \xi) \leq m_{(\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ),\cap}(\ell_0, \xi) \leq 0.25 \qquad \forall \xi \in \mathcal{A} + \delta\mathbb{B} \ . \tag{15}$$

Let $\delta_1 \in (0, \delta)$ be such that $\alpha_1(\delta_1) \leq 0.25$ where $\alpha_1 \in \mathcal{K}_\infty$ satisfies (14) with $r = 1$. Define $\bar{r} := \max\{r, 1\}$. Note that $S_r = S_{\bar{r}} \cup ((\mathcal{A} + \mathbb{B})\backslash(\mathcal{A} + r\mathbb{B}^\circ))$ and $S_{\bar{r}} \subset S_1 := \mathbb{R}^n\backslash(\mathcal{A} + \mathbb{B}^\circ)$. It then follows from (14), (15), and Lemma 1, that, for all $r > 0$ and $\xi \in \mathcal{A} + \delta_1\mathbb{B}$,

$$m_{\nu,S_r,\cap}(\ell_0, \xi) \leq m_{\nu,(\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ),\cap}(\ell_0, \xi) + m_{\nu,S_{\bar{r}},\cup}(\ell_0, \xi) \leq 0.25 + 0.25 = 0.5 \ . \tag{16}$$

Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ satisfy $\varepsilon_1 + \varepsilon_2 < 0.5$. Let $\bar{m}_{i,\cap} \in (0, 1)$ be a monotonically decreasing sequence such that $\sum_{i=0}^{\infty} \bar{m}_{i,\cap} \leq \varepsilon_1$. Let $\bar{m}_{i,\cup} \in (0, 1)$ be a monotonically decreasing sequence such that $\sum_{i=0}^{\infty} \bar{m}_{i,\cup} \leq \varepsilon_2$. For $i \in \mathbb{Z}_{\geq 0}$, let $r_i = \delta_1 2^i$ and pick ℓ_{i+1} sufficiently large so that

$$m_{S_{r_i},\cap}(\ell_{i+1}, x) \leq \bar{m}_{i,\cap} \qquad \forall x \in \mathcal{A} + r_{i+1}\mathbb{B} \ . \tag{17}$$

The values ℓ_i exist according to Proposition 5 and since $\mathcal{A}$ is globally asymptotically stable in probability for (1). Pick $\beta_i > r_{i+1}$ sufficiently large so that, with $\nu(s) = s$ for all $s \geq 0$,

$$m_{\nu,S_{\beta_i},\cup}(\ell_{i+1}, x) \leq \bar{m}_{i,\cup} \qquad \forall x \in \mathcal{A} + r_{i+1}\mathbb{B} \ . \tag{18}$$

The values β_i exist according to Lemma 4. Without loss of generality, we assume $i \mapsto \beta_i$ is strictly increasing. Let $\nu \in \mathcal{K}_\infty$ satisfy $\nu(s) \leq s$ for all $s \geq 0$ and $\nu(\beta_i) < r_i$ for all $i \in \mathbb{Z}_{\geq 0}$, which is possible since the values r_i are positive, strictly increasing, and unbounded.

Define $K_i := \overline{S_{r_i}\backslash S_{\beta_i}} = (\mathcal{A} + \beta_i\mathbb{B}) \setminus (\mathcal{A} + r_i\mathbb{B}^\circ)$, and note that, since $\nu(\beta_i) < r_i$,

$$\max_{g \in \mathcal{A} + \nu(\beta_i)\mathbb{B}} \mathbb{I}_{K_i}(g) = 0 \ . \tag{19}$$

Now we show that, for all $\xi \in \mathcal{A} + \beta_i\mathbb{B}$ and all $k \in \mathbb{Z}_{\geq 0}$,

$$m_{\nu,K_i,\cap}(k, \xi) = m_{K_i,\cap}(k, \xi) \ . \tag{20}$$

The equality (20) holds for $k = 0$ and all $\xi \in \mathbb{R}^n$. Suppose (20) holds for some $k \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + \beta_i \mathbb{B}$. Using (19), $G(\xi, v) \subset G_\nu(\xi, v)$ for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$, and $K_i \subset \mathcal{A} + \beta_i \mathbb{B}$, it follows that if $\xi \in \mathcal{A} + \beta_i \mathbb{B}$ then

$$\begin{aligned}\max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{K_i,\cap}(k, g) &\leq \max_{g \in G_\nu(\xi,v)} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) \\ &\leq \max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) + \max_{g \in \mathcal{A} + \nu(|\xi|_{\mathcal{A}})\mathbb{B}} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) \\ &= \max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) = \max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{K_i,\cap}(k, g) \ .\end{aligned}$$

Thus $\max_{g \in G_\nu(\xi,v)} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) = \max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{K_i,\cap}(k, g)$ for all $\xi \in \mathcal{A} + \beta_i \mathbb{B}$. In turn,

$$\begin{aligned}m_{\nu,K_i,\cap}(k+1, \xi) &= \int_{\mathbb{R}^m} \max_{g \in G_\nu(\xi,v)} \mathbb{I}_{K_i}(g) m_{\nu,K_i,\cap}(k, g) \mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g \in G(\xi,v)} \mathbb{I}_{K_i}(g) m_{K_i,\cap}(k, g) \mu(dv) = m_{K_i,\cap}(k+1, \xi) \quad \forall \xi \in \mathcal{A} + \beta_i \mathbb{B}.\end{aligned}$$

Hence, by induction, (20) holds for all $k \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + \beta_i \mathbb{B}$.

Using Lemmas 1 and 6, (17), (18) and (20), we get, for each $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_{i+1} \mathbb{B}$,

$$\begin{aligned}m_{\nu,S_{r_i},\cap}(\ell_{i+1}, \xi) &\leq m_{\nu,S_{\beta_i},\cup}(\ell_{i+1}, \xi) + m_{\nu,\overline{S_{r_i} \setminus S_{\beta_i}},\cap}(\ell_{i+1}, \xi) \\ &\leq \bar{m}_{i,\cup} + m_{K_i,\cap}(\ell_{i+1}, \xi) \\ &\leq \bar{m}_{i,\cup} + m_{S_{r_i},\cap}(\ell_{i+1}, \xi) \leq \bar{m}_{i,\cup} + \bar{m}_{i,\cap} \ .\end{aligned} \tag{21}$$

Next, we claim that for all $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i \mathbb{B}$,

$$m_{\nu,S_r,\cap}\left(\sum_{j=0}^{i} \ell_j, \xi\right) \leq 0.5 + \sum_{j=0}^{i-1} (\bar{m}_{j,\cup} + \bar{m}_{j,\cap}) \ . \tag{22}$$

The bound (22) holds for $i = 0$ and all $\xi \in \mathcal{A} + r_0 \mathbb{B} = \mathcal{A} + \delta_1 \mathbb{B}$ according to (16). Now suppose the bound (22) holds for some $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i \mathbb{B}$. Then, using (22) and (21) with this i, and Lemma 3, we get for all $\xi \in \mathcal{A} + r_{i+1} \mathbb{B}$,

$$\begin{aligned}m_{\nu,S_r,\cap}\left(\sum_{j=0}^{i+1} \ell_j, \xi\right) &\leq m_{\nu,S_{r_i},\cap}(\ell_{i+1}, \xi) + \sup_{\zeta \in \mathbb{R}^n \setminus S_{r_i}} m_{\nu,S_r,\cap}\left(\sum_{j=0}^{i} \ell_j, \zeta\right) \\ &\leq m_{\nu,S_{r_i},\cap}(\ell_{i+1}, \xi) + \sup_{\zeta \in \mathcal{A} + r_i \mathbb{B}} m_{\nu,S_r,\cap}\left(\sum_{j=0}^{i} \ell_j, \zeta\right) \\ &\leq \bar{m}_{i,\cup} + \bar{m}_{i,\cap} + 0.5 + \sum_{j=0}^{i-1} (\bar{m}_{j,\cup} + \bar{m}_{j,\cap}) = 0.5 + \sum_{j=0}^{i} (\bar{m}_{j,\cup} + \bar{m}_{j,\cap}) \ .\end{aligned}$$

It now follows by induction that for each $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i \mathbb{B}$, (22) holds. In turn, we conclude that, for each $r > 0$ and each $\xi \in \mathbb{R}^n$, $\widehat{m}_{\nu,S_r,\cap}(\xi) \leq 0.5 + \sum_{j=0}^{\infty} (\bar{m}_{j,\cup} + \bar{m}_{j,\cap}) =: \varrho < 1$. This bound establishes almost sure global asymptotic recurrence of $\mathcal{A}$ for (3).

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IX. The effect of pertubations on reachability and viability probabilities

We use the subscript "ρ" for probabilities for $x^+ \in G_\rho(x,v) := G(x+\rho\mathbb{B}, v)+\rho\mathbb{B}$ where $\rho > 0$.

