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LQ-OPTIMAL SAMPLED-DATA CONTROL UNDER STOCHASTIC DELAYS: GRIDDING APPROACH FOR STABILIZABILITY AND DETECTABILITY*

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Abstract. We solve a linear quadratic optimal control problem for sampled-data systems with stochastic delays. The delays are stochastically determined by the last few delays. The proposed optimal controller can be efficiently computed by iteratively solving a Riccati difference equation, provided that a discrete-time Markov jump system equivalent to the sampled-data system is stochastically stabilizable and detectable. Sufficient conditions for these notions are provided in the form of linear matrix inequalities, from which stabilizing controllers and state observers can be constructed.

Key words. time delay, optimal control, Markov process, stochastic stabilizability, stochastic detectability

AMS subject classifications. 49J15, 93C57, 93E15

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1. Introduction.

Communication delays occur in networked control systems due to signal processing and congestion in busy channels. Such delays are generally time-varying, and if their range is large, control methods developed for systems with constant delays in [7, 12] may not be suitable. With a rapid development of communication technologies, control under time-varying delays has received extensive attention in recent decades, as surveyed in [17, 40].

One approach to compensate for time-varying delays is the virtual time-delay method developed, e.g., for bilateral control [24]. In this method, the maximum value of delays is assumed to be known, and control signals are updated when the maximum time of delays has passed. This method keeps the apparent delays constant but may degrade the performance of networked control systems if the maximum delay is quite larger than the average delay. Another approach is to measure delays by time-stamped messages and exploit these measurements in the control algorithms, as in Figure 1. An example of this scenario can be found in interarea power systems [31]. Controllers using time-stamped information in this way are delay-dependent, and stabilization by such controllers has been studied in [8, 18, 19, 20, 25, 31, 33, 36, 41] and references therein. Time-stamped messages are also used for linear quadratic (LQ) control in [23, 26, 27, 34] and for model predictive control in [35] under stochastically time-varying delays.

In addition to the earlier studies mentioned above, the authors in [11, 21, 22, 38, 39] have developed design methods of LQ controllers for scenarios where the mea-

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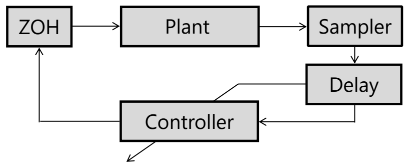

Fig. 1. *Closed-loop system.*

surements of random delays are not available. However, most of those syntheses of optimal controllers require online computation and are based on assumptions that the delays can take only finitely many values or be independent and identically distributed random variables. In practice, communication delays take a continuum of values and need to be modeled by more general stochastic processes, such as Markov processes in [3, 15, 16]. A notable exception can be found in [23], in which the authors have presented an offline computation method of delay-dependent LQ controllers for systems with continuous-valued Markovian delays. The formulation in [23] requires a solution of a nonlinear vector integral equation called the Riccati integral equation and ignores the intersample behavior of the closed-loop system.

In this paper, we study delay-dependent LQ control for sampled-data linear systems. The advantages of the proposed method are twofold. First, our delay model is more general than that in the above previous studies. Indeed, in the model we consider, the present delay is determined by the last few delays like in an autoregressive models (see, e.g., Chapter 9 of [29] for autoregressive models), and hence our delay model belongs to a class of higher-order Markov models. Second, we can efficiently compute an LQ control law that takes into account the intersample behavior. This controller is obtained by iteratively solving a Riccati difference equation.

A key step in the construction of the controller consists of reducing the original sampled-data problem into an LQ problem for a discrete-time Markov jump system whose jumps are modeled by a Markov chain taking values in a general Borel space. In [5], the reduced LQ problem has been solved under the assumption that the plant and the LQ criterion satisfy appropriately defined stochastic stabilizability and detectability notions. However, there has been relatively little work on the test of these properties. Using the results on stochastic stability in [4], we obtain novel sufficient conditions for stochastic stabilizability and detectability in terms of linear matrix inequalities (LMIs). From these results, we can also construct stabilizing controllers and state observers. The proposed method is inspired by the gridding methods for establishing the stability of networked control systems with aperiodic sampling and time-varying delays in [9, 10, 13, 18, 28]. Moreover, we show that the sufficient conditions can be arbitrarily tight under certain assumptions.

