Discrete Approximation to Continuous Anisotropic Shortest-Path Problem

Application to Minimum-Risk Path Planning for Groups of UAVs

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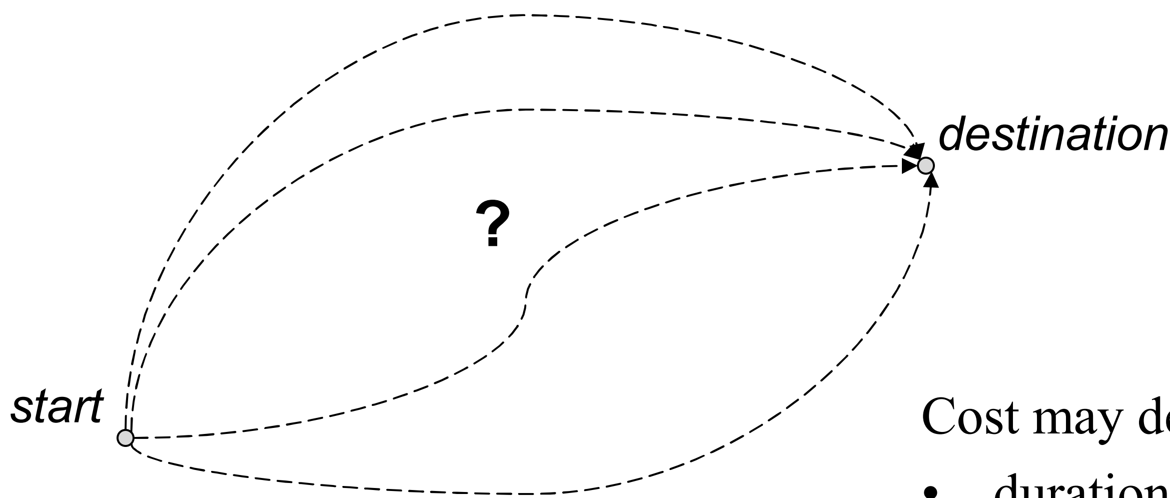
Continuous Routing Problem

Problem: find the minimum-cost path from the start to the destination point



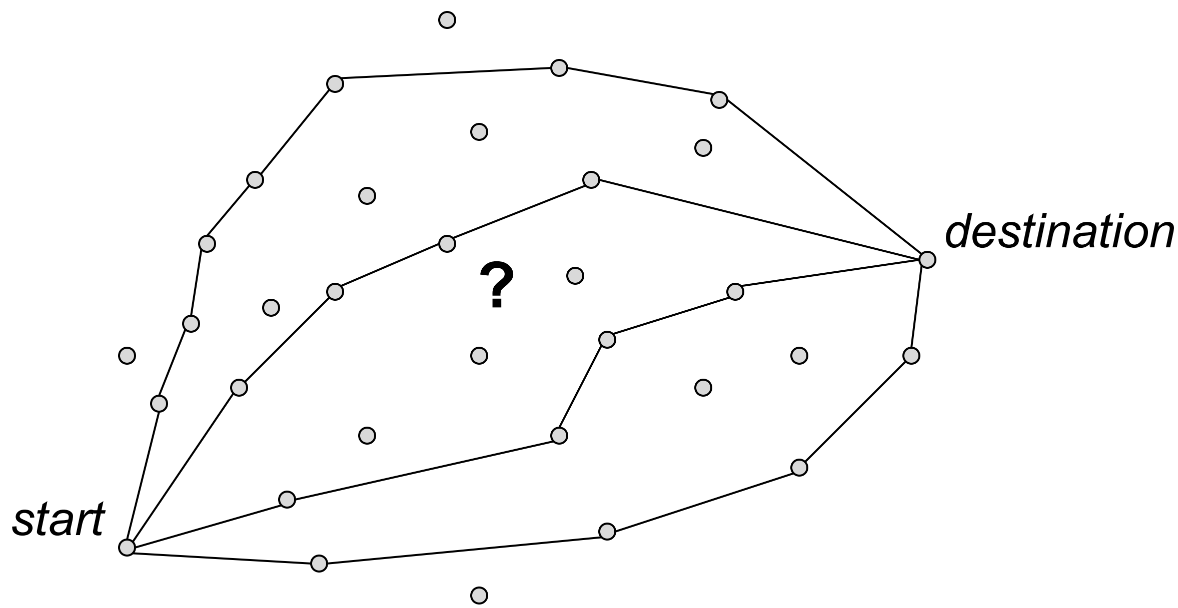
Cost may depend on

- duration of path
- fuel consumed along path
- probability of detection by adversary

It is often helpful to approximate this problem on a ***discrete domain*** *by using a* ***graph****.*

Continuous to Discrete

Paths consist of straight line segments between graph vertices.