A. Perturbed Reachability

Claim 1: For each closed set $S_1 \subset \mathbb{R}^n$, compact set $S_2 \subset \mathbb{R}^n$, $k \in \mathbb{Z}_{\geq 0}$, and $\varepsilon > 0$ there exists $\rho > 0$ such that, for all $\xi \in S_2 + \rho\mathbb{B}$,

$$\int_{\mathbb{R}^m} \max_{g \in G_\rho(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{S_1,\cup}(k,g)\right\} \mu(dv) \leq \max_{\zeta \in (\xi+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cup}(k+1,\zeta) + \varepsilon \ .$$

Proof: If the claim is false then for some closed set S_1, compact set S_2, $k \in \mathbb{Z}_{\geq 0}$, $\varepsilon > 0$, and each $i \in \mathbb{Z}_{\geq 1}$, there exists $\xi_i \in S_2 + (1/i)\mathbb{B}$ such that

$$\int_{\mathbb{R}^m} \max_{g \in G_{1/i}(\xi_i,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{S_1,\cup}(k,g)\right\} \mu(dv) > \max_{\zeta \in (\xi_i+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cup}(k+1,\zeta) + \varepsilon \ . \tag{23}$$

Without loss of generality, we assume that the sequence ξ_i converges to some $\xi \in S_2$, and so, for i large enough, $\xi \in (\xi_i + \varepsilon\mathbb{B}) \cap S_2$. By the outer semicontinuity of G for fixed v and Fatou's Lemma, with $\phi(k,g) := \max\left\{\mathbb{I}_{S_1}(g), m_{S_1,\cup}(k,g)\right\}$ we have

$$\begin{aligned}\limsup_{i\to\infty} \int_{\mathbb{R}^m} \max_{g \in G_{1/i}(\xi_i,v)} \phi(k,g)\mu(dv) &\leq \int_{\mathbb{R}^m} \max_{g \in G(\xi,v)} \phi(k,g)\mu(dv) = m_{S_1,\cup}(k+1,\xi) \\ &\leq \max_{\zeta \in (\xi_i+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cup}(k+1,\zeta) \ .\end{aligned}$$

This bound contradicts (23) for sufficiently large i and thus establishes the claim. ■

Lemma 7: For each closed set $S_1 \subset \mathbb{R}^n$ such that $\overline{\mathbb{R}^n \backslash S_1}$ is compact, each compact set $S_2 \subset \mathbb{R}^n$, $\ell \in \mathbb{Z}_{\geq 0}$, and $\varepsilon > 0$ there exists $\rho > 0$ such that, for all $k \in \{0, \ldots, \ell\}$,

$$m_{\rho,S_1,\cup}(k,\xi) \leq \max_{\zeta \in (\xi+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cup}(k,\zeta) + \varepsilon \qquad \forall \xi \in S_2 + \rho\mathbb{B} \ . \tag{24}$$

Proof: The proof exploits the fact that, for a closed set $S_1 \subset \mathbb{R}^n$, positive real numbers ρ_1 and ρ_2, and an upper semicontinuous function $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$\max_{g \in G_{\rho_1}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \max_{\zeta \in g+\rho_2\mathbb{B}} \phi(\zeta)\right\} \leq \max_{g \in G_{\rho_1+\rho_2}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \phi(g)\right\} \ . \tag{25}$$

To see that this fact holds, let $g^* \in G_{\rho_1}(\xi,v)$ and $\zeta^* \in g^* + \rho_2\mathbb{B}$ satisfy

$$\max_{g \in G_{\rho_1}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \max_{\zeta \in g+\rho_2\mathbb{B}} \phi(\zeta)\right\} = \max\left\{\mathbb{I}_{S_1}(g^*), \phi(\zeta^*)\right\} \ . \tag{26}$$

Since both ζ^* and g^* belong to $G_{\rho_1+\rho_2}(\xi,v)$, we have

$$\max\left\{\mathbb{I}_{S_1}(g^*), \phi(\zeta^*)\right\} \leq \max_{g \in G_{\rho_1+\rho_2}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \phi(g)\right\} \ . \tag{27}$$

Combining (26) and (27) gives (25).

Let S_1 be closed, S_2 compact, and $\ell \in \mathbb{Z}_{\geq 0}$. By the definitions of $m_{\rho,S_1,\cup}(0,\xi)$ and $m_{S_1,\cup}(0,\xi)$, the bound (24) holds, both with S_2 and also with $S_3 := \overline{\mathbb{R}^n \backslash S_1}$ in place of S_2, for $k = 0$ with $\varepsilon = 0$ for any $\rho > 0$. Suppose for some $k \in \{0, \ldots, \ell\}$ that for each $\varepsilon_k > 0$ there exists $\rho_k > 0$ such that (24) holds, for both S_2 and also with S_3 in place of S_2, with $\varepsilon = \varepsilon_k$ and $\rho = \rho_k$.

Let $\varepsilon_{k+1} > 0$ be arbitrary. Let $\rho > 0$ satisfy the conclusion of Claim 1 for $\varepsilon = 0.5\varepsilon_{k+1}$ for the given S_2 and also for $S_2 = S_3$. Pick $\varepsilon_k = 0.5 \min\{\varepsilon_{k+1}, \rho\}$. Pick $\rho_k > 0$ such that (24) holds, for both S_2 and also with $S_2 = S_3$, with $\varepsilon = \varepsilon_k$ and $\rho = \rho_k$. Pick $\rho_{k+1} = \min\{\rho_k, 0.5\rho\}$. Now let $\xi \in S_i + \rho_{k+1}\mathbb{B}$, where $i \in \{2, 3\}$, and note that

$$
\begin{aligned}
m_{\rho_{k+1},S_1,\cup}(k+1,\xi) &= \int_{\mathbb{R}^m} \max_{g \in G_{\rho_{k+1}}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{\rho_{k+1},S_1,\cup}(k,g)\right\} \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{\rho_k,S_1,\cup}(k,g)\right\} \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \left(\max_{\zeta \in (g+\varepsilon_k\mathbb{B}) \cap \overline{\mathbb{R}^n \backslash S_1}} m_{S_1,\cup}(k,\zeta) + \varepsilon_k\right)\right\} \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), \left(\max_{\zeta \in g+\varepsilon_k\mathbb{B}} m_{S_1,\cup}(k,\zeta) + \varepsilon_k\right)\right\} \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho+\varepsilon_k}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{S_1,\cup}(k,g)\right\} \mu(dv) + \varepsilon_k \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{\rho}(\xi,v)} \max\left\{\mathbb{I}_{S_1}(g), m_{S_1,\cup}(k,g)\right\} \mu(dv) + 0.5\varepsilon_{k+1} \\
&\leq \max_{\zeta \in (\xi+0.5\varepsilon_{k+1}\mathbb{B}) \cap S_i} m_{S_1,\cup}(k+1,\zeta) + 0.5\varepsilon_{k+1} + 0.5\varepsilon_{k+1} \\
&\leq \max_{\zeta \in (\xi+\varepsilon_{k+1}\mathbb{B}) \cap S_i} m_{S_1,\cup}(k+1,\zeta) + \varepsilon_{k+1} \ .
\end{aligned}
$$

The result of the lemma now follows by induction. ■

B. Perturbed Viability

Claim 2: For each pair of compact sets $S_1, S_2 \subset \mathbb{R}^n$, each $k \in \mathbb{Z}_{\geq 0}$, and each $\varepsilon > 0$ there exists $\rho > 0$ such that, for all $\xi \in S_2 + \rho\mathbb{B}$,

$$
\int_{\mathbb{R}^m} \max_{g \in G_\rho(\xi,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g) \mu(dv) \leq \max_{\zeta \in (\xi+\varepsilon\mathbb{B}) \cap S_2} m_{S_1,\cap}(k+1,\zeta) + \varepsilon \ .
$$

Proof: Suppose the claim is false, in other words, for some pair of compact sets $S_1, S_2 \subset \mathbb{R}^n$, some $k \in \mathbb{Z}_{\geq 0}$, some $\varepsilon > 0$, and each $i \in \mathbb{Z}_{\geq 0}$, there exists $\xi_i \in S_2 + (1/i)\mathbb{B}$ such that

$$
\int_{\mathbb{R}^m} \max_{g \in G_{1/i}(\xi_i,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g) \mu(dv) > \max_{\zeta \in (\xi_i+\varepsilon\mathbb{B}) \cap S_2} m_{S_1,\cap}(k+1,\zeta) + \varepsilon \ . \tag{28}
$$

Without loss of generality, suppose the sequence ξ_i converges to some $\xi \in S_2$, and so, for i large enough, $\xi \in (\xi_i + \varepsilon\mathbb{B}) \cap S_2$. By the outer semicontinuity of G for fixed v and Fatou's Lemma

$$\begin{aligned}\limsup_{i\to\infty} \int_{\mathbb{R}^m} \max_{g\in G_{1/i}(\xi_i,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g)\mu(dv) &\le \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g)\mu(dv) \\ &= m_{S_1,\cap}(k+1,\xi) \le \max_{\zeta\in(\xi_i+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cap}(k+1,\zeta)\ .\end{aligned}$$

This bound contradicts (28) for sufficiently large i, establishing the claim. ■

Lemma 8: For each pair of compact sets $S_1, S_2 \subset \mathbb{R}^n$, each $\ell \in \mathbb{Z}_{\ge 0}$, and each $\varepsilon > 0$ there exists $\rho > 0$ such that, for all $k \in \{0,\dots,\ell\}$,

$$m_{\rho,S_1,\cap}(k,\xi) \le \max_{\zeta\in(\xi+\varepsilon\mathbb{B})\cap S_2} m_{S_1,\cap}(k,\zeta) + \varepsilon \qquad \forall \xi \in S_2 + \rho\mathbb{B}\ . \tag{29}$$

Proof: This proof exploits the fact that, for a closed set $S_1 \subset \mathbb{R}^n$, positive real numbers ρ_1 and ρ_2, and an upper semicontinuous function $\phi : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$,

$$\max_{g\in G_{\rho_1}(\xi,v)} \mathbb{I}_{S_1}(g) \max_{\zeta\in(g+\rho_2\mathbb{B})\cap S_1} \phi(\zeta) \le \max_{g\in G_{\rho_1+\rho_2}(\xi,v)} \mathbb{I}_{S_1}(g)\phi(g)\ . \tag{30}$$

Indeed, let $g^* \in G_{\rho_1}(\xi,v)$ and $\zeta^* \in (g^* + \rho_2\mathbb{B}) \cap S_1$ satisfy

$$\max_{g\in G_{\rho_1}(\xi,v)} \mathbb{I}_{S_1}(g) \max_{\zeta\in(g+\rho_2\mathbb{B})\cap S_1} \phi(\zeta) = \mathbb{I}_{S_1}(g^*)\phi(\zeta^*)\ . \tag{31}$$

Since $\zeta^* \in S_1$ and $\zeta^* \in G_{\rho_1+\rho_2}(\xi,v)$, we get $\mathbb{I}_{S_1}(g^*)\phi(\zeta^*) = \mathbb{I}_{S_1}(\zeta^*)\phi(\zeta^*) \le \max_{g\in G_{\rho_1+\rho_2}(\xi,v)} \mathbb{I}_{S_1}(g)\phi(g)$. Combining this bound with (31) gives (30).

