An Algebraic Geometry Approach to Compute Strategically Equivalent Bimatrix Games ⋆

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Abstract: In this paper, a class of bimatrix games having the same Nash equilibria of a given game, either in pure or in mixed policies, is characterized. Such a goal is reached by computing the set of all the polynomials that are monotone strictly increasing in a given interval and by borrowing techniques from algebraic geometry to find solutions to a set of polynomial equalities.

Keywords: Game theory, Nash games, Algebraic approaches, Computational methods

1. INTRODUCTION

Noncooperative game theory deals with the problem of determining the best strategy for *multiple decision makers* (usually called *players*) in a way such that each agent pursues his own objective that can be possibly conflicting with the others (Myerson, 1991). Such a framework has attracted recent interest for economic aspects such as electronic commerce (Guttman et al., 1998), selfish routing in networks (Roughgarden, 2005), and algorithmic mechanism design (Nisan and Ronen, 1999). The concept of solution in such a scenario, in which multiple objectives are minimized at the same time, is not intuitive. Among the first proposed concept of solution (Borel, 1921; von Neumann, 1928; von Neumann and Morgenstern, 1944), the so called *Nash equilibrium* (Cournot and Fisher, 1897), is one of the most ubiquitous in the game theory literature.

In this paper, we consider the class of two–players games in which the set of actions (usually called *pure strategies*) of each player is finite, usually known as *bimatrix games* (Lemke and Howson, 1964; Mangasarian, 1964). A *mixed strategy* for such games is a collection of non–negative numbers, which have unit sum and are in one to one correspondence with pure strategies, with the understanding that each of this number corresponds to the probability that the player uses the corresponding action (Aumann and Hart, 1992). In Nash (1951), it is proved that every finite game has an equilibrium point in mixed policies, while in Mangasarian and Stone (1964), it is shown that, for such games, a point is a Nash equilibrium if and only if it is a solution of a single programming problem with linear constraints and a quadratic objective function that has a global maximum of zero.

The problem of finding any Nash equilibrium, even for games involving only two players, has been recently proved to be complete in the PPAD (polynomial parity argument, directed version) class (Mehta, 2014; Papadimitriou, 1992). This fact has leaded to the definition of *ε–Nash equilibrium*, *i.e.*, a strategy profile such that no deviating player could achieve a cost lower than the nominal one minus ε (Spirakis, 2015). In Chen et al. (2006), it is shown that, if ε is smaller than a given constant, dependent on the dimension of the game, the problem of computing an ε-Nash equilibrium of a bimatrix game is still PPAD-complete. However, by allowing higher values for the constant ε, it can be shown that an ε-Nash can be computed in polynomial time (Kontogiannis et al., 2006).

Due to the high computational complexity of the available algorithms to obtain Nash equilibria, it seems of interest to deal with an inverse problem, *i.e.*, to fix a bimatrix game, whose Nash equilibria are known, and to compute a set of bimatrix games that admit the same Nash equilibria. Such a goal is pursued here, considering *strategically equivalent games* (Başar and Olsder, 1999). This objective is reached by characterizing the set of all the polynomials that are monotone strictly increasing in a given interval and by solving, through computational algebraic geometry, a set of equalities involving these polynomials. It is worth to remark that similar techniques have been used in Possieri and Sassano (2015, 2016) to characterize the Nash equilibria of linear and nonlinear dynamical games.

2. BIMATRIX GAMES

Consider a two–player game G in which two players P_1 and P_2 are allowed to select policies within *action spaces* Γ_1 and Γ_2 (Hespanha, 2016). Letting $\gamma \in \Gamma_1$, $\sigma \in \Gamma_2$ be the actions of the two players, the task of player P_i, $i = 1, 2$, is to minimize the cost function

$$G_i : \Gamma_1 \times \Gamma_2 \to \mathbb{R},$$

without caring of the cost function of the other player. Note that, however, the two cost functions may be conflicting, leading to a competitive game.

A pair of policies $(\gamma^\star, \sigma^\star)$, with $\gamma^\star \in \Gamma_1$ and $\sigma^\star \in \Gamma_2$ is a *Nash equilibrium* of the game G if

$$\begin{aligned} G_1(\gamma^\star, \sigma^\star) &\leqslant G_1(\gamma, \sigma^\star), \qquad \forall \gamma \in \Gamma_1, \\ G_2(y^\star, \sigma^\star) &\leqslant G_2(\gamma^\star, \sigma), \qquad \forall \sigma \in \Gamma_2. \end{aligned}$$

⋆ This work is partially supported by the U.S. Army Research Laboratory and the U.S. Office of Naval Research under MURI grant No. N00014-16-1-2710.

A *pure bimatrix game* is a game where the action spaces of the two players are finite, i.e., $\Gamma_1 = \{1, \dots, m\}$ and $\Gamma_2 = \{1, \dots, n\}$, $m, n \in \mathbb{Z}$, $m \geqslant 1$, $n \geqslant 1$. The outcomes for the two players can be identified, in such games, by two matrices $A^G, B^G \in \mathbb{R}^{m \times n}$ such that

$$\begin{aligned} a^G_{ij} &= G_1(i,j), \qquad i = 1, \dots, m,\ j = 1, \dots, n, \\ b^G_{ij} &= G_2(i,j), \qquad i = 1, \dots, m,\ j = 1, \dots, n, \end{aligned}$$

where a^G_{ij} (respectively, b^G_{ij}) denotes the (i,j)–th entry of the matrix A^G (respectively, B^G). With such an encoding, the actions of players can be identified as

P_1 selects a row of the matrices A^G and B^G,
P_2 selects a column of the matrices A^G and B^G.