The remainder of the paper is organized as follows. We provide the problem statement in section 2. Section 3 is devoted to reducing our optimal control problem to an LQ problem for discrete-time Markov jump systems. In section 4, we recall the general results in [5] on the equivalent discrete-time LQ problem. Section 5 addresses the derivation of sufficient conditions for stochastic stabilizability and detectability. In section 6, we illustrate the proposed method with a numerical simulation of a batch reactor.

Notation. Let $\mathbb{Z}_+$, $\mathbb{R}^{n\times m}$, and $\mathbb{C}^{n\times m}$ denote the set of nonnegative integers and the sets of real and complex matrices with size $n \times m$, respectively. For a real matrix M, let us denote its transpose by $M^\top$. The Euclidean norm of $v \in \mathbb{R}^n$ is denoted by $\|v\| := (v^\top v)^{1/2}$ and the corresponding induced norm of a matrix $M \in \mathbb{R}^{m\times n}$ by $\|M\| := \sup\{\|Mv\| : v \in \mathbb{R}^n,\ \|v\| = 1\}$. For simplicity, we write a partitioned symmetric matrix

$$\begin{bmatrix} Q & W \\ W^\top & R \end{bmatrix} \quad \text{as} \quad \begin{bmatrix} Q & W \\ \star & R \end{bmatrix}.$$

Let $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ be a Borel space; that is, let $\mathcal{M}$ be a Borel subset of a complete and separable metric space and $\mathcal{B}(\mathcal{M})$ be its Borel σ-algebra. In this article, $\mathcal{M}$ is a compact subset of $\mathbb{R}^p$ except in section 4. For a σ-finite measure μ on $\mathcal{M}$, we denote by $\mathbb{H}_1^{n\times m}$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{R}^{n\times m}$ that are measurable and integrable in $\mathcal{M}$, and similarly by $\mathbb{H}_{\sup}^{n\times m}$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{R}^{n\times m}$ that are measurable and essentially bounded in $\mathcal{M}$. For $P \in \mathbb{H}_{\sup}^{n\times m}$, we define a norm $\|P\|_\infty$ by the essential supremum of the function $\|P\| : \mathcal{M} \to [0, \infty)$. For simplicity, we will write $\mathbb{H}_1^n := \mathbb{H}_1^{n\times n}$, $\mathbb{H}_{\sup}^n := \mathbb{H}_{\sup}^{n\times n}$, and

$$\mathbb{H}_{\sup}^{n+} := \{P \in \mathbb{H}_{\sup}^n : P(\phi) \geq 0 \quad \text{for } \mu\text{-almost every } \phi \in \mathcal{M}\}.$$

Additionally, we denote by $\mathbb{H}_{1,\mathbb{C}}^n$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{C}^{n\times m}$ that are measurable and integrable in $\mathcal{M}$ and by $\mathbb{H}_{\sup,\mathbb{C}}^{n+}$ the space of matrix-valued functions $P(\bullet) : \mathcal{M} \to \mathbb{C}^{n\times m}$ that are measurable and essentially bounded in $\mathcal{M}$ and satisfy $P(\phi) \geq 0$ for μ-almost every $\phi \in \mathcal{M}$. For a bounded linear operator T on a Banach space, let $r_\sigma(T)$ denote the spectral radius of T.

2. Problem statement. Consider the following linear continuous-time plant:

$$\dot{x}(t) = A_c x(t) + B_c u(t), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input of the plant. This plant is connected to a controller through a time-driven sampler with period $h > 0$ and an event-driven zero-order hold as shown in Figure 1.

The state x is measured at each sampling time $t = kh$ ($k \in \mathbb{Z}_+$), and the controller receives the sampled state $x(kh)$ at time $t = kh + \tau_k$, where $\tau_k > 0$ is a sensor-to-controller delay. We assume that the delay τ_k becomes known to the controller at the time $t = kh + \tau_k$ when the sampled state $x(kh)$ arrives. One way to measure the delays is to mark every output of the sampler with a time stamp and then to compute the difference between the value of the time stamp and the present time of a clock in the controller. Through the zero-order hold, the discrete-time signal u_k generated from the controller is transformed to the continuous-time signal

$$u(t) = \begin{cases} u_{-1}, & 0 \leq t < \tau_0, \\ u_k, & kh + \tau_k \leq t < (k+1)h + \tau_{k+1},\ k \in \mathbb{Z}_+, \end{cases} \tag{2}$$

where u_{-1} is an initial state of the zero-order hold.