Let $S_1, S_2 \subset \mathbb{R}^n$ be given compact sets and $\ell \in \mathbb{Z}_{\ge 0}$. By the definitions of $m_{\rho,S_1,\cap}(0,\xi)$ and $m_{S_1,\cap}(0,\xi)$, the bound (29) holds, both with S_2 and also with S_1 in place of S_2, for $k = 0$ with $\varepsilon = 0$ for any $\rho > 0$. Suppose for some $k \in \{0,\dots,\ell\}$ that for each $\varepsilon_k > 0$ there exists $\rho_k > 0$ such that (29) holds, for both S_2 and also with S_1 in place of S_2, with $\varepsilon = \varepsilon_k$ and $\rho = \rho_k$.

Let $\varepsilon_{k+1} > 0$ be arbitrary. Let $\rho > 0$ satisfy the conclusion of Claim 2 for $\varepsilon = 0.5\varepsilon_{k+1}$ for the given S_2 and also for $S_2 = S_1$. Pick $\varepsilon_k = 0.5\min\{\varepsilon_{k+1},\rho\}$. Pick $\rho_k > 0$ such that (29) holds, for both S_2 and also with $S_2 = S_1$, with $\varepsilon = \varepsilon_k$ and $\rho = \rho_k$. Pick $\rho_{k+1} = \min\{\rho_k, 0.5\rho\}$. Now

let $\xi \in S_i + \rho_{k+1}\mathbb{B}$, where $i \in \{1,2\}$, and note that

$$\begin{aligned}
m_{\rho_{k+1},S_1,\cap}(k+1,\xi) &= \int_{\mathbb{R}^m} \max_{g \in G_{\rho_{k+1}}(\xi,v)} \mathbb{I}_{S_1}(g) m_{\rho_{k+1},S_1,\cap}(k,g)\mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho}(\xi,v)} \mathbb{I}_{S_1}(g) m_{\rho_k,S_1,\cap}(k,g)\mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho}(\xi,v)} \mathbb{I}_{S_1}(g) \left(\max_{\zeta \in (g+\varepsilon_k\mathbb{B})\cap S_1} m_{S_1,\cap}(k,\zeta) + \varepsilon_k \right) \mu(dv) \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{0.5\rho+\varepsilon_k}(\xi,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g)\mu(dv) + \varepsilon_k \\
&\leq \int_{\mathbb{R}^m} \max_{g \in G_{\rho}(\xi,v)} \mathbb{I}_{S_1}(g) m_{S_1,\cap}(k,g)\mu(dv) + 0.5\varepsilon_{k+1} \\
&\leq \max_{\zeta \in (\xi+0.5\varepsilon_{k+1}\mathbb{B})\cap S_i} m_{S_1,\cap}(k+1,\zeta) + 0.5\varepsilon_{k+1} + 0.5\varepsilon_{k+1} \\
&\leq \max_{\zeta \in (\xi+\varepsilon_{k+1}\mathbb{B})\cap S_i} m_{S_1,\cap}(k+1,\zeta) + \varepsilon_{k+1} \ .
\end{aligned}$$

The result of the lemma now follows by induction. ■

Corollary 3: For each closed set $S \subset \mathbb{R}^n$, compact set $K \subset \mathbb{R}^n$, $\ell \in \mathbb{Z}_{\geq 0}$, and $\varepsilon > 0$ there exists $\rho > 0$ such that, for all $k \in \{0,\ldots,\ell\}$,

$$m_{\rho,S,\cap}(k,\xi) \leq \max_{\zeta \in (\xi+\varepsilon\mathbb{B})\cap K} m_{S,\cap}(k,\zeta) + \varepsilon \qquad \forall \xi \in K + \rho\mathbb{B} \ .$$

Proof: Since $G_\rho(x,v) \subset G_1(x,v)$ for all $\rho \in (0,1]$, we can use Lemma 4 with $\mathcal{A} = \{0\}$ and Lemma 6 to assert that for each $\varepsilon > 0$ there exists $R > 0$ such that, for all $\rho \in (0,1]$,

$$m_{\rho,\mathbb{R}^n\setminus R\mathbb{B}^\circ,\cup}(k,\xi) \leq \varepsilon \qquad \forall k \in \{0,\ldots,\ell\} \ , \ \xi \in K + \mathbb{B} \ .$$

Then defining $S_1 := S \cap R\mathbb{B}$ and $S_2 := K$ and applying Lemma 8 and Lemma 1, we can assert that, for each $\ell \in \mathbb{Z}_{\geq 0}$ and each $\varepsilon > 0$ there exists $\rho \in (0,1]$ such that, for all $k \in \{0,\ldots,\ell\}$,

$$m_{\rho,S,\cap}(k,\xi) \leq \max_{\zeta \in (\xi+\varepsilon\mathbb{B})\cap K} m_{S,\cap}(k,\zeta) + \varepsilon \qquad \xi \in K + \rho\mathbb{B} \ .$$

This bound is the result of the corollary. ■

The next result combines Corollary 3 and Proposition 5.

Corollary 4: If the compact set $\mathcal{A} \subset \mathbb{R}^n$ is almost surely globally asymptotically recurrent for (1) then for each pair of positive real numbers $r_{i+1} > r_i$ and each $\varepsilon > 0$ there exist $\ell \in \mathbb{Z}_{\geq 0}$ and $\rho > 0$ such that $m_{\rho,\mathbb{R}^n\setminus(\mathcal{A}+r_i\mathbb{B})^\circ,\cap}(\ell,\xi) \leq \varepsilon$ for all $\xi \in \mathcal{A} + r_{i+1}\mathbb{B}$.

X. Proof of Theorem 3

Throughout this proof, we use the subscript "ρ" to refer to probabilities for the system (4). The proof is split into two parts. In the first part, we construct a continuous function $\rho_{out} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is zero for all $x \in \mathcal{A} + 2\mathbb{B}$ and positive otherwise. We refer to this case as "perturbations out to infinity". In the second part, we construct a continuous function $\rho_{in} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is zero for all $x \in \mathbb{R}^n \backslash (\mathcal{A} + 3\mathbb{B}^\circ)$ and for $x \in \mathcal{A}$ and is positive otherwise. We refer to this case as "perturbations in toward the set $\mathcal{A}$". The result then follows by taking $\rho(x) = \min\{\rho_{in}(x), \rho_{out}(x)\}$. Throughout the proof, for each $r > 0$, $S_r := \mathbb{R}^n \backslash (\mathcal{A} + r\mathbb{B}^\circ)$.

1) Perturbations out to infinity: This part of the proof is very similar to the proof of Theorem 2. Due to the assumed global asymptotic stability in probability for (1) and Lemma 6, there exists $\delta > 0$ and, for each $r > 0$, there exists $\alpha_r \in \mathcal{K}_\infty$ such that, for each continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying $\rho(x) = 0$ for all $x \in \mathcal{A} + 2\mathbb{B}$,

$$\widehat{m}_{\rho, S_r, \cup}(\xi) \leq \alpha_r(|\xi|_{\mathcal{A}}) \qquad \forall \xi \in \mathcal{A} + \delta\mathbb{B} \ . \tag{32}$$

Hence, the set $\mathcal{A}$ is stable in probability regardless of how we choose ρ on $\mathbb{R}^n \backslash (\mathcal{A} + 2\mathbb{B})$. We show that there exists a continuous, nonnegative function ρ that is positive on $\mathbb{R}^n \backslash (\mathcal{A} + 2\mathbb{B})$ and preserves almost sure global asymptotic recurrence of $\mathcal{A}$. That is, there exists such a function ρ such that $\widehat{m}_{\rho, S_r, \cap}(\xi) = 0$ for each $r > 0$ and all $\xi \in S_r$. Due to Proposition 5, it is enough to show that for each $r > 0$ there exists $\varrho_r \in [0, 1)$ such that $\widehat{m}_{\rho, S_r, \cap}(\xi) \leq \varrho_r$ for all $\xi \in S_r$.