Bimatrix games also admit *mixed policies*, in the sense that P_1 (respectively, P_2) chooses a probability distribution on the selection of the rows (respectively, columns) of the matrices A^G and B^G. In this framework, a *mixed policy for* P_1 is a m–tuple $(y_1, \dots, y_m)$ such that $y_i \geqslant 0$, $i = 1, \dots, m$, and $\sum_{i=1}^m y_i = 1$, where y_i is the probability that P_1 uses the action i, $i = 1, \dots, m$. The set of all the mixed policies for P_1 is denoted $\mathcal{Y}$. On the other hand, a *mixed policy for* P_2 is a n–tuple $(z_1, \dots, z_n)$ such that $z_j \geqslant 0$, $j = 1, \dots, n$, and $\sum_{j=1}^n z_j = 1$, where z_j is the probability that player P_2 uses the action j, $j = 1, \dots, n$. The set of all the mixed policies for P_2 is denoted $\mathcal{Z}$. When dealing with mixed policies, a pair of policies $(y^\star, z^\star)$, with $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ is a *mixed Nash equilibrium* if

$$\begin{aligned} (y^\star)^\top A^G z^\star &\leqslant y^\top A^G z^\star, \qquad \forall y \in \mathcal{Y}, \\ (y^\star)^\top B^G z^\star &\leqslant y^\top B^G z, \qquad \forall z \in \mathcal{Z}. \end{aligned}$$

In Nash (1951), it is proved that every bimatrix game has at least one mixed Nash equilibrium. Moreover, in Mangasarian (1964), it is shown that $(y^\star, z^\star)$ is a Nash equilibrium if and only if it is a solution to

$$\begin{aligned} &\text{minimize } y^\top (A^G + B^G) z - p - q \text{ subject to:} \\ &\quad A^G z \dot{\geqslant} p\mathbb{1}, \quad (B^G)^\top y \dot{\geqslant} q\mathbb{1}, \\ &\quad y \in \mathcal{Y}, \quad z \in \mathcal{Z}, \end{aligned} \tag{1}$$

where $\dot{\geqslant}$ denotes the entry wise inequality $\geqslant$, and $\mathbb{1}$ denotes a vector of ones. Additionally, the minimum of the cost function in the problem given in (1) is exactly zero.

Consider two generic games G and H, with the same actions spaces Γ_1 and Γ_2, and such that:

- $G_1(\gamma, \sigma)$ and $G_2(\gamma, \sigma)$ are the outcomes in the game G for player P_1 and P_2, respectively.
- $H_1(\gamma, \sigma)$ and $H_2(\gamma, \sigma)$ are the outcomes in the game H for player P_1 and P_2, respectively.

The games G and H are *weakly strategically equivalent* if there exists two monotone strictly increasing scalar functions ϕ and ψ such that, $\forall \gamma \in \Gamma_1, \forall \sigma \in \Gamma_2$,

$$H_1(\gamma, \sigma) = \phi(G_1(\gamma, \sigma)), \quad H_2(\gamma, \sigma) = \psi(G_2(\gamma, \sigma)).$$

By Lemma 9.1 of Hespanha (2016), if G and H are strategically equivalent, they have the same Nash equilibria in pure policies. If, in addition, the games G and H admit the same Nash equilibria in mixed policies, then G and H are said to be *strongly strategically equivalent*. The following theorem characterizes weak and strong strategic equivalence for bimatrix games.

Theorem 1. *Let G be a bimatrix game identified by matrices $A^G, B^G \in \mathbb{R}^{m \times n}$ and let H be a bimatrix game identified by matrices A^H, B^H.*
i) If there exist monotonically strictly increasing functions ϕ, ψ such that for $i = 1, \dots, m$, $j = \dots, n$,

$$a^G_{ij} = \phi(a^H_{ij}), \quad b^G_{ij} = \psi(b^H_{ij}), \tag{2}$$

then G and H are weakly strategically equivalent.
ii) If, in addition, the functions ϕ and ψ are affine, G and H are strongly strategically equivalent.

One of the main objectives of this paper is formalized in the following problem.

Problem 1. Let a bimatrix game G, identified by matrices A^G and B^G be given.
(1) Find a family of games $\mathcal{F}_m(G)$ such that every game $\bar{G} \in \mathcal{F}_m(G)$ is strongly strategically equivalent to G.
(2) Find a family of games $\mathcal{F}_p(G)$ such that every game $\bar{G} \in \mathcal{F}_p(G)$ is weakly strategically equivalent to G.

If one is able to solve Problem 1(1), one obtains a set of games that, by Theorem 1.ii), have the same Nash equilibria of G in mixed policies. On the other hand if one is able to solve Problem 1(2), one obtains a set of games that, by Theorem 1.i) and Lemma 9.1 of Hespanha (2016), have the same Nash equilibria of G in pure policies. This feature is practically desirable in the process of designing of bimatrix game, because, by computing such families, one can design games with desired Nash equilibria, either in pure or in mixed policies, by computing the Nash equilibria of a single bimatrix game. Note that the families $\mathcal{F}_m(G)$ and $\mathcal{F}_p(G)$ are nonempty, because they contain G.

Hence, by letting G be a game having Nash equilibria $(y^\star_l, z^\star_l)$, $l = 1, \dots, L$, by solving Problem 1(1), we can compute a family of games that have $(y^\star_l, z^\star_l)$, $l = 1, \dots, L$, as Nash equilibria in mixed policies. On the other hand, by letting A and B be two matrices such that there exist pairs of indexes $(i^\star_l, j^\star_l)$, $l = 1, \dots, L$, such that $a_{i^\star_l j^\star_l} \leqslant a_{i, j^\star_l}$ and $b_{i^\star_l j^\star_l} \leqslant b_{i^\star_l, j}$, for all i, j, then, by solving Problem 1(2), we can compute a family of games that have $(i^\star_l, j^\star_l)$, $l = 1, \dots, L$, as Nash equilibria in pure policies.

Example 1. Consider the bimatrix game G defined by

$$A^G = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B^G = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}. \tag{3}$$

By solving the problem given in (1), one has that $\{(1/2, 1/2), (1, 0), (0, 1)\}$ is the set of all the Nash equilibria in mixed policies of game G. On the other hand, it can be easily checked that $\{(1, 1), (2, 2)\}$ is the set of all the Nash equilibria of game H in pure policies.

Hence, by solving Problem 1(1) with respect to G, one can obtain a family of games that have $(1/2, 1/2), (1, 0), (0, 1)$ as Nash equilibria in mixed policies, without actually computing the Nash equilibria of these games. On the other hand, if we are able to solve Problem 1(2), one can obtain a family of games that have the pairs $(1, 1)$ and $(2, 2)$ as pure Nash equilibria. △

3. ALGEBRAIC GEOMETRY NOTIONS

In this section, algebraic geometry notions are recalled following the exposition in Cox et al. (2015). These concepts are used in Section 5 to compute a solution to Problem 1.