Throughout this paper, we fix the probability space $(\Omega, \mathcal{F}, P)$. We assume that the delay $\{\tau_k : k \in \mathbb{Z}_+\}$ is smaller than one sampling period and that the latest delay is stochastically determined by the last few delays. We specifically assume that the delay sequence $\{\tau_k : k \in \mathbb{Z}_+\}$ is a higher-order Markov chain. For some known $p \in \mathbb{N}$, define a delay vector ϕ_k by

$$\phi_k := \begin{bmatrix} \tau_k \\ \vdots \\ \tau_{k-p+1} \end{bmatrix} \qquad \forall k \in \mathbb{Z}_+, \tag{3}$$

where $\tau_{-p+1}, \dots, \tau_{-1} < h$ are the time delays associated with the sampling instants $t = (-p+1)h, \dots, -h$.

Assumption 2.1 (higher-order Markovian delays). The sequence $\{\phi_k : k \in \mathbb{Z}_+\}$ in (3) is a time-homogeneous Markov chain taking values in $\mathcal{M} := [\tau_{\min}, \tau_{\max}]^p \subset [0, h)^p$ and having transition probability kernel $\mathcal{G}(\bullet, \bullet)$ with a density $g(\bullet, \bullet)$ with respect to a σ-finite measure μ on $\mathcal{M}$, so that for every $k \in \mathbb{Z}_+$ and every Borel set $\mathcal{B}$ of $\mathcal{M}$,

$$\mathcal{G}(\phi, \mathcal{B}) := P(\phi_{k+1} \in \mathcal{B} | \phi_k = \phi) = \int_{\mathcal{B}} g(\phi, \ell)\mu(d\ell).$$

The choice of the dimension p depends on the accuracy of delay models and computational cost. As the dimension p increases, we may obtain more accurate models of delays. However, a large p requires substantial computational resources for optimal controllers. Moreover, the gridding method presented in section 5 suffers from the curse of dimensionality in the case of large p.

Define

$$\xi_0 := \begin{bmatrix} x_0 \\ u_{-1} \end{bmatrix}. \tag{4}$$

Let $\hat{\mu}$ be a probability measure on $\mathbb{R}^{n+m} \times \mathcal{M}$. We assume that the pair of the initial state and delay (ξ_0, ϕ_0) has a distribution $\widehat{\mu}$. Define $\widehat{\mu}_{\mathcal{M}}$ by $\widehat{\mu}_{\mathcal{M}}(\mathcal{B}) := \widehat{\mu}(\mathbb{R}^{n+m} \times \mathcal{B}) = P(\phi_0 \in \mathcal{B})$ for all Borel sets $\mathcal{B}$ of $\mathcal{M}$. We place the following mild assumption on the initial distribution $\widehat{\mu}$.

Assumption 2.2 (initial distribution). The initial distribution $\widehat{\mu}$ of (ξ_0, ϕ_0) satisfies (A1) $E(\|\xi_0\|^2) < \infty$ and (A2) $\widehat{\mu}_{\mathcal{M}}$ is absolutely continuous with respect to μ.

The assumption of absolute continuity guarantees the existence of the Radon–Nikodym derivative of $\widehat{\mu}_{\mathcal{M}}$.