Approximation accuracy and *computation time* depend heavily on the choice of graph.

- *Where to place vertices?*
- *How to connect edges?*

Can we achieve paths that are arbitrarily close to optimal?

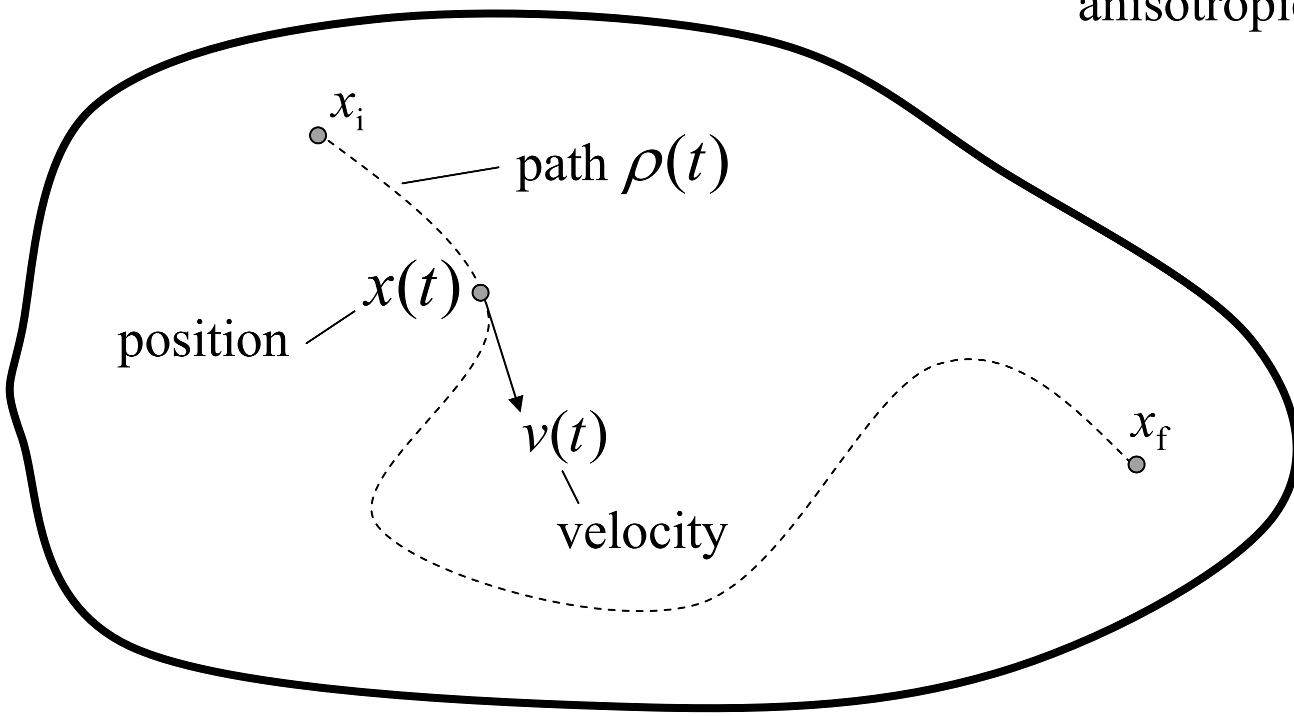
Continuous Weighted Anisotropic Shortest Path Problem

Problem: find the minimum-cost path from the start to the destination point

$$\rho^* = \arg\min_{\rho \in \mathcal{P}} J[\rho] \qquad J[\rho] := \int_0^T \ell(x(t), v(t))\,dt$$

$$\mathcal{P} := \{\rho \in C \ : \|\dot{\rho}(t)\| = 1 \text{ and } \|\ddot{\rho}(t)\| \leq A\}$$

anisotropic cost function



No computationally efficient method is known for solving this continuous problem.

Related Work

Discrete approximations to continuous anisotropic shortest-path problems

- **Rowe and Ross, 1990.**
  - Ground vehicles moving on hilly terrain described by polygonal mesh (2½D)
  - Exact algorithm with worst-case complexity $O(n^n)$.
- **Lanthier, Maheshwari, and Sack, 1999.**
  - Polynomial time approximation algorithms.
- **Kim and Hespanha, 2003.**
  - Approximation error can be made arbitrarily small.
  - Discretization based on nonuniform sampling.
- **Jia and Varaiya, 2004**.
  - Grid-based discretization with shifting.
- **Zabarankin, Uryasev, and Murphey, 2006.**
  - Constrained shortest-path (CSP) problem in 3D.
  - Incorporates ellipsoid model of radar cross sections of UAVs.
- **Carlyle, Royset, and Wood, 2007.**
  - Accounts for many types of mission constraints and threats.
  - Solves using a fast new CSP algorithm.