Since $G_{\rho,\nu}(x, v) = G_\nu(x, v)$ for all $(x, v) \in (\mathcal{A} + 2\mathbb{B}) \times \mathbb{R}^m$ and $\mathcal{A}$ is globally asymptotically stability in probability for (3), it follows from Proposition 5 and Lemma 5 that for each $r > 0$ there exists $\ell_0 \in \mathbb{Z}_{\geq 0}$ such that, for each function ρ that is zero on $\mathcal{A} + 2\mathbb{B}$,

$$m_{\rho, (\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ), \cap}(\ell_0, \xi) \leq m_{(\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ), \cap}(\ell_0, \xi) \leq 0.25 \qquad \forall \xi \in \mathcal{A} + \delta\mathbb{B} \ . \tag{33}$$

Let $\delta_1 \in (0, \min\{\delta, 1\})$ be such that $\alpha_1(\delta_1) \leq 0.25$ where $\alpha_1 \in \mathcal{K}_\infty$ satisfies (32) with $r = 1$. Define $\bar{r} := \max\{r, 1\}$. Note that $S_r = S_{\bar{r}} \cup ((\mathcal{A} + \mathbb{B}) \backslash (\mathcal{A} + r\mathbb{B}^\circ))$ and $S_{\bar{r}} \subset S_1 := \mathbb{R}^n \backslash (\mathcal{A} + \mathbb{B}^\circ)$. It then follows from (32), (33), Lemma 1 and Lemma 5 that, for all $r > 0$ and $\xi \in \mathcal{A} + \delta_1\mathbb{B}$, and for each function ρ that is zero on $\mathcal{A} + 2\mathbb{B}$,

$$\begin{aligned} m_{\rho, S_r, \cap}(\ell_0, \xi) &\leq m_{\rho, (\mathcal{A}+\mathbb{B})\backslash(\mathcal{A}+r\mathbb{B}^\circ), \cap}(\ell_0, \xi) + m_{\rho, S_{\bar{r}}, \cup}(\ell_0, \xi) \\ &\leq 0.25 + \widehat{m}_{\rho, S_1, \cup}(\xi) \leq 0.25 + \alpha_1(\delta_1) \leq 0.5 \ . \end{aligned} \tag{34}$$

Let $\varepsilon_1 \in (0, 0.5)$. Let $\bar{m}_{i,\cap} \in (0,1)$ be a monotonically decreasing sequence such that $\sum_{i=0}^{\infty} \bar{m}_{i,\cap} \leq \varepsilon_1$. For $i \in \mathbb{Z}_{\geq 0}$, let $r_i = \delta_1 2^i$ and, using Corollary 4, pick ℓ_{i+1} sufficiently large and $\rho_i > 0$ sufficiently small so that

$$m_{\rho_i, S_{r_i}, \cap}(\ell_{i+1}, x) \leq \bar{m}_{i,\cap} \qquad \forall x \in \mathcal{A} + r_{i+1}\mathbb{B} \ . \tag{35}$$

Without loss of generality, assume $i \mapsto \rho_i$ is monotonically nonincreasing. Let $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be continuous, positive on $\mathbb{R}^n \backslash (\mathcal{A} + 2\mathbb{B})$, and such that $\rho(\xi) = 0$ for all $\xi \in \mathcal{A} + 2\mathbb{B}$ and

$$\rho(\xi) \leq \min\left\{0.5|\xi|_{\mathcal{A}}, \rho_i\right\} \qquad \forall \xi \in S_{0.5r_i} \ . \tag{36}$$

For example, pick $\rho(\xi) := \inf_{z \in \mathbb{R}^n} \left(\widetilde{\rho}(z) + |z - \xi|\right)$ with $\widetilde{\rho}(z) := \min\left\{\max\left\{0, 0.5|z|_{\mathcal{A}} - 1\right\}, \widehat{\rho}(z)\right\}$ and $\widehat{\rho}(z) = \rho_i$ for all $z \in (\mathcal{A} + 0.5r_{i+1}\mathbb{B}) \backslash (\mathcal{A} + 0.5r_i\mathbb{B}^\circ)$. It follows from $\rho(g) \leq 0.5|g|_{\mathcal{A}}$ in (36) that $(g + \rho(g)\mathbb{B}) \cap S_{r_i} \neq \varnothing$ implies $g \in S_{0.5r_i}$ and thus, since $S_{r_i} \subset S_{0.5r_i}$,

$$G_{\rho,\nu}(x, v) \cap S_{r_i} \subset G_{\rho_i,\nu}(x, v) \qquad \forall (x, v) \in S_{r_i} \times \mathbb{R}^m \ . \tag{37}$$

It then follows from (37), (35), and Lemma 5 that

$$m_{\rho, S_{r_i}, \cap}(\ell_{i+1}, x) \leq \bar{m}_{i,\cap} \qquad \forall x \in (\mathcal{A} + r_{i+1}\mathbb{B}) \backslash (\mathcal{A} + r_i\mathbb{B}^\circ) \ . \tag{38}$$

Now let $r > 0$ be arbitrary. We claim that, for all $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i\mathbb{B}$,

$$m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i} \ell_j, \xi\right) \leq 0.5 + \sum_{j=0}^{i-1} \bar{m}_{j,\cap} \ . \tag{39}$$

The bound (39) holds for $i = 0$ and all $\xi \in \mathcal{A} + r_0\mathbb{B} = \mathcal{A} + \delta_1\mathbb{B}$ according to (34). Now suppose the bound (39) holds for some $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i\mathbb{B}$. Then the bound (39) holds with i replaced by $i + 1$ for $\xi \in \mathcal{A} + r_i\mathbb{B}$ since

$$m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i+1} \ell_j, \xi\right) \leq m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i} \ell_j, \xi\right) \leq 0.5 + \sum_{j=0}^{i-1} \bar{m}_{j,\cap} \leq 0.5 + \sum_{j=0}^{i} \bar{m}_{j,\cap} \ .$$

Now consider $\xi \in (\mathcal{A} + r_{i+1}\mathbb{B}) \backslash (\mathcal{A} + r_i\mathbb{B}^\circ)$. Using (39) and (38), and Lemma 3, we get

$$m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i+1} \ell_j, \xi\right) \ \leq \ m_{\rho, S_{r_i}, \cap}\left(\ell_{i+1}, \xi\right) + \sup_{\zeta \in \mathbb{R}^n \backslash S_{r_i}} m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i} \ell_j, \zeta\right)$$

$$\leq m_{\rho, S_{r_i}, \cap}\left(\ell_{i+1}, \xi\right) + \sup_{\zeta \in \mathcal{A} + r_i\mathbb{B}} m_{\rho, S_r, \cap}\left(\sum_{j=0}^{i} \ell_j, \zeta\right) \leq \bar{m}_{i,\cap} + 0.5 + \sum_{j=0}^{i-1} \bar{m}_{j,\cap} = 0.5 + \sum_{j=0}^{i} \bar{m}_{j,\cap} \ .$$

It follows by induction that (39) holds for each $i \in \mathbb{Z}_{\geq 0}$ and all $\xi \in \mathcal{A} + r_i\mathbb{B}$. In turn, it follows that $\widehat{m}_{\rho, S_r, \cap}(\xi) \leq 0.5 + \sum_{j=0}^{\infty} \bar{m}_{j,\cap} =: \varrho < 1$ for each $r > 0$ and each $\xi \in \mathbb{R}^n$ as required.

2) Perturbations in toward the set $\mathcal{A}$: We construct a continuous perturbation that satisfies $\rho(\xi) = 0$ for all $\xi \in \mathcal{A} \cup (\mathbb{R}^n \backslash (\mathcal{A} + 3\mathbb{B}^\circ))$. The nominal system is the one constructed under "perturbations out to infinity", for which $\mathcal{A}$ is globally asymptotically stable in probability.

For each $i \in \mathbb{Z}_{\geq 0}$ define $r_{i,a} := (3)(2^{-i})$, $\ell_{i,a} := 2^i$, and $\varepsilon_{i,a} := 2^{-i}$. Then using stability in probability for (3) and Lemma 7, let $\rho_{i,a} > 0$ and $\delta_{i,a} \in (0, r_{i,a}]$ be such that, for each $i \in \mathbb{Z}_{\geq 0}$,

$$m_{\rho_{i,a}, S_{r_{i,a}}, \cup}(\ell_{i,a}, \xi) \leq \varepsilon_{i,a} \qquad \forall \xi \in \mathcal{A} + \delta_{i,a}\mathbb{B} \ . \tag{40}$$

Without loss of generality, assume that $i \mapsto \rho_{i,a}$ and $i \mapsto \delta_{i,a}$ are nonincreasing. Let $\rho_a : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be continuous and satisfy $\rho_a(\xi) = 0$ for all $\xi \in \mathcal{A} \cup (\mathbb{R}^n \backslash (\mathcal{A} + 3\mathbb{B}^\circ))$, $\rho_a(\xi) > 0$ otherwise, and $\rho_a(\xi) \leq \rho_{i,a}$ for all $\xi \in \mathcal{A} + r_{i,a}\mathbb{B}$. It follows, for all $(x, v) \in (\mathcal{A} + r_{i,a}\mathbb{B}) \times \mathbb{R}^m$, that either $G_{\rho_a,\nu}(x, v) \subset G_{\rho_{i,a},\nu}(x, v)$ or $G_{\rho_{i,a},\nu}(x, v) \cap S_{r_{i,a}} \neq \varnothing$. From Lemma 6, $\delta_{i,a} \in (0, r_{i,a}]$, and (40),

$$m_{\rho_a, S_{r_{i,a}}, \cup}(\ell_{i,a}, \xi) \leq \varepsilon_{i,a} \qquad \forall \xi \in \mathcal{A} + \delta_{i,a}\mathbb{B} \ . \tag{41}$$

Let $\bar{m}_{\cup,i}$ and $\bar{m}_{\cap,i}$ be monotonically nonincreasing, summable sequences of positive numbers. For $i \in \mathbb{Z}_{\geq 0}$, define $r_i := (3)(2^{-i})$. Using global asymptotic stability in probability, Corollary 4, and Lemma 7, pick $\delta_i \in (0, r_i)$ sufficiently small, then $\ell_i \in \mathbb{Z}_{\geq 0}$ sufficiently large, and then $\rho_i > 0$ sufficiently small so that $i \mapsto \delta_i$ and $i \mapsto \rho_i$ are nonincreasing and, for all $\xi \in \mathcal{A} + \delta_i\mathbb{B}$,

$$m_{\rho_i, (\mathcal{A}+r_i\mathbb{B}) \cap S_{\delta_{i+1}}, \cap}(\ell_i, \xi) \leq \bar{m}_{\cap,i} \ , \qquad m_{\rho_i, S_{r_i}, \cup}(\ell_i, \xi) \leq \bar{m}_{\cup,i} \ . \tag{42}$$