Let $\{\mathcal{F}_k : k \in \mathbb{Z}_+\}$ denote a filtration, where $\mathcal{F}_k$ represents the σ-field generated by

$$\{u_{-1}, x(0), \phi_0, \dots, x(kh), \phi_k\} = \{\tau_{-p+1}, \dots, \tau_{-1}, u_{-1}, x(0), \tau_0, \dots, x(kh), \tau_k\}.$$

Set $\mathcal{U}_c$ as the class of control inputs $u = \{u_k : k \in \mathbb{Z}_+\}$ such that u_k is $\mathcal{F}_k$ measurable and the controlled system (1) and (2) satisfies $E(\|x(t)\|^2) \to 0$ as $t \to \infty$ and $E(\|u_k\|^2) \to 0$ as $k \to \infty$ for every initial distribution $\widehat{\mu}$ satisfying Assumption 2.2. For all $u \in \mathcal{U}_c$, we consider the infinite-horizon continuous-time quadratic cost functional $\mathcal{J}_c$ defined by

$$\mathcal{J}_c(\widehat{\mu}, u) := E\left(\int_0^\infty x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt\right), \tag{5}$$

where $Q_c \geq 0$ and $R_c > 0$ are weighting matrices with appropriate dimensions.

In this paper, we study the following LQ problem.

PROBLEM 2.3. *Consider a sampled-data system* (1) *and* (2)*, and let Assumptions* 2.1 *and* 2.2 *hold. Find an optimal control law* $u^{\mathrm{opt}} \in \mathcal{U}_c$ *that achieves* $\mathcal{J}_c(\widehat{\mu}, u^{\mathrm{opt}}) = \inf_{u \in \mathcal{U}_c} \mathcal{J}_c(\widehat{\mu}, u)$ *for every initial distribution* $\widehat{\mu}$ *satisfying Assumption* 2.2.

Remark 2.4. In this paper, we impose the following two assumptions on delays: (i) The communication channel from the controller to the actuator has no delays. (ii) Delays are smaller than one sampling period h. If the controller-to-actuator delays

τ_{ca} are deterministic (see, e.g., [40, section 2.3.2] for this situation), then we can also apply the proposed method in the presence of the controller-to-actuator delays by using the total delays $\tau_k + \tau_{ca}$ instead of the sensor-to-controller delays τ_k. To deal with delays larger than one sampling period h, we can employ the technique presented in section II of [2], but a stochastic model on delays becomes complicated. Therefore, we here assume that delays are smaller than one sampling period h.

3. Reduction to discrete-time LQ problem. In this section, we transform Problem 2.3 to an LQ problem of discrete-time Markov jump linear systems.

Consider the sampled-data system (1) and (2). We define

$$\xi_k := \begin{bmatrix} x(kh) \\ u_{k-1} \end{bmatrix}. \tag{6}$$

Then the dynamics of ξ can be described by the following discrete-time Markov jump linear system:

$$\xi_{k+1} = A(\phi_k)\xi_k + B(\phi_k)u_k, \tag{7}$$

where, for every vector $\phi \in \mathcal{M}$ whose the first element is given by τ, we define

$$A(\phi) := \begin{bmatrix} A_d & B_d - \Gamma(\tau) \\ 0 & 0 \end{bmatrix}, \quad B(\phi) := \begin{bmatrix} \Gamma(\tau) \\ I \end{bmatrix}, \tag{8a}$$

$$A_d := e^{A_c h},\ B_d := \int_0^h e^{A_c s} B_c ds,\ \Gamma(\tau) := \int_0^{h-\tau} e^{A_c s} B_c ds. \tag{8b}$$

By definition, the matrices A and B satisfy $A \in \mathbb{H}^n_{\sup}$ and $B \in \mathbb{H}^{n\times m}_{\sup}$. This delay-dependent discrete-time system is widely used for the analysis of time-delay systems, e.g., in [9, 10, 18, 23, 26, 27, 34].

Let $\{\mathcal{F}^d_k : k \in \mathbb{Z}_+\}$ denote a filtration, where $\mathcal{F}^d_k$ represents the σ-field generated by

$$\{\xi_0, \phi_0, \dots, \xi_k, \phi_k\}.$$

We denote by $\mathcal{U}_d$ the discrete-time counterpart of $\mathcal{U}_c$, defined as the class of control inputs $\{u_k : k \in \mathbb{Z}_+\}$ such that u_k is $\mathcal{F}^d_k$ measurable and $E(\|\xi_k\|^2) \to 0$ as $k \to \infty$ for every initial distribution $\widehat{\mu}$. The following result establishes that these classes of control inputs are equal.

LEMMA 3.1. *For the sampled-data system* (1), (2) *and its discretization* (7)*, we obtain* $\mathcal{U}_c = \mathcal{U}_d$.