Discrete Approximation to Continuous Shortest Path

Previous Result (Kim, Hespanha, 2003):

$\mathcal{P}_{\mathcal{X}}-$ set of all piecewise linear paths connecting points in $\mathcal{X}$

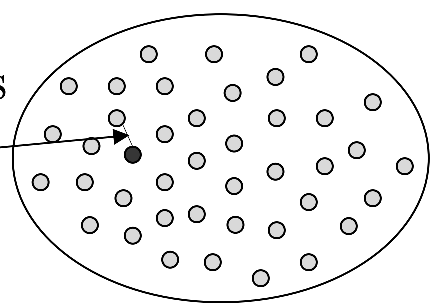
For every $A, \epsilon > 0$, there exists a finite set $\mathcal{X}$ of points such that for every initial and final points x_i, x_f,

$$\inf_{\rho \in \mathcal{P}_{\mathcal{X}}} J[\rho] \leq \inf_{\rho \in \mathcal{P}} J[\rho] + \epsilon$$

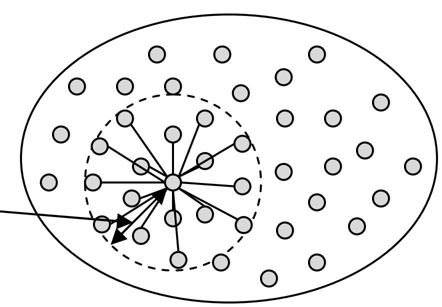
.

Given ϵ, one can compute constants ϵ_x and ϵ_δ, and the graph must satisfy:

Distance less than ϵ_x



Connect to all vertices within a distance ϵ_δ



Discrete Approximation to Continuous Shortest Path

New Result (Honeycomb sampling theorem):

For every $A, \epsilon > 0$, there exists

1. a finite convex partition of the region of interest and
2. a finite set $\mathcal{X}$ of points in the boundaries of the partition

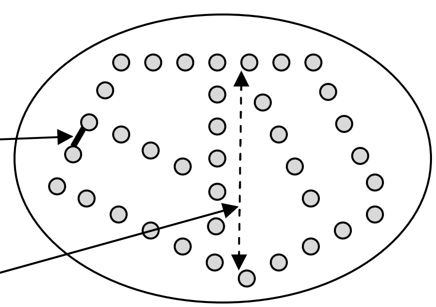
such that for every initial and final points x_i, x_f,

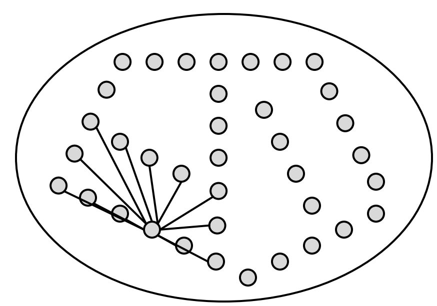
$$\inf_{\rho \in \mathcal{P}_{\mathcal{X}}} J[\rho] \leq \inf_{\rho \in \mathcal{P}} J[\rho] + \epsilon.$$

Given ϵ, one can compute constants ϵ_x and ϵ_δ, and the graph must satisfy:

Distance less than ϵ_x

Diameter less than ϵ_δ





Connect all vertices in the same partition element

Computational Complexity

Complexity of all-pairs shortest path algorithm: $O(mn + n \ \log n)$

$(n = \text{number of vertices}, m = \text{number of edges})$

Previous sampling theorem

$$n = \left(\frac{A_R}{\varepsilon_x^2}\right) \qquad m = O\left(n \cdot \frac{\pi\varepsilon_\partial^2}{A_R} n\right) = O\left(\frac{A_R \pi \varepsilon_\partial^2}{\varepsilon_x^4}\right)$$

Complexity: $$O\left(\frac{A_R^2 \pi \varepsilon_\partial^2}{\varepsilon_x^6} + \frac{A_R^2}{\varepsilon_x^4} \log\left[\frac{A_R}{\varepsilon_x^2}\right]\right)$$