Let $x = [\, x_1 \ \cdots \ x_n \,]^\top$. Let $\mathbb{K}$ be a field (in the following either $\mathbb{R}$ or $\mathbb{C}$). Let $\mathbb{K}[x_1, \dots, x_n]$ (briefly, $\mathbb{K}[x]$) denote the

ring of all the scalar polynomials in x with coefficients in $\mathbb{K}$. Given a set $\{p_1, \dots, p_s\}$ of polynomials in $\mathbb{K}[x]$, the *ideal* generated by $p_1, \dots, p_s$ is

$$\langle p_1, \dots, p_s \rangle := \{q : \exists h_i \in \mathbb{K}[x] \text{ s.t. } q = \textstyle\sum_{i=1}^{s} h_i p_i\};$$

while the *affine variety* defined by $p_1, \dots, p_s$ is the set

$$\boldsymbol{V}(p_1, \dots, p_s) = \{x \in \mathbb{K}^n : p_j(x) = 0,\ j = 1, \dots, s\}.$$

Given an ideal $\mathcal{I}$ in $\mathbb{K}[x]$, the *affine variety of* $\mathcal{I}$ is

$$\boldsymbol{V}(\mathcal{I}) = \{x \in \mathbb{K}^n : p(x) = 0,\ \forall p \in \mathcal{I}\}.$$

Let $\mathbb{Z}_{\geqslant 0}$ denote the set of all integers greater than or equal to zero. Let $\mathbb{Z}^n_{\geqslant 0}$ denote the set of all the n–tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_i \in \mathbb{Z}_{\geqslant 0}$, $i = 1, \dots, n$. Define the set of monomials $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. A *monomial order* on $\mathbb{K}[x]$, denoted $>$, is a total, well ordering relation on the set of monomials x^α. The *lexicographic monomial ordering* (briefly, *lex order*), denoted $>_\ell$, is defined as: $x^\alpha >_\ell x^\beta$ if the first nonzero entry of $\alpha - \beta := (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$, is positive.

Let a monomial order on $\mathbb{K}[x]$ be fixed. The *leading monomial* of a polynomial $f \in \mathbb{K}[x]$, denoted $\mathrm{LM}(f)$, is the largest monomial appearing in f, while the *leading term* of f, denoted $\mathrm{LT}(f)$ is the term cx^α, where $x^\alpha = \mathrm{LM}(f)$. Given an ideal $\mathcal{I}$ in $\mathbb{K}[x]$, a finite set $\mathcal{G} = \{g_1, \dots, g_l\} \subset \mathcal{I}$ is a *Groebner basis* of $\mathcal{I}$ if $\langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_l) \rangle = \langle \mathrm{LT}(\mathcal{I}) \rangle$, where $\mathrm{LT}(\mathcal{I}) = \{cx^\alpha : \exists f \in \mathcal{I} \text{ such that } \mathrm{LT}(f) = cx^\alpha\}$.

Let $\overline{\mathbb{K}}$ denote the *algebraic closure* of $\mathbb{K}$, *i.e.*, the algebraic extension of $\mathbb{K}$ that is algebraically closed (for instance, $\overline{\mathbb{R}} = \mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C}$). By Zorn's Lemma, it can be shown that every field has an algebraic closure and that the algebraic closure of a field is unique up to an isomorphism that fixes every member of $\mathbb{K}$ (Atiyah and Macdonald, 1969). Let $\mathcal{I}$ be an ideal in $\mathbb{K}[x]$ and let $\boldsymbol{V}_{\overline{\mathbb{K}}}(\mathcal{I}) \subset \overline{\mathbb{K}}^n$ be defined as

$$\boldsymbol{V}_{\overline{\mathbb{K}}}(\mathcal{I}) = \{x \in \overline{\mathbb{K}}^n :\ q(x) = 0,\ \forall q \in \mathcal{I}\}.$$

Let $\mathcal{I}$ be an ideal in $\mathbb{K}[x]$. The *j–th elimination ideal of* $\mathcal{I}$ is $\mathcal{I}_j := \mathcal{I} \cap \mathbb{K}[x_{j+1}, \dots, x_n]$. The following two theorems are well known (Cox et al., 2015, Ch. 3, Thm. 2 and 3) and states the triangularization property of any Groebner basis and how a Groebner bases can be used to compute solutions to systems of polynomial equalities.

Theorem 2. *Let an ideal $\mathcal{I}$ in $\mathbb{K}[x]$ be given and let $\mathcal{G}$ be a Groebner basis of $\mathcal{I}$ according to the lex order with $x_1 >_\ell x_2 >_\ell \dots >_\ell x_n$. Then, for every $j \in \{0, 1, \dots, n\}$, a Groebner basis $\mathcal{G}_j$ of the j–th elimination ideal of $\mathcal{I}$ is*

$$\mathcal{G}_j = \mathcal{G} \cap \mathbb{K}[x_{j+1}, \dots, x_n].$$

Theorem 3. *Let $\mathcal{I} = \langle p_1, \dots, p_s \rangle$ be an ideal in $\mathbb{C}[x]$ and let $\mathcal{I}_1$ be the first elimination ideal of $\mathcal{I}$. For each $1 \leqslant i \leqslant s$, let $c_i \in \mathbb{C}[x_2, \dots, x_n]$, $c_i \neq 0$, be such that $p_i = c_i x_1^{N_i} +$ terms in which x_1 has degree striclty lower than N_i, for some $N_i \geqslant 0$. Assume that there exists a partial solution $(\bar{x}_2, \dots, \bar{x}_n) \in \boldsymbol{V}(\mathcal{I}_1)$. If $(\bar{x}_2, \dots, \bar{x}_n) \notin \boldsymbol{V}(c_1, \dots, c_s)$, then there exists $\bar{x}_1$ such that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in \boldsymbol{V}(\mathcal{I})$.*

4. MONOTONE STRICTLY INCREASING UNIVARIATE POLYNOMIALS

In this section, necessary and sufficient conditions such that an univariate polynomial p is monotone strictly increasing in a given interval are stated. These conditions are used in Section 5.2 to compute a solution to Problem 1(2).

Let p be an univariate polynomial in $\mathbb{R}[s]$ and let p' denote the derivative of p with respect to s. A polynomial p is *monotone strictly increasing* if $a > b$ implies $p(a) > p(b)$.

An univariate polynomial p in $\mathbb{R}[s]$ is a *sum of squares* (briefly, *SOS*) if there exist $q_1, \dots, q_m \in \mathbb{R}[s]$ such that $p = \sum_{i=1}^{m} q_i^2$. The following proposition is taken from Marshall (2008) and Menini and Tornambe (2015).