Proof. Since the filtrations $\{\mathcal{F}_k\}$ and $\{\mathcal{F}^d_k\}$ are equal by definition, it is enough to prove that the following two conditions are equivalent:

1. $\lim_{k\to\infty} E(\|\xi_k\|^2) = 0$.
2. $\lim_{t\to\infty} E(\|x(t)\|^2) = 0$ and $\lim_{k\to\infty} E(\|u_k\|^2) = 0$.

The statement 2 ⇒ 1 follows directly from the definition of ξ_k. To prove the converse statement, we note that for the system dynamics (1) and (2), there exist constants $M_1, M_2, M_3 > 0$ such that

$$\|x(kh+\theta)\| \le M_1\|x(kh)\| + M_2\|u_{k-1}\| + M_3\|u_k\| \qquad \forall \theta \in [0,h). \tag{9}$$

Therefore,

$$\begin{aligned}\|x(kh+\theta)\|^2 \le M_1^2\|x(kh)\|^2 + M_2^2\|u_{k-1}\|^2 + M_3^2\|u_k\|^2 + 2M_1M_2\|x(kh)\| \cdot \|u_{k-1}\| \\ + 2M_2M_3\|u_{k-1}\| \cdot \|u_k\| + 2M_3M_1\|u_k\| \cdot \|x(kh)\| \qquad \forall \theta \in [0,h).\end{aligned}$$

Since

$$ab \le \frac{a^2+b^2}{2} \qquad \forall a,b \ge 0,$$

applying this inequality to the terms $\|x(kh)\|\cdot\|u_{k-1}\|$, $\|u_{k-1}\|\cdot\|u_k\|$, and $\|u_k\|\cdot\|x(kh)\|$, we obtain

$$\|x(kh+\theta)\|^2 \le N_1\|x(kh)\|^2 + N_2\|u_{k-1}\|^2 + N_3\|u_k\|^2 \qquad \forall \theta \in [0,h)$$

for appropriate constants $N_1, N_2, N_3 > 0$. It then follows that if $E(\|\xi_k\|^2) \to 0$ as $k \to \infty$, then $E(\|x(t)\|^2) \to 0$ as $t \to \infty$, which completes the proof. ∎

Since the integrand $x(t)^\top Q_c x(t) + u(t)^\top R_c u(t)$ of the cost functional $\mathcal{J}_c$ in (5) is nonnegative for every $t \ge 0$, the cost functional $\mathcal{J}_c$ can be expressed as the following (discrete-time) summation:

$$\mathcal{J}_c = E\left(\sum_{k=0}^{\infty} \mathcal{J}_k\right),$$

where

$$\mathcal{J}_k := \int_{kh}^{(k+1)h} x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt \qquad \forall k \in \mathbb{Z}_+. \tag{10}$$

For every vector $\phi \in \mathcal{M}$ whose first element is given by τ, we define the matrices Q, W, and R by

$$Q(\phi) := \begin{bmatrix} Q_{11}(\tau) & Q_{12}(\tau) \\ \star & Q_{22}(\tau) \end{bmatrix}, \tag{11a}$$

$$W(\phi) := \int_\tau^h \begin{bmatrix} \alpha(\theta)^\top Q_c \gamma(\tau,\theta) \\ \beta(\theta)^\top Q_c \gamma(\tau,\theta) - \gamma(\tau,\theta)^\top Q_c \gamma(\tau,\theta) \end{bmatrix} d\theta, \tag{11b}$$

$$R(\phi) := (h-\tau)R_c + \int_\tau^h \gamma(\tau,\theta)^\top Q_c \gamma(\tau,\theta) d\theta, \tag{11c}$$

$$Q_{11}(\tau) := \int_0^h \alpha(\theta)^\top Q_c \alpha(\theta) d\theta, \tag{11d}$$

$$Q_{12}(\tau) := \int_0^h \alpha(\theta)^\top Q_c \beta(\theta) d\theta - \int_\tau^h \alpha(\theta)^\top Q_c \gamma(\tau,\theta) d\theta, \tag{11e}$$