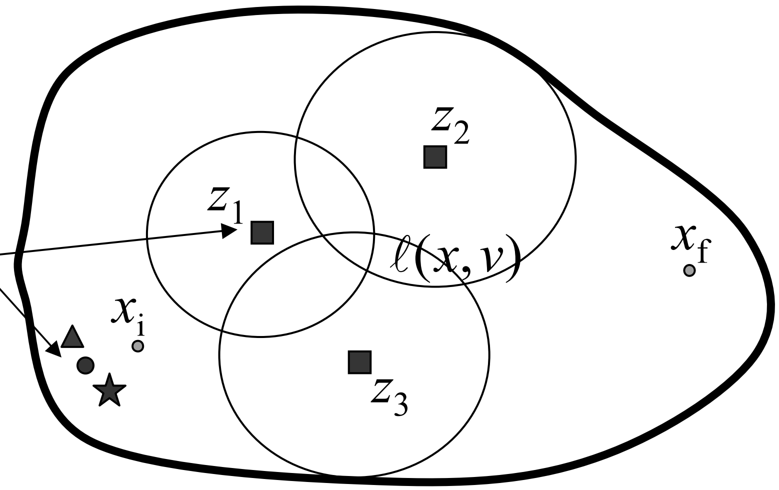
Honeycomb sampling theorem

$$n = O\left(\frac{\pi\varepsilon_\delta}{\varepsilon_x} \cdot \frac{A_R}{\pi\left(\varepsilon_\delta / 2\right)^2}\right) = O\left(\frac{4A_R}{\varepsilon_x \varepsilon_\delta}\right) \qquad m = O\left(n \cdot \frac{\pi\varepsilon_\delta}{\varepsilon_x}\right) = O\left(\frac{4A_R\pi}{\varepsilon_x^2}\right)$$

Complexity: $$O\left(\frac{16\pi A_R^2}{\varepsilon_x^3 \varepsilon_\delta} + \frac{4A_R^2}{\varepsilon_x^2 \varepsilon_\delta^2} \log\left[2\frac{A_R}{\varepsilon_x \varepsilon_\delta}\right]\right)$$

Minimum-Risk Path Planning for Groups of UAVs

- Heterogeneous group of M UAVs
- K Surface-to-Air Missile (SAM) sites with radar sensors
- Goal: travel from x_i to x_f with minimum cost



Probability of j^{th} UAV being detected by i^{th} SAM in time interval dt: $\eta_{ij}(x,v,z_i)dt$

anisotropic cost function $$\ell(x,v) := \sum_{i,j=1}^{K,M} \eta(x,v,z_i)$$

Probability that some UAV will be detected along path ρ on time interval $[0,T]$:

$$p_{\text{detect}} := 1 - \exp\left[\int_0^T \ell(\rho(t),\dot{\rho}(t))dt\right]$$

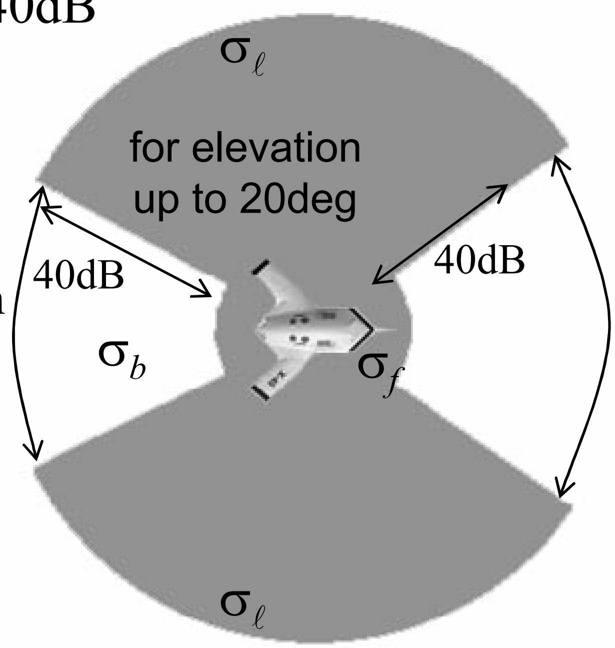
Pareto-optimal paths can be found by solving: $$\min_{\rho} \int_0^T \ell(\rho(t),\dot{\rho}(t))dt$$

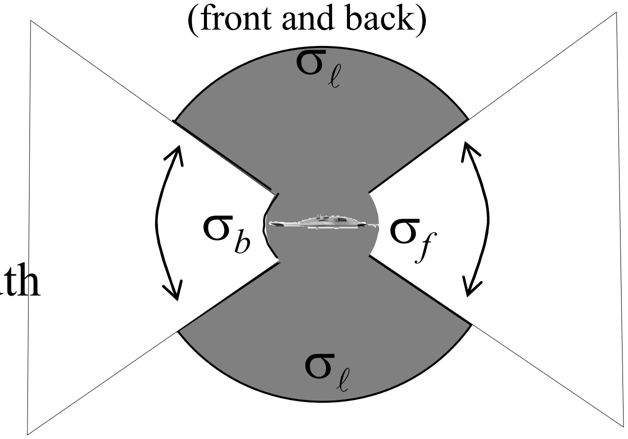
Radar Cross Section

RCS (σ) $\equiv$ ratio between backscatter power reflected by UAV towards radar and power intercepted by the UAV (density per unit of solid angle)