Proposition 1. *An polynomial p in $\mathbb{R}[s]$ is greater than or equal to zero for all s in $\mathbb{R}$ if and only if it is an SOS. Moreover, if p is an SOS, then there exists two polynomials $q_1, q_2 \in \mathbb{R}[s]$ such that $p = q_1^2 + q_2^2$.*

The following technical lemma gives necessary and sufficient conditions for the polynomial p being monotone strictly increasing on an interval $[a, b]$.

Lemma 1. *Let an interval $[a, b]$, with $b > a$ be given. Let p be an univariate polynomial in $\mathbb{R}[s]$ of degree $M + 1$ and let $p' = \sum_{k=0}^{M} \kappa_k s^k$. Define $z \in \mathbb{R}[r]$,*

$$z := (1 + r^2)^M \textstyle\sum_{k=0}^{M} \kappa_k (a + (b - a)\frac{r^2}{1+r^2})^k. \tag{4}$$

The polynomial p is monotone strictly increasing in $[a, b]$ if and only if z is a sum of two squares.

Corollary 1. *Let the assumptions of Lemma 1 hold; p is monotone strictly increasing in $[a, b]$ if and only if there exist $q_1 = \sum_{k=0}^{M} \delta_k r^k$, $q_2 = \sum_{k=0}^{M} \epsilon_k r^k$ such that $z = q_1^2 + q_2^2$.*

Let z be the polynomial given in (4), let θ_d, $d = 0, \dots, 2M$, be such that $z = \sum_{d=0}^{2M} \theta_d r^d$ and let $\hat{\theta}_d$, $d = 0, \dots, 2M$, be such that $\hat{z} := q_1^2 + q_2^2 = \sum_{d=0}^{2M} \hat{\theta}_d r^d$, with $q_1 = \sum_{k=0}^{M} \delta_k r^k$, $q_2 = \sum_{k=0}^{M} \epsilon_k r^k$. Note that, letting $\kappa = [\,\kappa_0 \ \cdots \ \kappa_M\,]^\top$, $\delta = [\,\delta_0 \ \cdots \ \delta_M\,]^\top$, $\epsilon = [\,\epsilon_0 \ \cdots \ \epsilon_M\,]^\top$, one has that $\theta_d \in \mathbb{R}[\kappa]$, $\hat{\theta}_d \in \mathbb{R}[\delta, \epsilon]$, and $z = \hat{z}$ if and only if $\theta_d = \hat{\theta}_d$, $d = 0, \dots, 2M$, for some κ, δ, and ϵ. Moreover, since r appears in z only with even powers, we have that $\theta_d = 0$, for d odd, $d < 2M$. Hence, to compute a parametrization of the set of all the polynomials strictly increasing in $[a, b]$, we have to compute the solution in κ to the system of algebraic equation $\hat{\theta}_d = \theta_d$, d even, and $\hat{\theta}_d = 0$, d odd, $d \in \{0, \dots, 2M\}$. Note that $\hat{\theta}_d = 0$, d odd, is a set of algebraic equation only in δ, ϵ. Thus, the solution to this equations is a set of algebraic constraint on δ, ϵ and a set of expressions of the parameters in κ as functions of δ, ϵ.

Let $\mathcal{S}^d := \{(m, l) : m, l \in \mathbb{Z}_{\geqslant 0}, m \leqslant M, l \leqslant M, m + l = d\}$. It can be easily checked that, for $d = 0, \dots, 2M$,

$$\hat{\theta}_d = \textstyle\sum_{(m,l) \in \mathcal{S}^d} \delta_m \delta_l + \epsilon_m \epsilon_l. \tag{5}$$

On the other hand, by (4), one has that the coefficient κ_k in z multiplies, for $k = 0, \dots, M$, the term

$$\textstyle\sum_{m=0}^{M-k} \sum_{l=0}^{k} \binom{M-k}{m} \binom{k}{l} a^{k-l} b^l r^{2(m+l)}.$$

Hence, by defining the set $\mathcal{R}_k^d = \{(m, l) : m, l \in \mathbb{Z}_{\geqslant 0}, m \leqslant M - k, l \leqslant k, 2(m + l) = d\}$, one has that

$$\theta_d = \textstyle\sum_{k=0}^{M} \kappa_k \sum_{(m,l) \in \mathcal{R}_k^d} \binom{M-k}{m} \binom{k}{l} a^{k-l} b^l.$$

Note that $\theta_d = 0$ for d odd, because $\mathcal{R}_k^d = \emptyset$ for d odd. Hence, to compute a solution to $\hat{\theta}_d = \theta_d$, d even, define $T^M \in \mathbb{R}^{M+1 \times M+1}$, $w^M \in \mathbb{R}^{M+1}$,

$$T_{ij}^M = \textstyle\sum_{(m,l) \in \mathcal{R}_{j-1}^{2(i-1)}} \binom{M-(j-1)}{m} \binom{(j-1)}{l} a^{(j-1)-l} b^l, \tag{6a}$$

$$w_i^M = \textstyle\sum_{(m,l) \in \mathcal{S}^{2(i-1)}} \delta_m \delta_l + \epsilon_m \epsilon_l, \tag{6b}$$

where T^M_{ij} is the (i,j)–th entry of matrix T^M and w_i is the i–th entry of vector w, $i,j = 1,\dots,M+1$.