$$\begin{aligned} Q_{22}(\tau) := \tau R_c &+ \int_0^h \beta(\theta)^\top Q_c \beta(\theta) d\theta + \int_\tau^h \gamma(\tau,\theta)^\top Q_c \gamma(\tau,\theta) d\theta \\ &- \int_\tau^h \left(\beta(\theta)^\top Q_c \gamma(\tau,\theta) + \gamma(\tau,\theta)^\top Q_c \beta(\theta)\right) d\theta, \end{aligned} \tag{11f}$$

where the functions α, β, and γ are given by

$$\alpha(\theta) := e^{A_c\theta}, \quad \beta(\theta) := \int_0^\theta e^{A_c s} B_c ds \qquad \forall \theta \in [0,h)$$

$$\gamma(\tau,\theta) := \int_0^{\theta-\tau} e^{A_c s} B_c ds \qquad \forall \tau \in [\tau_{\min}, \tau_{\max}],\ \forall \theta \in [0,h).$$

The next lemma shows that each $\mathcal{J}_k$ is a quadratic form on the state ξ_k and the input u_k of the discrete-time system Σ_d in (7).

LEMMA 3.2. *Let x and ξ be the solutions of the sampled-data system* (1) *and* (2) *and of the discrete-time system* (7) *with the initial state ξ_0 defined by* (4)*, respectively. Then $\mathcal{J}_k$ defined by* (10) *satisfies*

$$\mathcal{J}_k = \begin{bmatrix} \xi_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q(\phi_k) & W(\phi_k) \\ \star & R(\phi_k) \end{bmatrix} \begin{bmatrix} \xi_k \\ u_k \end{bmatrix} \qquad \forall k \in \mathbb{Z}_+, \tag{12}$$

where the matrices Q, W, and R are defined as in (11).

Proof. If $0 \leq \theta < \tau_k$, then

$$\begin{aligned} x(kh+\theta) &= \alpha(\theta)x_k + \beta(\theta)u_{k-1}, \\ u(kh+\theta) &= u_{k-1}, \end{aligned}$$

and we have

$$\begin{aligned} &x(kh+\theta)^\top Q_c x(kh+\theta) + u(kh+\theta)^\top R_c u(kh+\theta) \\ &\quad = x_k^\top \alpha(\theta)^\top Q_c \alpha(\theta) x_k + 2x_k^\top \alpha(\theta)^\top Q_c \beta(\theta) u_{k-1} + u_{k-1}^\top \beta(\theta)^\top Q_c \beta(\theta) u_{k-1} \\ &\qquad + u_{k-1}^\top R_c u_{k-1}. \end{aligned}$$

On the other hand, if $\tau_k \leq \theta < h$, then

$$\begin{aligned} x(kh+\theta) &= \alpha(\theta)x_k + (\beta(\theta) - \gamma(\tau_k,\theta))u_{k-1} + \gamma(\tau_k,\theta)u_k, \\ u(kh+\theta) &= u_k. \end{aligned}$$

Hence

$$\begin{aligned} &x(kh+\theta)^\top Q_c x(kh+\theta) + u(kh+\theta)^\top R_c u(kh+\theta) \\ &\quad = x_k^\top \alpha(\theta)^\top Q_c \alpha(\theta) x_k + 2x_k^\top \alpha(\theta)^\top Q_c (\beta(\theta) - \gamma(\tau_k,\theta)) u_{k-1} \\ &\qquad + 2x_k^\top \alpha(\theta)^\top Q_c \gamma(\tau_k,\theta) u_k + u_{k-1}^\top (\beta(\theta) - \gamma(\tau_k,\theta))^\top Q_c (\beta(\theta) - \gamma(\tau_k,\theta)) u_{k-1} \\ &\qquad + 2u_{k-1}^\top (\beta(\theta) - \gamma(\tau_k,\theta))^\top Q_c \gamma(\tau_k,\theta) u_k + u_k^\top \gamma(\tau_k,\theta)^\top Q_c \gamma(\tau_k,\theta) u_k + u_k^\top R_c u_k. \end{aligned}$$