RCS *changes significantly with the attitude of the UAV with respect to the radar*

$\sigma_f = \sigma_b = \sigma_\ell - 40\text{dB}$





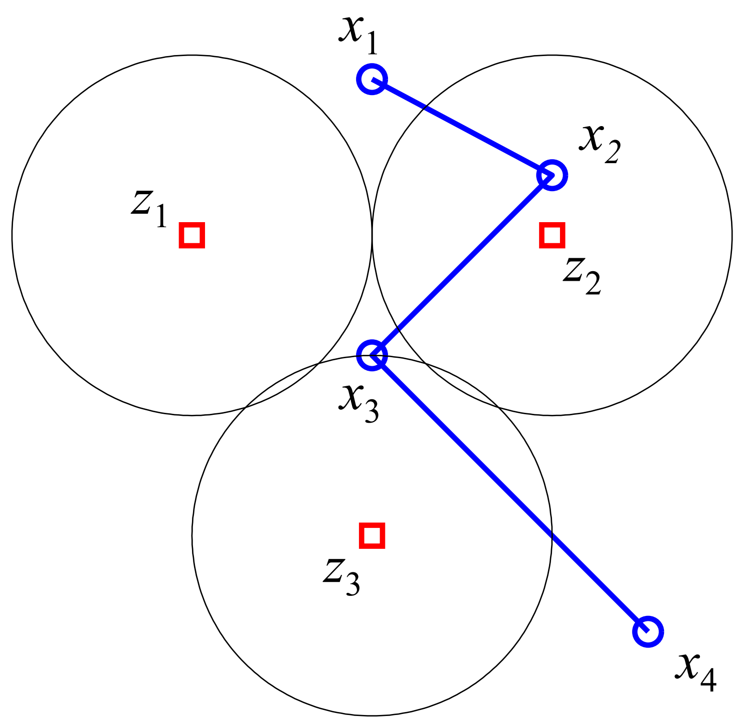
η_{ij} is a (nonlinear) function of

- Radar power
- RCS $\equiv \sigma(\theta_{el},\theta_{az})$
- distance from UAV to radar

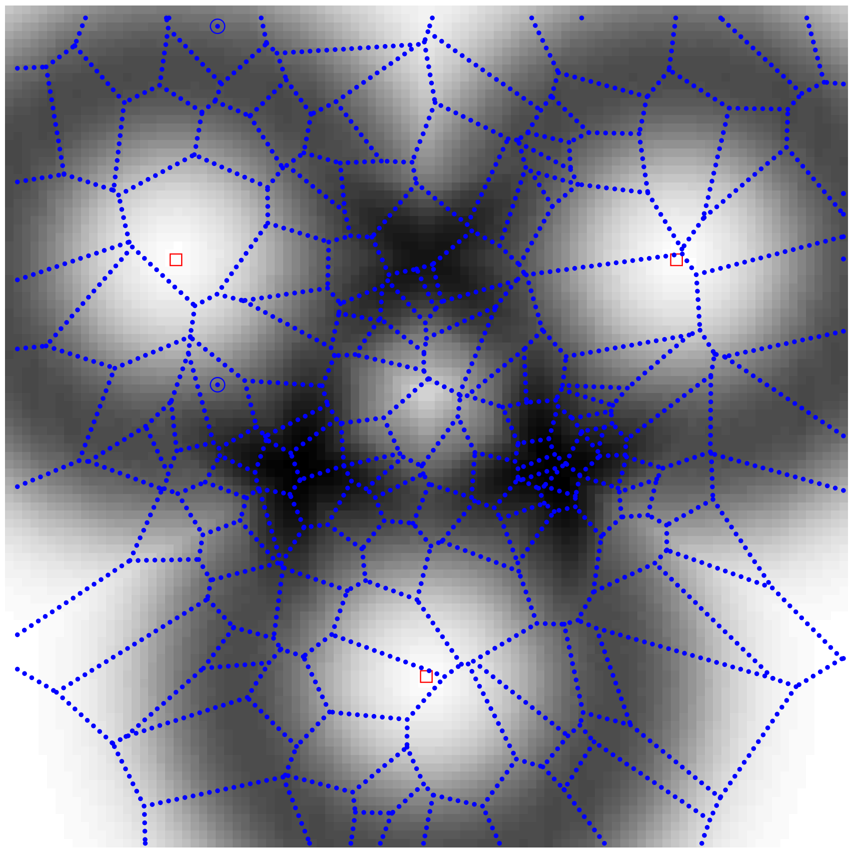
data from MICA OEP-1.3 scenario

Minimum-Risk Path Planning Scenario

Travel from x_1 to x_4, passing through waypoints x_2 and x_3 while minimizing the risk from SAM sites $z_{1,}$ z_2, and $z_{3.}$