Proposition 2. *If $b > a$, T^M is invertible for any M.*

By Proposition 2, a solution to the set of equalities $\hat\theta_d = \theta_d$, d even, in κ is given by $\kappa = (T^M)^{-1}w^M$. We still need to solve $\hat\theta_d = 0$, d odd, $1 \leqslant d \leqslant 2M-1$, that is a set of algebraic equalities in the unknowns δ, ϵ. Theorems 2 and 3 are the correct instruments to compute the solution to these inequalities. Namely, by defining $\xi = [\,\xi_1 \;\cdots\; \xi_{2M+2}\,]^\top$, where $\xi_i = \delta_{M+1-i}$, $i = 1,\dots,M+1$, $\xi_j = \epsilon_{2M+2-j}$, $j = M+1,\dots,2M+2$. and by letting

$$\mathcal{I}^M = \langle \hat\theta_1, \hat\theta_3, \dots, \hat\theta_{2M-1}\rangle, \tag{7}$$

in $\mathbb{R}[\xi]$, let $\mathcal{G}$ be a Groebner basis of $\mathcal{I}^M$ according to $\xi_1 >_\ell \xi_2 >_\ell \dots >_\ell \xi_{2M+2}$. Let $\mathcal{I}^M_i$ be the i–th elimination ideal of $\mathcal{I}^M$. By Theorem 2, a Groebner basis of $\mathcal{I}^M_i$ is given by $\mathcal{G}_i = \{g_{i,1}, g_{i,2}, \dots, g_{i,l_i}\} = \mathcal{G} \cap \mathbb{R}[\xi_{i+1},\dots,\xi_{2M+2}]$. Let $c_{i,j} \in \mathbb{R}[\xi_{i+2},\dots,\xi_{2M+2}]$ be such that $g_{i,j} = c_{i,j}\xi_{i+1}^{N_j} +$ terms in which ξ_{i+1} has degree striclty lower than N_j, for some $N_j \geqslant 0$, $j = 1,\dots,l_i$. Letting $(\bar a_{i+2},\dots,\bar a_{2M+2})$ be such that $(\bar a_{i+2},\dots,\bar a_{2M+2}) \in \boldsymbol{V}_{\mathbb{R}}(G_{i+1})$, if $(\bar a_{i+2},\dots,\bar a_{2M+2}) \notin \boldsymbol{V}_{\mathbb{R}}(c_{i,1},\dots,c_{i,l_i})$, then there exists $\bar a_{i+1}$ such that $(\bar a_{i+1},\bar a_{i+2},\dots,\bar a_{2M+2}) \in \boldsymbol{V}_{\mathbb{R}}(G_i)$. It is also possible to check what solutions such that $(\bar a_{i+2},\dots,\bar a_{2M+2}) \in \boldsymbol{V}_{\mathbb{R}}(c_{i,1},\dots,c_{i,l_i})$ extend to $(\bar a_{i+1},\bar a_{i+2},\dots,\bar a_{2M+2})$ by defining the ideal $\mathcal{J} = \mathcal{I}^M + \langle c_{i,1},\dots,c_{i,l_i}\rangle$ and by repeating such a procedure for the ideal $\mathcal{J}$. Hence, letting $\bar i$ be the smallest integer such that $\mathcal{G}_{\bar i} \neq \emptyset$, it is possible to iteratively apply this procedure backward to compute the set of all the solutions in $\boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^M)$. Therefore, the set of all the $p' \in \mathbb{R}[s]$ of degree M that are nonnegative in $[a,b]$ can be computed as

$$\begin{aligned} p' &= \textstyle\sum_{k=0}^{M} \kappa_k s^k, \\ \kappa &= (T^M)^{-1} w^M(\xi), \\ \xi &\in \boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^M) \cap \mathbb{R}^{2M+2}, \end{aligned} \tag{8}$$

where $\boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^M)$ can be obtained with the procedure given above. Note that the set $\boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^M) \cap \mathbb{R}^{2M+2}$ is nonempty, since the set $\{\xi \in \mathbb{R}^{2M+2} : \delta_d = 0, \epsilon_d = 0, d \text{ odd}\}$ is a subset of $\boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^M) \cap \mathbb{R}^{2M+2}$, because if $\delta_d = 0, \epsilon_d = 0$, then, by (5), $\hat\theta_d = 0$, for d odd. Thus, the set of all the monotone strictly increasing polynomials p in $[a,b]$ is

$$p = \textstyle\sum_{k=0}^{M+1} \bar\kappa_k s^k, \tag{9}$$

with $\bar\kappa_k = \frac{\kappa_{k-1}}{k}$, $k = 1,\dots,M+1$, $\bar\kappa_0 \in \mathbb{R}$, κ satisfies (8). Note that techniques similar to the ones presented in this section have been used in Menini et al. (2015) to achieve dead–beat regulation for mechanical jugglers.

5. COMPUTATION OF STRATEGICALLY EQUIVALENT BIMATRIX GAMES

In this section, a procedure to compute a solution to Problem 1 is given. Note that the conditions of Theorem 1 (namely, the equality relations in (2)) need to hold for $n \cdot m$ points (namely, the entries of matrices A^G and B^G). By the Lagrange interpolation formula (Bakhvalov, 1977), given a finite set of points $\mathcal{P} = \{\bar s_1,\dots,\bar s_m\}$ (usually called *sample points*), there exists a polynomial p of degree lower than or equal to $m-1$ such that, for any function $f : \mathbb{R} \to \mathbb{R}$, $p(\bar s) = f(\bar s)$, $\bar s \in \mathcal{P}$. In addition, letting $\bar m \leqslant m-1$ be the smallest integer such that there exists a polynomial p of degree $\bar m$ such that $p(\bar s) = f(\bar s)$, for all $\bar s \in \mathcal{P}$, one has that there exists a unique polynomial p of degree $\bar m$ (called *Lagrange form of f with respect to $\mathcal{P}$*) such that $p(\bar s) = f(\bar s)$, for all $\bar s \in \mathcal{P}$. The following example shows that, even if the function f is monotone strictly increasing in an interval $[a,b]$, the Lagrange form of f may not be monotone on $[a,b]$.

Example 2. Let $f = 3\cos(2s) + 7s$ and $[a,b] = [0,3]$. Function f is strictly increasing in $[a,b]$, because $f'(s) > 0$, for all s in $[a,b]$. Consider the sample points $\mathcal{P} = \{0, 1.5, 3\}$. The Lagrange form of f with respect to $\mathcal{P}$ is $p = -\frac{8}{3}s^2\sin^2\left(\frac{3}{2}\right)\cos(3) + s(4 + 4\cos(3) - \cos(6)) + 3$. Clearly, such a polynomial is not strictly increasing because $p'(s) < 0$, for $0 < s < \frac{-12-12\cos(3)+3\cos(6)}{4-8\cos(3)+4\cos(6)}$. However, by allowing higher order polynomials, we can use (8)–(9) to compute a strictly increasing polynomial q such that $q(\bar s) = f(\bar s)$, for all $\bar s \in \mathcal{P}$. Namely, letting $M = 2$, one has to compute a solution to