Substituting these equations into

$$\begin{aligned} \mathcal{J}_k = &\int_{kh}^{kh+\tau_k} \left(x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) \right) dt \\ &+ \int_{kh+\tau_k}^{(k+1)h} \left(x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) \right) dt, \end{aligned}$$

we obtain the desired expression (12). ∎

For the discrete-time Markov jump system (7), we define the infinite-horizon discrete-time quadratic cost functional $\mathcal{J}_{d1}$ by

$$\mathcal{J}_{d1}(\widehat{\mu}, u) := \sum_{k=0}^{\infty} E\left(\begin{bmatrix} \xi_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q(\phi_k) & W(\phi_k) \\ \star & R(\phi_k) \end{bmatrix} \begin{bmatrix} \xi_k \\ u_k \end{bmatrix} \right). \tag{13}$$

Then a solution to the LQ problem for sampled-data systems with stochastic delays can be obtained as a solution to the following problem for discrete-time Markov jump systems.

PROBLEM 3.3. *Consider a discrete-time Markov jump system* (7)*, and let Assumptions* 2.1 *and* 2.2 *hold. Find an optimal control law* $u^{\mathrm{opt}} \in \mathcal{U}_d$ *that achieves* $\mathcal{J}_{d1}(\widehat{\mu}, u^{\mathrm{opt}}) = \inf_{u \in \mathcal{U}_d} \mathcal{J}_{d1}(\widehat{\mu}, u)$ *for every initial distribution* $\widehat{\mu}$ *satisfying Assumption* 2.2.

LEMMA 3.4. *A control input* u^{opt} *is a solution to Problem* 2.3 *if and only if* u^{opt} *is also a solution to Problem* 3.3 *where the system matrices* A*,* B *and the weighting matrices* Q*,* W*,* R *are defined by* (8) *and* (11).

Proof. Since $\mathcal{U}_c = \mathcal{U}_d$ from Lemma 3.1, it follows that $u^{\mathrm{opt}} \in \mathcal{U}_c$ if and only if $u^{\mathrm{opt}} \in \mathcal{U}_d$. Let x and ξ be the solutions of the sampled-data system (1) and (2) and of the discrete-time system (7) with the initial state ξ_0 defined by (4), respectively. Since

$$x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) \geq 0 \qquad \forall t \geq 0,$$

we obtain

$$\begin{aligned}
\mathcal{J}_c(\widehat{\mu}, u) &= E\left(\int_0^\infty x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt\right) \\
&= E \sum_{k=0}^\infty \left(\int_{kh}^{(k+1)h} x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt\right) \\
&= \sum_{k=0}^\infty E\left(\int_{kh}^{(k+1)h} x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt\right).
\end{aligned}$$

It also follows from Lemma 3.2 that

$$\begin{aligned}
&\sum_{k=0}^\infty E\left(\int_{kh}^{(k+1)h} x(t)^\top Q_c x(t) + u(t)^\top R_c u(t) dt\right) \\
&\qquad = \sum_{k=0}^\infty E\left(\begin{bmatrix} \xi_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q(\phi_k) & W(\phi_k) \\ \star & R(\phi_k) \end{bmatrix} \begin{bmatrix} \xi_k \\ u_k \end{bmatrix}\right) = \mathcal{J}_{d1}(\widehat{\mu}, u).
\end{aligned}$$

Thus if a control input $u^{\mathrm{opt}} \in \mathcal{U}_c$ is a solution to Problem 2.3, then u^{opt} satisfies $u^{\mathrm{opt}} \in \mathcal{U}_d$ and is a solution to Problem 3.3 where the system matrices A, B and the weighting matrices Q, W, R are defined by (8), (11), and vice versa. $\square$

Let us next remove the cross term of the cost function $\mathcal{J}_{d1}$. To this end, as in the deterministic case [1, section 3.4], we transform u_k into $\bar{u}_k$ in the following way:

$$\bar{u}_k = u_k + R(\phi_k)^{-1} W(\phi_k)^\top \xi_k \qquad \forall k \in \mathbb{Z}_+. \tag{14}$$

Since $Q_c \geq 0$, $R_c > 0$, and $h - \tau_{\max} > 0$, it follows that $R(\phi)$ in (11) is invertible for all $\phi \in \mathcal{M}$. Therefore, the right-hand side of (14) is well-defined for all $\phi_k \in \mathcal{M}$.