Let $\rho_b : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function that is positive on $(\mathcal{A} + 3\mathbb{B}^\circ)\backslash\mathcal{A}$ and satisfies $\rho_b(\xi) \leq \min\{0.5|\xi|_{\mathcal{A}}, \rho_a(\xi)\}$ for all $\xi \in \mathbb{R}^n$ and $\rho_b(\xi) \leq \rho_i$ for all $\xi \in \mathcal{A} + 2r_i\mathbb{B}$. Thus

$$G_{\rho_b,\nu}(x, v) \cap (\mathcal{A} + r_i\mathbb{B}) \subset G_{\rho_i,\nu}(x, v) \qquad \forall (x, v) \in (\mathcal{A} + r_i\mathbb{B}) \times \mathbb{R}^m \tag{43}$$

so that, with Lemma 5, the containment $\delta_i \in (0, r_i)$, and the first inequality in (42), we have

$$m_{\rho_b, (\mathcal{A}+r_i\mathbb{B}) \cap S_{\delta_{i+1}}, \cap}(\ell_i, \xi) \leq \bar{m}_{\cap,i} \qquad \forall \xi \in \mathcal{A} + \delta_i\mathbb{B} \ . \tag{44}$$

Moreover, due to the bounds on ρ_b, we have $G_{\rho_b,\nu}(x, v) \subset G_{\rho_a,\nu}(x, v)$ for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$ and, for all $(x, v) \in (\mathcal{A}+r_i\mathbb{B})\times\mathbb{R}^m$, we have either $G_{\rho_b,\nu}(x, v) \subset G_{\rho_i,\nu}(x, v)$ of $G_{\rho_i,\nu}(x, v) \cap S_{r_i} \neq \varnothing$ so that, with Lemma 6, the second inequality in (42), and (41) we have

$$m_{\rho_b, S_{r_i}, \cup}(\ell_i, \xi) \ \leq \ \bar{m}_{\cup,i} \qquad \forall \xi \in \mathcal{A} + \delta_i\mathbb{B} \tag{45a}$$

$$m_{\rho_b, S_{r_{i,a}}, \cup}(\ell_{i,a}, \xi) \ \leq \ \varepsilon_{i,a} \qquad \forall \xi \in \mathcal{A} + \delta_{i,a}\mathbb{B} \ . \tag{45b}$$

The combination of (44) and (45a), together with Lemma 1 gives that

$$m_{\rho_b,S_{\delta_{i+1}},\cap}(\ell_i,\xi) \leq \bar{m}_{\cap,i} + \bar{m}_{\cup,i} \qquad \forall \xi \in \mathcal{A} + \delta_i \mathbb{B} \ . \tag{46}$$

Let $r > 0$ and let $i_0 \in \mathbb{Z}_{\geq 0}$ satisfy $r_{i_0} \leq r$. Let $\ell \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > 0$ be arbitrary, let $i^* \in \mathbb{Z}_{\geq 0}$ satisfy $\varepsilon_{i^*,a} \leq \varepsilon$, $\ell_{i^*,a} \geq \ell$ and $r_{i^*,a} \leq r$, and let $\widehat{i} \in \mathbb{Z}_{\geq i_0}$ satisfy $\delta_{\widehat{i}+1} \leq \delta_{i^*,a}$. We claim that

$$m_{\rho_b,S_r,\cup}(\ell,\xi) \leq \sum_{j=i+1}^{\widehat{i}} (2\bar{m}_{\cup,j} + \bar{m}_{\cap,j}) + \varepsilon \qquad \forall \xi \in \mathcal{A} + \delta_{i+1}\mathbb{B} \ , \qquad \forall i \in \left\{ i_0, \ldots, \widehat{i} \right\} \ . \tag{47}$$

For $i = \widehat{i}$, (47) follows from (45b) with $i = i^*$. Now suppose (47) holds for some $i \in \left\{ i_0 + 1, \ldots, \widehat{i} \right\}$. Apply Lemma 2 with $S_1 = S_{\delta_{i+1}}$, $k = \ell_i$ and use (45a) and (46), to get

$$m_{\rho_b,S_r,\cup}(\ell,\xi) \leq 2\bar{m}_{\cup,i} + \bar{m}_{\cap,i} + \sum_{j=i+1}^{\widehat{i}} (2\bar{m}_{\cup,j} + \bar{m}_{\cap,j}) + \varepsilon \qquad \forall \xi \in \mathcal{A} + \delta_i \mathbb{B} \ .$$

Now (47) follows by induction. Stability in probability then follows from (47) since $\ell \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > 0$, and hence $\widehat{\ell}$, in (47) are arbitrary and the bound tends to zero as $i \to \infty$ since the sequences $\bar{m}_{\cup,j}$ and $\bar{m}_{\cap,j}$ are summable.

Finally, we establish almost sure global asymptotic recurrence of $\mathcal{A}$. Let $\bar{m}_{i,\cap}$, $\bar{m}_{i,\cup}$ be monotonically nonincreasing sequences of positive real numbers with $\sum_{i=0}^{\infty} \bar{m}_{\cap,i} \leq 0.25$ and $\sum_{i=0}^{\infty} \bar{m}_{\cup,i} \leq 0.25$. Set $r_0 = \infty$ and $r_i := 3(2^{-i+1})$ for $i \in \mathbb{Z}_{\geq 1}$. Pick $\delta_i \in (0, r_i)$, $\ell_i \in \mathbb{Z}_{\geq 0}$, and $\rho_i > 0$ so that

$$m_{\rho_i,(\mathcal{A}+r_i\mathbb{B})\cap S_{\delta_{i+1}},\cap}(\ell_i,\xi) \leq \bar{m}_{\cap,i} \ , \qquad m_{\rho_b,S_{r_i},\cup}(\ell_i,\xi) \leq \bar{m}_{\cup,i} \qquad \forall \xi \in \mathcal{A} + \delta_i \mathbb{B} \tag{48}$$

and $i \mapsto \delta_i$ and $i \mapsto \rho_i$ are nonincreasing. Since $S_{r_0} = \varnothing$, we can take $\delta_0 = 3$. Let $\rho_c : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function that is positive on $(\mathcal{A} + 3\mathbb{B}^\circ) \backslash \mathcal{A}$ and satisfies $\rho_c(\xi) \leq \rho_b(\xi)$ for all $\xi \in \mathbb{R}^n$ and $\rho_c(\xi) \leq \rho_i$ for all $\xi \in \mathcal{A} + 2r_i\mathbb{B}$. Like in (46), we conclude from (48) that

$$m_{\rho_c,S_{\delta_{i+1}},\cap}(\ell_i,\xi) \leq \bar{m}_{\cap,i} + \bar{m}_{\cup,i} \qquad \forall \xi \in \mathcal{A} + \delta_i \mathbb{B} \ . \tag{49}$$

Let $r > 0$ be given. Let $\delta \in (0, r)$ be such that $\widehat{m}_{\rho_b,S_r,\cup}(\xi) \leq 0.25$ for all $\xi \in \mathcal{A} + \delta\mathbb{B}$. Let j be the smallest nonnegative integer such that $\delta_j < \delta$. We claim that, for each $i \in \{0, \ldots, j\}$,

$$m_{\rho_c,S_r,\cap}\left(1 + \sum_{k=i}^{j-1} \ell_i, \xi\right) \leq \sum_{k=i}^{j-1} (\bar{m}_{\cap,i} + \bar{m}_{\cup,i}) + 0.25 \qquad \forall \xi \in \mathcal{A} + \delta_i \mathbb{B} \ . \tag{50}$$

The case $i = j$ uses $\widehat{m}_{\rho_b,S_r,\cup}(\xi) \leq 0.25$ for $\xi \in \mathcal{A} + \delta\mathbb{B}$, $\delta_j < \delta$, and Lemma 1 with $S_1 = \varnothing$. Suppose (50) holds for some $i \in \{1, \ldots, j\}$. Let $\xi \in \mathcal{A} + \delta_{i-1}\mathbb{B}$. Using Lemma 3 and (49),

$$m_{\rho_c,S_r,\cap}\left(1 + \sum_{k=i-1}^{j-1} \ell_i, \xi\right) \leq m_{\rho_c,S_{\delta_i},\cap}(\ell_{i-1}, \xi) + \sup_{\zeta \in \mathcal{A}+\delta_i\mathbb{B}} m_{\rho_c,S_r,\cap}\left(1 + \sum_{k=i}^{j-1} \ell_i, \zeta\right)$$

$$\leq \bar{m}_{\cap,i-1} + \bar{m}_{\cup,i-1} + \sum_{k=i}^{j-1} (\bar{m}_{\cap,i} + \bar{m}_{\cup,i}) + 0.25 = \sum_{k=i-1}^{j-1} (\bar{m}_{\cap,i} + \bar{m}_{\cup,i}) + 0.25 \ .$$

Now (50) follows by induction for all $i \in \{0, \ldots, j\}$. From (50), we conclude using $\sum_{i=0}^{\infty} \bar{m}_{\cap,i} \leq 0.25$ and $\sum_{i=0}^{\infty} \bar{m}_{\cup,i}$ that $\widehat{m}_{\rho_c,S_r,\cap}(\xi) \leq 0.75$ for all $\xi \in \mathcal{A} + 3\mathbb{B}$. Since $\widehat{m}_{\rho_c,S_3,\cap}(\xi) = 0$ for all $\xi \in S_3 := \mathbb{R}^n \backslash (\mathcal{A} + 3\mathbb{B}^\circ)$, it follows from Lemma 3 with $S_1 = S_3$ and k and ℓ arbitrarily large that $\mathcal{A}$ is almost surely globally asymptotically recurrent.