$$\begin{aligned} f(\bar s) &= \bar\kappa_0 + \textstyle\sum_{k=1}^{3} \frac{\kappa_{k-1}}{k}\bar s^k, \quad \forall \bar s \in \mathcal{P}, \\ \kappa &= (T^2)^{-1} w^2(\xi), \\ \xi &\in \boldsymbol{V}_{\mathbb{R}}(\mathcal{I}^2) \cap \mathbb{R}^8, \end{aligned} \tag{10}$$

in $\bar\kappa_0$, ξ. Note that this is a set of algebraic equalities in ξ that can be solved by using Theorems 2 and 3. A solution to this problem is given by

$$q(s) = 1.066s^3 - 2.171s^2 + 3.878s + 3.$$

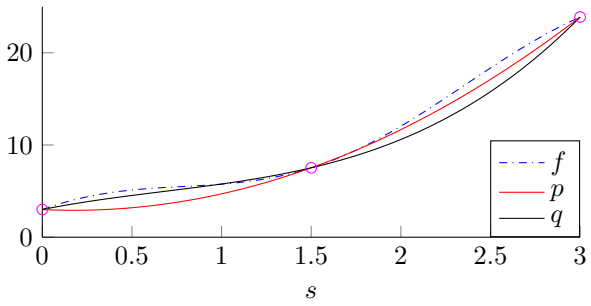

Fig. 1. Functions f, p, and q in Example 2.

Figure 1 depicts the functions f, p, and q in $[0,3]$. △

As shown in Example 2, even if there exists a function f such that $\bar s_i > \bar s_j$ implies $f(\bar s_i) > f(\bar s_j)$, the Lagrange form of f may be not strictly increasing. However, by allowing higher order polynomials, it may be possible to compute a monotone strictly increasing polynomial p such that $f(\bar s) = p(\bar s)$, for all $\bar s \in \mathcal{P}$, where $\mathcal{P}$ is a finite set of points. By this reasoning, to actually compute a solution to Problem 1, we will consider only polynomial functions.

5.1 Strongly strategically equivalent games

Let a bimatrix game G, identified by $m \times n$ matrices A^G and B^G be given. Consider a game H identified by

$$A^H = [a^H_{ij}]_{m\times n}, \quad B^H = [b^H_{ij}]_{m\times n}.$$

By Theorem 1.ii), if there exist two affine monotone strictly increasing functions ϕ and ψ such that

$$a^G_{ij} = \phi(a^H_{ij}), \quad b^G_{ij} = \psi(b^H_{ij}),$$

for $i = 1,\dots,m$, $j = \dots,n$, then the games G and H are strongly strategically equivalent. Hence, by letting $\phi = \phi_0 + \phi_1 s$, with $\phi_1 > 0$, $\psi = \psi_0 + \psi_1 s$, with $\psi_1 > 0$,

we have that H is strategically equivalent to G if, for $i = 1, \ldots, m$, $j = \ldots, n$,

$$a_{ij}^G = \phi_0 + \phi_1 a_{ij}^H, \quad b_{ij}^G = \psi_0 + \psi_1 b_{ij}^H$$

Note that, by Anderson et al. (1975), the inequality condition $\phi_1 > 0$ ($\psi_1 > 0$) is equivalent to the equality condition $\phi_1\omega_1^2 - 1 = 0$ (respectively, $\psi_1\omega_2^2 - 1 = 0$), where ω_1 (respectively, ω_2) is an auxiliary variable. Hence, to compute a family $\mathcal{F}_m(G)$ one has to solve the following set of algebraic equations:

$$\begin{array}{rr} a_{ij}^G - \phi_0 - \phi_1 a_{ij}^H = 0, & b_{ij}^G - \psi_0 - \psi_1 b_{ij}^H = 0, \\ \phi_1\omega_1^2 - 1 = 0, & \phi_2\omega_2^2 - 1 = 0, \end{array} \quad (11)$$

in the variables $a_{ij}^H, b_{ij}^H, \psi_0, \psi_1, \phi_0, \phi_1, \omega_1, \omega_2$. Note that (11) is a set of polynomial equalities in such indeterminates. Theorems 2 and 3 are correct instruments to solve this problem. Hence, define the ideal $\mathcal{I}$ on $\mathbb{R}[a_{ij}^H, b_{ij}^H, \psi_0, \psi_1, \phi_0, \phi_1, \omega_1, \omega_2]$,

$$\mathcal{I} = \langle a_{ij}^G - \phi_0 - \phi_1 a_{ij}^H, b_{ij}^G - \psi_0 - \psi_1 b_{ij}^H, \phi_1\omega_1^2 - 1, \phi_2\omega_2^2 - 1\rangle,$$

and compute a Groebner basis $\mathcal{G}$ of $\mathcal{I}$ according to $a_{11}^H >_\ell a_{12}^H >_\ell \cdots >_\ell a_{mn}^H >_\ell b_{11}^H >_\ell b_{12}^H >_\ell \cdots >_\ell b_{mn}^H >_\ell \phi_1 >_\ell \psi_1 >_\ell \phi_0 >_\ell \psi_0 >_\ell \omega_1 >_\ell \omega_2$. Note that, by Theorem 2, $\mathcal{G} \cap \mathbb{R}[\omega_2]$ is a Groebner basis of $\mathcal{I} \cap \mathbb{R}[\omega_2]$, $\mathcal{G} \cap \mathbb{R}[\omega_1, \omega_2]$ is a Groebner basis of $\mathcal{I} \cap \mathbb{R}[\omega_1, \omega_2]$ and so on. Hence, by recursively applying the technique given in Theorem 3, one can obtain the set of all the solutions to (11). The following example shows how to apply such a procedure for the bimatrix game given in Example 1.