Honeycomb Sampling

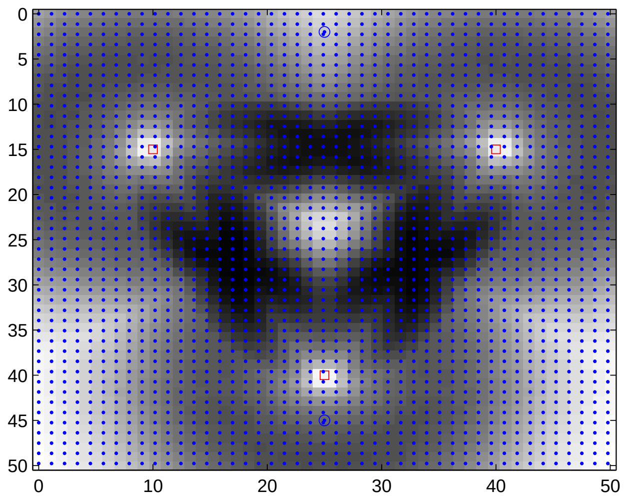


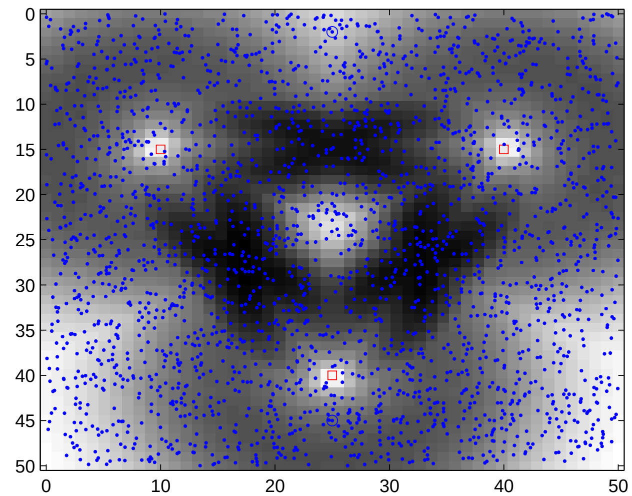
Shading represents $\sup_v \|g_x + A\, g_v\|$ (dark = high)