XI. Proof of Propositions 1-8

A. Proof of Proposition 1

i) Let $\phi \in \mathcal{S}_{c,m}(x)$ and $\zeta \in \mathbb{R}^n$. For each $i \in \mathbb{Z}_{\geq 0}$ define $\phi^c_{i+1} : (\mathbb{R}^m)^{i+1} \backslash \text{dom } \phi_{i+1} \to \mathbb{R}^n$ by $\phi^c_{i+1}(v_0, \ldots, v_i) := \zeta$ for all $(v_0, \ldots, v_i) \in (\mathbb{R}^m)^{i+1} \backslash \text{dom } \phi_{i+1}$. This function is measurable since it is constant on its domain, which is measurable by virtue of ϕ_{i+1} being measurable. Consider ϕ_{i+1} and ϕ^c_{i+1} to be set-valued mappings defined on $(\mathbb{R}^m)^{i+1}$ that are empty valued outside of their domains. For each $i \in \mathbb{Z}_{\geq 0}$ define $\phi^e_{i+1} : (\mathbb{R}^m)^{i+1} \rightrightarrows \mathbb{R}^n$ as $\phi^e_{i+1}(v_0, \ldots, v_i) := \phi_{i+1}(v_0, \ldots, v_i) \cup \phi^c_{i+1}(v_0, \ldots, v_i)$. According to [15, Proposition 14.11(b)], ϕ^e_{i+1} is measurable. By construction, ϕ^e_{i+1} is a (single-valued) function defined on $(\mathbb{R}^m)^{i+1}$ that is equal to ϕ_{i+1} on $\text{dom } \phi_{i+1}$. Define $\mathbf{x}_{i+1}(\omega) := \phi^e_{i+1}(\mathbf{v}_0(\omega), \ldots, \mathbf{v}_i(\omega))$ so that, in particular, the conditions (2) hold. According to [18, Theorem 1.4, p. 601], $\mathbf{x}_{i+1}$ is a random variable; according to [18, Theorem 1.5, p. 603], $\mathbf{x}$ is adapted to the natural filtration of $\mathbf{v}$. It follows from the properties of ϕ that $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$.

ii) Follows from [18, Theorem 1.5, p. 603] and the properties of functions $\phi \in \mathcal{S}_{c,m}(x)$.

B. Proof of Proposition 2

Since $\nu \in \mathcal{K}_\infty$, M_ν is outer semicontinuous and locally bounded. Therefore, G_ν is locally bounded. Since the graph of the union of two set-valued mappings is equal to the union of their graphs, it follows that $x \mapsto G_\nu(x, v)$ is outer semicontinuous. Finally, it follows from [15, Proposition 14.11(a)] that $v \mapsto G_\nu(x, v)$ and $v \mapsto \text{graph}(G_\nu(\cdot, v))$ are measurable.

C. Proof of Proposition 3

The set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined as $H(\xi) := \{\xi\} + \rho(\xi)\mathbb{B}$ is outer semicontinuous and locally bounded, since ρ is continuous, and $G_{\rho,\nu}(x, v) = H(G_\nu(H(x), v))$. Thus, according [15, Prop. 5.52(a),(b)], $G_{\rho,\nu}$ satisfies the first and last conditions of Standing Assumption 1.

The measurability of $v \mapsto G_{\rho,\nu}(x, v)$, which is the second condition of Standing Assumption 1, follows from two applications of [15, Theorem 14.13(b)]. The random variable v plays the role of the variable t in each application of the theorem. In the first application, the constant (with respect to v) mapping H plays the role of S and G plays the role of M. In the second application, $G_\nu(H(x), \cdot)$ plays the role of $S(\cdot)$ and $H(\cdot)$ plays the role of $M(t, \cdot)$.

Finally, consider the measurability of $v \mapsto \text{graph}(H(G_\nu(H(\cdot), v)))$. The proof of [15, Prop. 5.52(b)] shows that, for set-valued mappings $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the graph of $T \circ S$ is equal to the domain of the set-valued mapping R given as $(x, w) \mapsto S(x) \cap T^{-1}(w)$, and the graph of R, up to a reordering of coordinates, is equal to the set $[\text{graph}(S) \times \mathbb{R}^n] \cap [\mathbb{R}^n \times \text{graph}(T)]$. Let H play the role of S and $G_\nu(\cdot, v)$ play the role of T. By assumption, $\text{graph}(T)$ is measurable, while $\text{graph}(S)$ is constant and thus measurable. From [15, Theorem 14.11(a)], the graph of R is measurable. Since the domain of R is the projection of the graph of R, [15, Theorem14.13] gives that the domain of R is measurable. In turn, $v \mapsto \text{graph}(G_\nu(H(\cdot), v)))$ is measurable. Applying this idea again with $S(\cdot) = G_\nu(H(\cdot), v)))$ and $T = H$ gives that $v \mapsto \text{graph}(H(G_\nu(H(\cdot), v)))$ is measurable, which is the third condition of Standing Assumption 1.

D. Proof of Proposition 4

Let $(k, x) \in \mathbb{Z}_{\geq 1} \times \mathbb{R}^n$. Due to [17, Prop. 8], there exists $\phi \in \mathcal{S}_{c,m}(x)$ satisfying $\phi_{i+1} \in \text{argmax}_{g \in G(\phi_i, v_i)} \mathbb{I}_S(g) m_{S,\cap}(k - (i + 1), g)$ for all $i \in \{0, \ldots, k - 1\}$, where the dependence of ϕ_{i+1} on $(v_0, \ldots, v_i)$ is suppressed. Let $\mathbf{x}^\phi$ be the random process generated from ϕ according to Proposition 1 and let $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ be arbitrary. We claim that,[3] for each $\ell \in \{1, \ldots, k\}$,

$$\mathbb{E}\left[\left(\prod_{j=1}^{\ell} \mathbb{I}_S(\mathbf{x}_j)\right) m_{S,\cap}(k - \ell, \mathbf{x}_\ell)\right] \leq m_{S,\cap}(k, x) \tag{51}$$

with equality when $\mathbf{x} = \mathbf{x}^\phi$. Using $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ and the definitions of $\mathbf{x}^\phi$ and $m_{S,\cap}(k, x)$,

$$\mathbb{E}\left[\mathbb{I}_S(\mathbf{x}_1) m_{S,\cap}(k - 1, \mathbf{x}_1)\right] \leq \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \mathbb{I}_S(g) m_{S,\cap}(k - 1, g) \mu(dv) = m_{S,\cap}(k, x)$$

[3]When computing the expected value, we set $\mathbb{I}_S(\mathbf{x}_j(\omega)) = 0$ for $\omega \notin \text{dom } \mathbf{x}_j$ and $m_{S,\cap}(k - \ell, \mathbf{x}_\ell(\omega)) = 0$ for $\omega \notin \text{dom } \mathbf{x}_\ell$.

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with equality when $\mathbf{x} = \mathbf{x}^\phi$. So (51) holds for $\ell = 1$, with equality when $\mathbf{x} = \mathbf{x}^\phi$. Now suppose (51) holds for some $\ell \in \{1, \ldots, k-1\}$, with equality when $\mathbf{x} = \mathbf{x}^\phi$. By the definition of $m_{S,\cap}$ and the properties of $\mathbf{x}$, we have

$$\begin{array}{rcl} m_{S,\cap}(k-\ell, \mathbf{x}_\ell) & \equiv & \displaystyle\int_{\mathbb{R}^m} \max_{g \in G(\mathbf{x}_\ell, v)} \mathbb{I}_S(g) m_{S,\cap}(k-(\ell+1), g)\mu(dv) \\ & \overset{\text{wpo}}{\geq} & \mathbb{E}\left[\mathbb{I}_S(\mathbf{x}_{\ell+1}) m_{S,\cap}(k-(\ell+1), \mathbf{x}_{\ell+1}) | \mathcal{F}_\ell\right] \end{array} \tag{52}$$

with equivalence when $\mathbf{x} = \mathbf{x}^\phi$. Using (51), (52), we get

$$\begin{aligned} &\mathbb{E}\left[\left(\prod_{j=1}^{\ell+1} \mathbb{I}_S(\mathbf{x}_j)\right) m_{S,\cap}(k-(\ell+1), \mathbf{x}_{\ell+1})\right] \\ &\qquad = \mathbb{E}\left[\left(\prod_{j=1}^{\ell} \mathbb{I}_S(\mathbf{x}_j)\right) \mathbb{E}\left[\mathbb{I}_S(\mathbf{x}_{\ell+1}) m_{S,\cap}(k-(\ell+1), \mathbf{x}_{\ell+1}) | \mathcal{F}_\ell\right]\right] \\ &\qquad \leq \mathbb{E}\left[\left(\prod_{j=1}^{\ell} \mathbb{I}_S(\mathbf{x}_j)\right) m_{S,\cap}(k-\ell, \mathbf{x}_\ell)\right] \leq m_{S,\cap}(k, x) \end{aligned}$$

with equality for $\mathbf{x} = \mathbf{x}^\phi$. Thus, by induction, (51) holds for all $\ell \in \{1, \ldots, k\}$ with equality for $\mathbf{x} = \mathbf{x}^\phi$. The case $\ell = k$ in (51) gives $\mathbb{E}\left[\prod_{j=1}^k \mathbb{I}_S(\mathbf{x}_j)\right] \leq m_{S,\cap}(k, x) = \mathbb{E}\left[\prod_{j=1}^k \mathbb{I}_S(\mathbf{x}_j^\phi)\right]$. Since $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ is arbitrary and $\mathbb{P}(\cap_{j=1}^k \mathbf{x}_j^{-1}(S)) = \mathbb{E}\left[\prod_{j=1}^k \mathbb{I}_S(\mathbf{x}_j)\right]$, this case establishes the result.

E. Proof of Proposition 5

We start by proving the following preliminary result:

Proposition 15: For each closed set $S \subset \mathbb{R}^n$, the function $\widehat{m}_{S,\cap}$ is upper semicontinuous and $\widehat{m}_{S,\cap}(\xi) = \int_{\mathbb{R}^m} \max_{g \in G(\xi,v)} \mathbb{I}_S(g) \widehat{m}_{S,\cap}(g) \mu(dv)$.