Example 1 (mixed policies). Consider the bimatrix game given in Example 1. A Groebner basis of the ideal $\mathcal{I}$ for this game is $\mathcal{G} = \{g_1, \ldots, g_{10}\}$, with

$$\begin{array}{ll} g_1 = \omega_2^2\psi_1 - 1 & g_2 = \omega_1^2\phi_1 - 1 \\ g_3 = b_{2,2}^H + \omega_2^2\psi_0 + 2\omega_2^2 & g_4 = b_{2,1}^H + \omega_2^2\psi_0 - 2\omega_2^2 \\ g_5 = b_{1,2}^H + \omega_2^2\psi_0 - 3\omega_2^2 & g_6 = b_{1,1}^H + \omega_2^2\psi_0 + \omega_2^2 \\ g_7 = a_{2,2}^H + \omega_1^2\phi_0 + \omega_1^2 & g_8 = a_{2,1}^H + \omega_1^2\phi_0 \\ g_9 = a_{1,2}^H + \omega_1^2\phi_0 - \omega_1^2 & g_{10} = a_{1,1}^H + \omega_1^2\phi_0 + 2\omega_1^2. \end{array}$$

Note that we have $\mathcal{G} \cap \mathbb{C}[\phi_0, \psi_0, \omega_1, \omega_2] = \emptyset$, hence $\boldsymbol{V}(\mathcal{I} \cap \mathbb{C}[\phi_0, \psi_0, \omega_1, \omega_2]) = \mathbb{C}^4$ is a partial solution. To compute which partial solution $c \in \mathbb{C}^4$ extends to a solution to our problem, consider $\mathcal{G} \cap \mathbb{C}[\psi_1, \phi_0, \psi_0, \omega_1, \omega_2] = g_1$. By considering that $\mathrm{LT}(g_1) = \omega_2^2\psi_1$, we can extend the solution for $\omega_2 \neq 0$. Note also that $g_1 = 0$ has not a solution when $\omega_2 = 0$. Consider now $\mathcal{G} \cap \mathbb{C}[\phi_1, \psi_1, \phi_0, \psi_0, \omega_1, \omega_2] = \{g_1, g_2\}$. By considering that $\mathrm{LT}(g_2) = \omega_1^2\phi_1$ and $\mathrm{LT}(g_2) = (\omega_2^2\psi_1)\phi_1^0$, we can extend the solution when $\omega_1 \neq 0$. Note that, when $\omega_1 \neq 0$, there exists no solution. On the other hand, by considering that $\mathrm{LT}(g_3) = b_{2,2}^H$, $\mathrm{LT}(g_4) = b_{2,1}^H$, $\ldots$, $\mathrm{LT}(g_{10}) = a_{1,1}^H$, we have that the other solutions can be always extended. Note that the solutions obtained by applying Theorem 3 are in $\mathbb{C}$. However, once that the set of all solutions $\mathcal{S} \subset \mathbb{C}^{2mn+4}$ has been computed, one can easily obtain the subset of real solutions to the system of equalities, by computing $\mathcal{S} \cap \mathbb{R}^{2mn+4}$. Hence, a solution to Problem 1(1) for the game given in Example 1 is

$$A^H = \begin{bmatrix} -\phi_0\omega_1^2 - 2\omega_1^2 & \omega_1^2 - \omega_1^2\phi_0 \\ -\omega_1^2\phi_0 & -\phi_0\omega_1^2 - \omega_1^2 \end{bmatrix},$$

$$B^H = \begin{bmatrix} -\psi_0\omega_2^2 - \omega_2^2 & 3\omega_2^2 - \omega_2^2\psi_0 \\ 2\omega_2^2 - \omega_2^2\psi_0 & -\psi_0\omega_2^2 - 2\omega_2^2 \end{bmatrix},$$

with $\omega_1, \omega_2, \phi_0, \psi_0 \in \mathbb{R}$, $\omega_1 \neq 0$, $\omega_2 \neq 0$.

5.2 Weakly strategically equivalent games

Let a bimatrix game G, identified by $m \times n$ matrices A^G and B^G be given. Consider a game H identified by

$$A^H = [a_{ij}^H]_{m\times n}, \quad B^H = [b_{ij}^H]_{m\times n}.$$

By Theorem 1.i), if there exist two monotone strictly increasing functions ϕ and ψ such that (2) holds, then the games are weakly strategically equivalent. Hence, let $\underline{\varpi}_A = \min_i \min_j A_{ij}^G$, $\underline{\varpi}_B = \min_i \min_j B_{ij}^G$, $\overline{\varpi}_A = \max_i \max_j A_{ij}^G$ and $\overline{\varpi}_B = \max_i \max_j B_{ij}^G$. We can find a family of bimatrix games H solving Problem 1(2) by computing a set of functions ϕ and ψ that are monotone strictly increasing in the intervals $[\underline{\varpi}_A, \overline{\varpi}_A]$ and $[\underline{\varpi}_B, \overline{\varpi}_B]$, respectively. By allowing only polynomial functions of finite grade M, we can use (8)–(9) to compute a set of bimatrix games that are weakly strategically equivalent to the given game G. Namely, we say that two games G and H are *M–strategically equivalent* if there exist ϕ and ψ in $\mathbb{R}[s]$ of degree $M + 1$ in s such that

a) $a_{ij}^H = \phi(a_{ij}^G)$, $b_{ij}^H = \psi(b_{ij}^G)$, for all i, j;

b) ϕ and ψ are monotone strictly increasing in the intervals $[\underline{\varpi}_A, \overline{\varpi}_A]$ and $[\underline{\varpi}_B, \overline{\varpi}_B]$, respectively.

Note that, if the polynomial p is monotone strictly increasing in $[a, b]$, then, by the Implicit Function Theorem (Fleming, 1977), p is monotone and strictly increasing in $[a, b]$. In addition, the inverse function p^{-1} is invertible (Binmore, 1982). Hence, the set of all the games H that are M–strategically equivalent can be obtained as follows. Define the ideal $\mathcal{I}_M$ as in (7) and compute the set

$$\Xi = \boldsymbol{V}_{\overline{\mathbb{R}}}(\mathcal{I}^M) \cap \mathbb{R}^{2M+2},$$

as detailed in Section 4. Assume that $\underline{\varpi}_A \neq \overline{\varpi}_A$ and $\underline{\varpi}_B \neq \overline{\varpi}_B$ (the case $\underline{\varpi}_A = \overline{\varpi}_A$ or $\underline{\varpi}_B = \overline{\varpi}_B$ is considered at the end of this section). Define the matrix T_A^M (respectively, T_B^M) by using (6a) with a substituted by $\underline{\varpi}_A$ (respectively, $\underline{\varpi}_B$) and b substituted by $\overline{\varpi}_A$ (respectively, $\underline{\varpi}_B$). Hence, compute $\mu \in \mathbb{R}^M[\xi]$, $\nu \in \mathbb{R}^M[\xi]$,

$$\mu = (T_A^M)^{-1} w^M(\xi), \quad \nu = (T_B^M)^{-1} w^M(\xi).$$

Note that T_A^M and T_M^B are invertible by Proposition 2. Thus, define polynomials in $\mathbb{R}[s, \xi]$

$$\begin{array}{l} \phi(s, \xi, c_1) = c_1 + \sum_{k=1}^{M+1} \frac{\mu_{k-1}(\xi)}{k} s^k, \\ \psi(s, \xi, c_2) = c_2 + \sum_{k=1}^{M+1} \frac{\nu_{k-1}(\xi)}{k} s^k, \end{array}$$

where c_1, c_2 are additional parameters in $\mathbb{R}$. Therefore, by Corollary 1, the set of all the M–strategically equivalent games to G are the bimatrix games H identified by matrices A^H and B^H such that

$$\begin{array}{rr} a_{ij}^H = \phi(a_{ij}^G, \xi, c_1), & b_{ij}^H = \psi(b_{ij}^G, \xi, c_2), \\ \xi \in \Xi, & c_1, c_2 \in \mathbb{R}. \end{array} \quad (12)$$