$$g_x := \nabla_x \ell(x, v) \qquad g_v := \nabla_v \ell(x, v)$$

Comparison of Sampling Algorithms

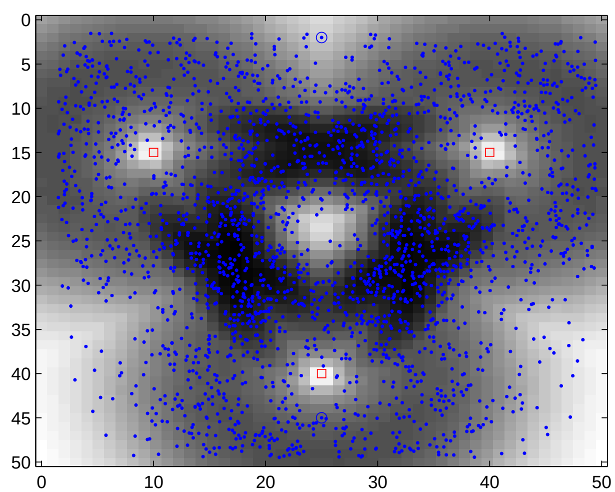
Grid sampling

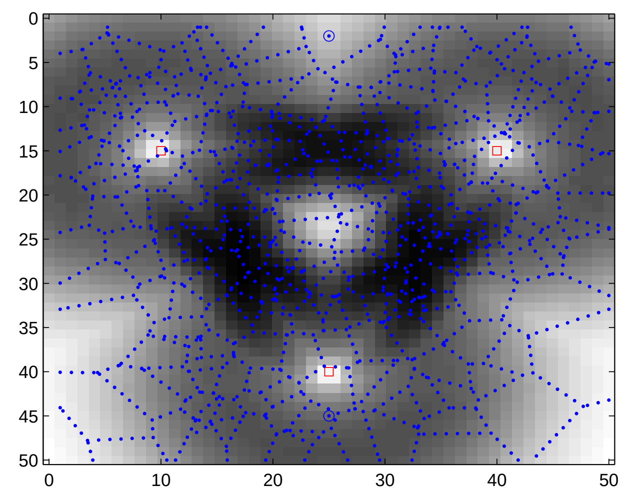




Uniform random sampling

Random gradient-based sampling



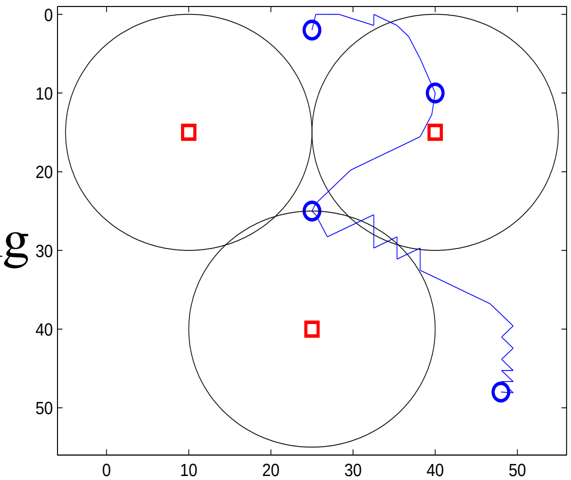


Honeycomb sampling

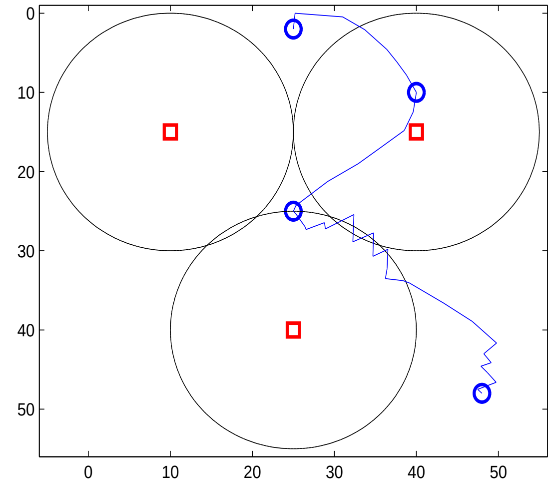
Comparison of Sampling Algorithms

Cost = 27.0

Grid sampling



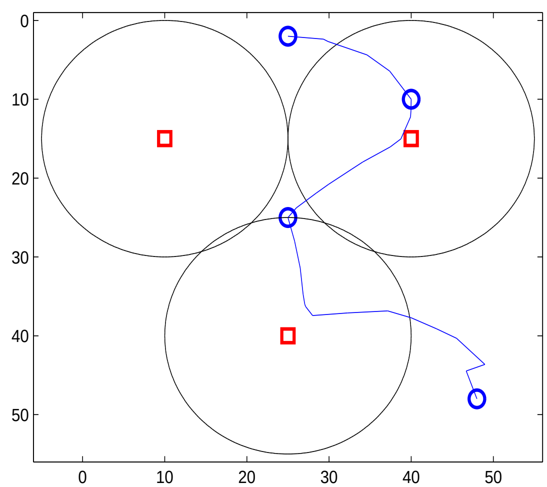
Cost = 27.0



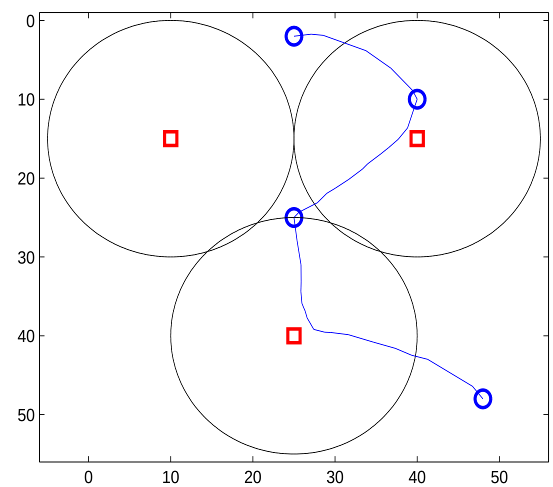
Uniform random sampling

Cost = 27.4

Random gradient-based sampling



Cost = 27.3



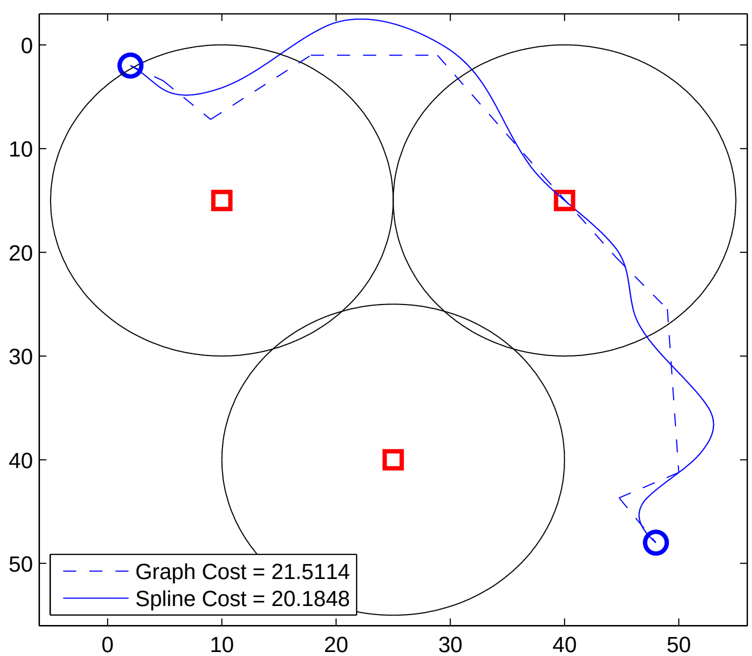
Honeycomb sampling

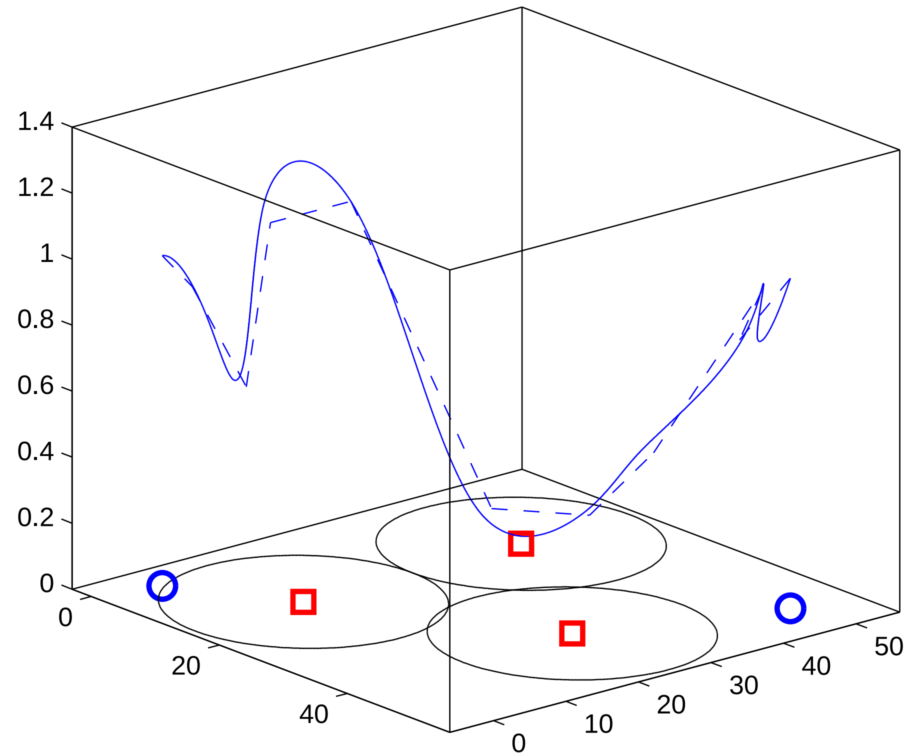
Comparison of Sampling Algorithms

Method	Cost	Vertices	Edges	Computation Time (s)
Grid sampling	27.0	3490	221000	12.4
Uniform random sampling	27.0	4450	372000	21.7
Gradient-based random sampling	27.4	4450	511000	29.7
Honeycomb sampling	27.3	3820	92400	5.3

Spline Smoothing Optimization

$$\rho_i(t) = a_i + b_i(t - t_i) + c_i(t - t_i) \;\; + d_i(t - t_i) \;\; , \qquad\qquad i \in \{1, \ldots, N\}$$





Original cost	21.5
Cost of spline path	20.2

Conclusions and Future Work

Conclusions

- When using a discrete approximation to a continuous routing problem, the sampling and graph construction process is a crucial step in terms of achieving high accuracy with low computation.
- For the continuous weighted anisotropic shortest path problem, the Honeycomb sampling theorem provides a method for computing a path arbitrarily close to optimal through discrete methods, which can be performed with significantly less computation than previous methods.

Future Work

- Apply graph reduction techniques to further reduce computation
- Improve post-processing optimization algorithm