Proof: Let the sequence x_j converge to x as $j \to \infty$. Let $i^* \in \mathbb{Z}_{\geq 0}$ be arbitrary. Then, using upper semicontinuity of $m_{S,\cap}(i^*, \cdot)$ and the monotonicity of $m_{S,\cap}(\cdot, x_j)$,

$$\limsup_{j\to\infty} \left(\lim_{i\to\infty} m_{S,\cap}(i, x_j)\right) \leq \limsup_{j\to\infty} \left(m_{S,\cap}(i^*, x_j)\right) \leq m_{S,\cap}(i^*, x) \ .$$

Then, since $i^* \in \mathbb{Z}_{\geq 0}$ is arbitrary, $\limsup_{j\to\infty} \left(\lim_{i\to\infty} m_{S,\cap}(i, x_j)\right) \leq \lim_{i^*\to\infty} m_{S,\cap}(i^*, x)$, which is the desired upper semicontinuity since $\widehat{m}_{S,\cap}(\xi) := \lim_{i\to\infty} m_{S,\cap}(i, \xi)$. Next we claim that

$$\lim_{j\to\infty} \max_{g \in G(\xi,v)} m_{S,\cap}(j, g) = \max_{g \in G(\xi,v)} \lim_{j\to\infty} m_{S,\cap}(j, g) \ . \tag{53}$$

Let $\varepsilon > 0$. Let $g_j \in G(\xi, v)$ satisfy $\max_{g \in G(\xi,v)} m_{S,\cap}(j, g) = m_{S,\cap}(j, g_j)$. Without loss of generality, perhaps by identifying a subsequence, assume that g_j converges to some $g^* \in G(\xi, v)$. Let j^* be such that $m_{S,\cap}(j^*, g^*) \leq \lim_{j\to\infty} m_{S,\cap}(j, g^*) + \varepsilon$. Then

$$\begin{aligned} \lim_{j\to\infty} \max_{g \in G(\xi,v)} m_{S,\cap}(j, g) &= \lim_{j\to\infty} m_{S,\cap}(j, g_j) \leq \limsup_{j\to\infty} m_{S,\cap}(j^*, g_j) \leq m_{S,\cap}(j^*, g^*) \\ &\leq \lim_{j\to\infty} m_{S,\cap}(j, g^*) + \varepsilon \leq \max_{g \in G(\xi,v)} \lim_{j\to\infty} m_{S,\cap}(j, g) + \varepsilon \ . \end{aligned}$$

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Since $\varepsilon > 0$ is arbitrary, $\lim_{j\to\infty} \max_{g\in G(\xi,v)} m_{S,\cap}(j,g) \leq \max_{g\in G(\xi,v)} \lim_{j\to\infty} m_{S,\cap}(j,g)$. Conversely, it is clear that $\max_{g\in G(\xi,v)} \lim_{j\to\infty} m_{S,\cap}(j,g) \leq \lim_{j\to\infty} \max_{g\in G(\xi,v)} m_{S,\cap}(j,g)$. Thus, (53) holds. Finally, using the Lebesgue dominated convergence theorem and (53),

$$\begin{aligned} \widehat{m}_{S,\cap}(\xi) = \lim_{j\to\infty} m_{S,\cap}(j+1,\xi) &= \lim_{j\to\infty} \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(j,g)\mu(dv) \\ = \int_{\mathbb{R}^m} \lim_{j\to\infty} \max_{g\in G(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(j,g)\mu(dv) &= \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g) \lim_{j\to\infty} m_{S,\cap}(j,g)\mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g) \widehat{m}_{S,\cap}(g)\mu(dv) \ , \end{aligned}$$

which is the stated result. ■

We present several corollaries of Proposition 15.

Corollary 5: Let $S \subset \mathbb{R}^n$ be closed and $\rho \in [0,1)$. If $\widehat{m}_{S,\cap}(\xi) \leq \rho$ for all $\xi \in S$ then $\widehat{m}_{S,\cap}(\xi) \leq \rho$ for all $\xi \in \mathbb{R}^n$.

Proof: By assumption, $\mathbb{I}_S(g)\widehat{m}_{S,\cap}(g) \leq \rho$ for all $g \in \mathbb{R}^n$. From Proposition 15, for each $\xi \in \mathbb{R}^n$, $\widehat{m}_{S,\cap}(\xi) = \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g)\widehat{m}_{S,\cap}(g)\mu(dv) \leq \int_{\mathbb{R}^m} \rho\mu(dv) = \rho$, giving the result. ■

Corollary 6: Let $S \subset \mathbb{R}^n$ be closed. For each $k \in \mathbb{Z}_{\geq 0}$,

$$\widehat{m}_{S,\cap}(\xi) \leq m_{S,\cap}(k,\xi) \cdot \sup_{\zeta\in S} \widehat{m}_{S,\cap}(\zeta) \qquad \forall \xi \in S \ . \tag{54}$$

Proof: The bound in (54) holds for $k = 0$ since $m_{S,\cap}(0,\xi) = 1$ for all ξ. Suppose the bound (54) holds for some $k \in \mathbb{Z}_{\geq 0}$. Using Proposition 15 and the definition of $m_{S,\cap}$, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned} \widehat{m}_{S,\cap}(\xi) = \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g)\widehat{m}_{S,\cap}(g)\mu(dv) &\leq \left(\int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(k,g)\mu(dv) \right) \sup_{\zeta\in S} \widehat{m}_{S,\cap}(\zeta) \\ &= m_{S,\cap}(k+1,\xi) \cdot \sup_{\zeta\in S} \widehat{m}_{S,\cap}(\zeta) \ . \end{aligned}$$

The corollary now follows by induction. ■

The next result comes from Corollary 6, taking the limit as $k \to \infty$ in (54), and Corollary 5.

Corollary 7: Let $S \subset \mathbb{R}^n$ be closed. Then $\widehat{m}_{S,\cap}(\xi) \leq \widehat{m}_{S,\cap}(\xi) \cdot \sup_{\zeta\in S} \widehat{m}_{S,\cap}(\zeta)$ for each $\xi \in S$. In particular, $\sup_{\zeta\in S} \widehat{m}_{S,\cap}(\zeta) < 1$ implies $\sup_{\zeta\in\mathbb{R}^n} \widehat{m}_{S,\cap}(\zeta) = 0$.

We are now ready to prove Proposition 5. The claim 2) $\Longrightarrow$ 3) follows from Corollary 7.

3) $\Longrightarrow$ 4): Suppose that there exist a compact set $K \subset \mathbb{R}^n$ and $\rho > 0$ and for each $i \in \mathbb{Z}_{\geq 0}$ there exists $\xi_i \in K$ such that $m_{S,\cap}(i,\xi_i) > \rho$. Without loss of generality, we can assume that the sequence ξ_i converges to a point $\xi \in K$. By assumption, there exists $i^* \in \mathbb{Z}_{\geq 0}$ such that

$m_{S,\cap}(i^*,\xi) \leq \rho/2$. Since $m_{S,\cap}(i^*,\cdot)$ is upper semicontinuous, for $i \in \mathbb{Z}_{\geq i^*}$ sufficiently large we have $m_{S,\cap}(i,\xi_i) \leq m_{S,\cap}(i^*,\xi_i) \leq \rho$. This contradiction establishes the result.

The claims 4) $\implies$ 1) and 1) $\implies$ 2) follow from Proposition 4 and the definition of almost sure global asymptotic recurrence of $\mathcal{A}$.

F. Proof of Proposition 6

Let $(k,x) \in \mathbb{Z}_{\geq 1} \times \mathbb{R}^n$. Due to [17, Proposition 8], there exists $\phi \in \mathcal{S}_{c,m}(x)$ satisfying $\phi_{i+1} \in \operatorname{argmax}_{g \in G(\phi_i,v_i)} \max\{\mathbb{I}_S(g), m_{S,\cup}(k-(i+1),g)\}$ for all $i \in \{0,\ldots,k-1\}$ and $\phi_{i+1} \in G(\phi_i,v_i)$ for $i \in \mathbb{Z}_{\geq k}$, where the dependence of ϕ_{i+1} on $(v_0,\ldots,v_i)$ is suppressed. Let $\mathbf{x}^\phi$ be the random process generated from ϕ according to Proposition 1 and let $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ be arbitrary. We claim that,[4] for each $\ell \in \{1,\ldots,k\}$,

$$\mathbb{E}\left[\max\left\{\max_{j\in\{1,\ldots,\ell\}} \mathbb{I}_S(\mathbf{x}_j), m_{S,\cup}(k-\ell,\mathbf{x}_\ell)\right\}\right] \leq m_{S,\cup}(k,x) \tag{55}$$

with equality when $\mathbf{x} = \mathbf{x}^\phi$. Using $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ and the definitions of $\mathbf{x}^\phi$ and $m_{S,\cup}(k,x)$,

$$\mathbb{E}\left[\max\{\mathbb{I}_S(\mathbf{x}_1), m_{S,\cup}(k-1,\mathbf{x}_1)\}\right] \leq \int_{\mathbb{R}^m} \max_{g\in G(x,v)} \max\{\mathbb{I}_S(g), m_{S,\cup}(k-1,g)\}\,\mu(dv) = m_{S,\cup}(k,x)$$

with equality when $\mathbf{x} = \mathbf{x}^\phi$. So (55) holds for $\ell = 1$, with equality when $\mathbf{x} = \mathbf{x}^\phi$. Now suppose (55) holds for some $\ell \in \{1,\ldots,k-1\}$, with equality when $\mathbf{x} = \mathbf{x}^\phi$. By the definition of $m_{S,\cup}$ and the properties of $\mathbf{x}$, we have