Assume now that $\underline{\varpi}_A = \overline{\varpi}_A$. Clearly, this implies that $a_{ij}^G = \bar{a}$, $\bar{a} \in \mathbb{R}$, for all i, j. Therefore, the set of games that are weakly strategically equivalent to G are such that B^H satisfies (12), while A^H is a constant matrix. The same can be argued for $\underline{\varpi}_B = \overline{\varpi}_B$. The following example shows how to apply such a procedure to the game given in Example 1.

Example 1 (pure policies). Consider the bimatrix game given in Example 1. Let $M = 2$. By applying the procedure given in Section 4, one can compute the set Ξ, that is the set of all the vectors

$$\xi = [\, \delta_0 \;\; \delta_1 \;\; \delta_2 \;\; \epsilon_0 \;\; \epsilon_1 \;\; \epsilon_2 \,]^\top \in \mathbb{R}^6 \setminus \mathbb{0}$$

such that (at least) one of the following conditions hold:

- $\epsilon_1 = 0 \wedge \delta_1 = 0 \wedge \delta_2 \neq 0$;
- $\epsilon_1 = 0 \wedge \delta_2 = 0 \wedge \delta_1 = 0$;
- $\epsilon_1 = 0 \wedge \delta_2 = 0 \wedge \delta_1 \neq 0 \wedge \delta_0 = 0$;
- $\epsilon_2 = 0 \wedge \delta_2 = 0 \wedge \delta_1 \neq 0 \wedge \delta_0 = -\frac{\epsilon_0 \epsilon_1}{\delta_1}$;
- $\epsilon_2 = 0 \wedge \epsilon_0 = 0 \wedge \delta_1 = 0 \wedge \delta_2 \neq 0$;
- $\epsilon_2 = 0 \wedge \epsilon_1 = 0 \wedge \delta_2 = 0 \wedge \delta_1 = 0$;
- $\delta_2 \neq 0 \wedge \delta_1 = -\frac{\epsilon_1 \epsilon_2}{\delta_2} \wedge \epsilon_2 \neq 0 \wedge \delta_0 = \frac{\delta_2 \epsilon_0}{\epsilon_2}$;
- $\epsilon_2 \neq 0 \wedge \epsilon_1 = 0 \wedge \delta_2 = 0 \wedge \delta_0 = 0$;
- $\epsilon_2 = 0 \wedge \epsilon_1 \neq 0 \wedge \epsilon_0 = 0 \wedge \delta_2 = 0 \wedge \delta_1 = 0$;

where $\wedge$ denotes the logical AND operator. Hence, one has that $\underline{\varpi}_A = -2$, $\overline{\varpi}_A = 1$, $\underline{\varpi}_B = -2$, $\overline{\varpi}_B = 3$,

$$w^2 = \begin{bmatrix} \delta_0^2 + \epsilon_0^2 \\ \delta_1^2 + 2\delta_0\delta_2 + \epsilon_1^2 + 2\epsilon_0\epsilon_2 \\ \delta_2^2 + \epsilon_2^2 \end{bmatrix},$$

$$(T_A^2)^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 4 \\ -2 & -1 & 4 \\ 1 & -1 & 1 \end{bmatrix}, \qquad (T_B^2)^{-1} = \frac{1}{25}\begin{bmatrix} 9 & 6 & 4 \\ -6 & 1 & 4 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore, the set of all the 2–strategically equivalent games H to the game given in Example 1 is the set of all the bimatrix games identified by A^H and B^H such that

$$a_{ij}^H = \phi(a_{ij}^G), \quad b_{ij}^H = \psi(b_{ij}^G), \quad i,j = 1,2,$$

with $\phi(s) = \frac{1}{27}(-\delta_1^2+(\delta_0-\delta_2)^2-\epsilon_1^2+(\epsilon_0-\epsilon_2)^2)s^3+\frac{1}{18}(-\delta_1^2-2(\delta_0-\delta_2)(\delta_0+2\delta_2)-\epsilon_1^2-2(\epsilon_0-\epsilon_2)(\epsilon_0+2\epsilon_2))s^2+\frac{1}{9}(2\delta_1^2+(\delta_0+2\delta_2)^2+2\epsilon_1^2+(\epsilon_0+2\epsilon_2)^2)+c_1$, $\psi(s) = \frac{1}{75}(-\delta_1^2+(\delta_0-\delta_2)^2-\epsilon_1^2+(\epsilon_0-\epsilon_2)^2)s^3+\frac{1}{50}(-6\delta_0^2+2\delta_2\delta_0+\delta_1^2+4\delta_2^2-6\epsilon_0^2+\epsilon_1^2+4\epsilon_2^2+2\epsilon_0\epsilon_2)s^2+\frac{1}{25}(6\delta_1^2+(3\delta_0+2\delta_2)^2+6\epsilon_1^2+(3\epsilon_0+2\epsilon_2)^2)+c_2$, where $\delta_0, \delta_1, \delta_2, \epsilon_0, \epsilon_1, \epsilon_2, c_1, c_2$ are free parameters in $\mathbb{R}$ such that (at least) one of the conditions of above holds.

6. CONCLUSIONS

A class of games that have the same Nash equilibrium, either in pure or in mixed policies, has been characterized. In such a way, a whole family of games that share the same Nash equilibria can be computed without requiring the explicit computation of the solution. Such an approach is of interest because it has been proved that the computation of a Nash equilibrium of a bimatrix game is generically PPAD complete, even if one allows ε–Nash equilibria.

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