$$\begin{array}{rcl} m_{S,\cup}(k-\ell,\mathbf{x}_\ell) & \equiv & \displaystyle\int_{\mathbb{R}^m} \max_{g\in G(\mathbf{x}_\ell,v)} \max\{\mathbb{I}_S(g), m_{S,\cup}(k-(\ell+1),g)\}\,\mu(dv) \\ & \overset{\text{wpo}}{\geq} & \mathbb{E}\left[\max\{\mathbb{I}_S(\mathbf{x}_{\ell+1}), m_{S,\cup}(k-(\ell+1),\mathbf{x}_{\ell+1})\}\,|\mathcal{F}_\ell\right] \end{array} \tag{56}$$

with equivalence when $\mathbf{x} = \mathbf{x}^\phi$. Using (55), (56), and the containments $\max_{j\in\{1,\ldots,\ell\}} \mathbb{I}_S(\mathbf{x}_j) \in \{0,1\}$, and $\max\{\mathbb{I}_S(\mathbf{x}_{\ell+1}), m_{S,\cup}(k-\ell,\mathbf{x}_{\ell+1})\} \in [0,1]$, we get

$$\begin{aligned} &\mathbb{E}\left[\max\left\{\max_{j\in\{1,\ldots,\ell+1\}} \mathbb{I}_S(\mathbf{x}_j), m_{S,\cup}(k-(\ell+1),\mathbf{x}_{\ell+1})\right\}\right] \\ &= \mathbb{E}\left[\max\left\{\max_{j\in\{1,\ldots,\ell\}} \mathbb{I}_S(\mathbf{x}_j), \mathbb{E}\left[\max\{\mathbb{I}_S(\mathbf{x}_{\ell+1}), m_{S,\cup}(k-(\ell+1),\mathbf{x}_{\ell+1})\}\,|\mathcal{F}_\ell\right]\right\}\right] \\ &\leq \mathbb{E}\left[\max\left\{\max_{j\in\{1,\ldots,\ell\}} \mathbb{I}_S(\mathbf{x}_j), m_{S,\cup}(k-\ell,\mathbf{x}_\ell)\right\}\right] \leq m_{S,\cup}(k,x) \end{aligned}$$

with equality for $\mathbf{x} = \mathbf{x}^\phi$. By induction, (55) holds for all $\ell \in \{1,\ldots,k\}$ with equality for $\mathbf{x} = \mathbf{x}^\phi$. The case $\ell = k$ in (55) gives $\mathbb{E}\left[\max_{j\in\{1,\ldots,k\}} \mathbb{I}_S(\mathbf{x}_j)\right] \leq m_{S,\cup}(k,x) = \mathbb{E}\left[\max_{j\in\{1,\ldots,k\}} \mathbb{I}_S(\mathbf{x}_j^\phi)\right]$. Since $\mathbf{x} \in \mathcal{S}_{\mathbf{a},\mathbf{r}}(x)$ is arbitrary and $\mathbb{P}\left(\cup_{j=1}^k \mathbf{x}_j^{-1}(S)\right) = \mathbb{E}\left[\max_{j\in\{1,\ldots,k\}} \mathbb{I}_S(\mathbf{x}_j)\right]$, the result follows.

[4]When computing the expected value, we set $\mathbb{I}_S(\mathbf{x}_j(\omega)) = 0$ for $\omega \notin \operatorname{dom} \mathbf{x}_j$ and $m_{S,\cup}(k-\ell,\mathbf{x}_\ell(\omega)) = 0$ for $\omega \notin \operatorname{dom} \mathbf{x}_\ell$.

G. Proof of Proposition 8

Using Propositions 5 and 7, we show that the bound in Proposition 8 is equivalent to the combination of the bound in Proposition 7 and almost sure global asymptotic recurrence of $\mathcal{A}$.

$\Longleftarrow$ To see that the bound in Proposition 8 implies the bound in Proposition 7, define $\alpha_\varepsilon(s) := \gamma_\varepsilon(s,0)+s$, note that $\alpha_\varepsilon \in \mathcal{K}_\infty$, set $k=0$, and note from the definition of $\widetilde{m}_{\mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cup}$ that $\widehat{m}_{\mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cup}(\xi) = \widetilde{m}_{\mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ),\cup}(0,\xi) \leq \gamma_\varepsilon(|\xi|_{\mathcal{A}},0) \leq \alpha_\varepsilon(|\xi|_{\mathcal{A}})$.

That the bound in Proposition 8 implies almost sure global asymptotic recurrence follows from the equivalence of item 4 and item 1 in Proposition 5 and the fact that, for each closed set $S \subset \mathbb{R}^n$ and each $k \in \mathbb{Z}_{\geq 0}$,

$$m_{S,\cap}(k+1,\xi) \leq \widetilde{m}_{S,\cup}(k,\xi) \qquad \forall \xi \in \mathbb{R}^n \ . \tag{57}$$

Indeed, according to Lemma 1 with $S_1 = \varnothing$ and the definitions of $\widehat{m}_{S,\cup}$ and $\widetilde{m}_{S,\cup}$, we have that $m_{S,\cap}(1,\xi) \leq m_{S,\cup}(1,\xi) \leq \widehat{m}_{S,\cup}(\xi) = \widetilde{m}_{S,\cup}(0,\xi)$ so that (57) holds for $k=0$. Now suppose (57) holds for some $k \in \mathbb{Z}_{\geq 0}$. Then

$$m_{S,\cap}(k+1,\xi) = \int_{\mathbb{R}^m} \max_{g \in G(\xi,v)} \mathbb{I}_S(g) m_{S,\cap}(k,g)\mu(dv) \leq \int_{\mathbb{R}^m} \max_{g\in G(\xi,v)} \widetilde{m}_{S,\cup}(k-1,g)\mu(dv) = \widetilde{m}_{S,\cup}(k,\xi).$$

By induction, (57) holds for all $k \in \mathbb{Z}_{\geq 0}$. With $S = \mathbb{R}^n\backslash(\mathcal{A}+\varepsilon\mathbb{B}^\circ)$, (57) gives $m_{S,\cap}(k+1,\xi) \leq \widetilde{m}_{S,\cup}(k,\xi) \leq \gamma_\varepsilon(|\xi|_{\mathcal{A}},k)$ for all $\xi \in \mathbb{R}^n$. Since $\gamma_\varepsilon \in \mathcal{KL}$, we have item 4 in Proposition 5.

$\Longrightarrow$ Due to the equivalence between item 4 and item 1 in Proposition 5, almost sure global asymptotic recurrence of $\mathcal{A}$ implies that, for each $r>0$ and $R>r$ there exists $\psi_{r,R} \in \mathcal{L}$ such that $m_{S_r,\cap}(\ell,\xi) \leq \psi_{r,R}(\ell)$ for all $(\ell,\xi) \in \mathbb{Z}_{\geq 0} \times (\mathcal{A}+R\mathbb{B})$, where $S_r := \mathbb{R}^n\backslash(\mathcal{A}+r\mathbb{B}^\circ)$. We claim that, for each $r>0$ and $R>0$, there exists $\widehat{\psi}_{r,R} \in \mathcal{L}$ such that $\widetilde{m}_{S_r,\cup}(\ell,\xi) \leq \widehat{\psi}_{r,R}(\ell)$ for all $(\ell,\xi) \in \mathbb{Z}_{\geq 0} \times (\mathcal{A}+R\mathbb{B})$. It is enough to prove that for each $\varepsilon>0$ and $R>0$ there exists $\ell > 0$ such that $\widetilde{m}_{S_r,\cup}(\ell,\xi) \leq \varepsilon$ for all $\xi \in \mathcal{A}+R\mathbb{B}$. For $\varepsilon>0$, let $\widetilde{r}>0$ satisfy $\alpha_r(\widetilde{r}) \leq 0.5\varepsilon$ where the functions $\alpha_r \in \mathcal{K}_\infty$ characterize stability in probability of $\mathcal{A}$. Let $\ell \in \mathbb{Z}_{\geq 0}$ be such that $\psi_{\widetilde{r},R}(\ell) \leq 0.5\varepsilon$. Using [17, Lemma 6], for $\xi \in \mathcal{A}+R\mathbb{B}$ we get $\widetilde{m}_{S_r,\cup}(\ell,\xi) \leq m_{S_{\widetilde{r}},\cap}(\ell,\xi) + \sup_{\zeta \in \mathcal{A}+\widetilde{r}\mathbb{B}} \widehat{m}_{S_r,\cup}(\zeta) \leq 0.5\varepsilon + 0.5\varepsilon = \varepsilon$ as desired. We can assume that $s \to \widehat{\psi}_{r,s}(\ell)$ is continuous and strictly increasing on $(0,\infty)$. For all $(s,\ell) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, define $\gamma_r(s,\ell) := \min\left\{\alpha_r(s), \widehat{\psi}_{r,s}(\ell)\right\}$ and note that $\gamma_r \in \mathcal{KL}$. Then the result follows from

$$\widetilde{m}_{S_r,\cup}(\ell,\xi) \leq \min\left\{\alpha_r\left(|\xi|_{\mathcal{A}}\right), \widehat{\psi}_{r,|\xi|_{\mathcal{A}}}(\ell)\right\} = \gamma_r\left(|\xi|_{\mathcal{A}},\ell\right) \qquad \forall (\ell,\xi) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \ .$$

XII. Conclusion

We have established that the existence of a continuous strict Lyapunov function that is smooth on $\mathbb{R}^n \backslash \mathcal{A}$ is necessary for global asymptotic stability in probability of the compact set $\mathcal{A}$ for a class of stochastic difference inclusions. To prove this result, we gave an equivalent characterization of global asymptotic stability in probability that entails uniform attractivity in probability and we established that global asymptotic stability in probability is robust to sufficiently small, state-dependent perturbations. The ideas presented here apply readily to other forms of stochastic stability, including the notion of recurrence, a property described thoroughly in [10]. Future work will describe the Lyapunov construction for this setting. In addition, we expect to be able to extend the contribution of this work to a large class of stochastic hybrid systems, merging our approach with the framework used in [7].

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