Define

$$\bar{A}(\phi) := A(\phi) - B(\phi) R(\phi)^{-1} W(\phi)^\top \qquad \forall \phi \in \mathcal{M}, \tag{15}$$

and let $C(\phi)$ and $D(\phi)$ be the matrices obtained from the following Cholesky decompositions:

$$C(\phi)^\top C(\phi) = Q(\phi) - W(\phi) R(\phi)^{-1} W(\phi)^\top \qquad \forall \phi \in \mathcal{M}, \tag{16a}$$

$$D(\phi)^\top D(\phi) = R(\phi) \qquad \forall \phi \in \mathcal{M}. \tag{16b}$$

These Cholesky decompositions are possible if the weighting matrix in Lemma 3.2 satisfies

$$\begin{bmatrix} Q(\phi) & W(\phi) \\ \star & R(\phi) \end{bmatrix} \geq 0 \qquad \forall \phi \in \mathcal{M}. \tag{17}$$

This is because $R(\phi) > 0$ for every $\phi \in \mathcal{M}$ and the Schur complement formula leads to

$$Q(\phi) - W(\phi)R(\phi)^{-1}W(\phi)^\top \geq 0 \qquad \forall \phi \in \mathcal{M}.$$

Under the transformation (14), we obtain the following result.

LEMMA 3.5. *Assume that the weighting matrices Q, W, R in* (11) *satisfy the inequality* (17)*. A control input u^{opt} is a solution to Problem* 3.3 *with the system* (7) *and the LQ cost* (13) *if and only if $\bar{u}^{\mathrm{opt}}$ with $\bar{u}_k^{\mathrm{opt}} = u_k^{\mathrm{opt}} + R(\phi_k)^{-1}W(\phi_k)^\top \xi_k$ is a solution to Problem* 3.3 *where the Markov jump system is given by*

$$\xi_{k+1} = \bar{A}(\phi_k)\xi_k + B(\phi_k)\bar{u}_k \tag{18}$$

and the LQ cost by

$$\mathcal{J}_{d2}(\widehat{\mu}, \bar{u}) := \sum_{k=0}^{\infty} E\left(\|C(\phi_k)\xi_k\|^2 + \|D(\phi_k)\bar{u}_k\|^2\right). \tag{19}$$

Here the matrices $\bar{A}$, C, and D are defined as in (15) *and* (16)*.*

Proof. Let ξ and $\bar{\xi}$ be the solutions of the difference equations (7) and (18) with the same Markov parameter ϕ. If $\bar{u}_k$ satisfies (14) and if $\xi_k = \bar{\xi}_k$, then

$$\begin{aligned} \xi_{k+1} &= A(\phi_k)\xi_k + B(\phi_k)u_k \\ &= A(\phi_k)\xi_k + B(\phi_k)(\bar{u}_k - R(\phi_k)^{-1}W(\phi_k)^\top \xi_k) \\ &= \bar{A}(\phi_k)\bar{\xi}_k + B(\phi_k)\bar{u}_k = \bar{\xi}_{k+1}. \end{aligned}$$

Therefore, if $\xi_0 = \bar{\xi}_0$, then $\xi_k = \bar{\xi}_k$ for every $k \in \mathbb{Z}_+$. Thus $u \in \mathcal{U}_d$ for the system (7) if and only if $\bar{u} \in \mathcal{U}_d$ for the system (18). Moreover, if (14) holds, then

$$\begin{aligned} \mathcal{J}_{d2}(\widehat{\mu}, \bar{u}) &= \sum_{k=0}^{\infty} E\left(\begin{bmatrix} \bar{\xi}_k \\ \bar{u}_k \end{bmatrix}^\top \begin{bmatrix} C^\top(\phi_k)C(\phi_k) & 0 \\ \star & D^\top(\phi_k)D(\phi_k) \end{bmatrix} \begin{bmatrix} \bar{\xi}_k \\ \bar{u}_k \end{bmatrix}\right) \\ &= \sum_{k=0}^{\infty} E\left(\begin{bmatrix} \bar{\xi}_k \\ \bar{u}_k \end{bmatrix}^\top \begin{bmatrix} Q(\phi_k) - W(\phi_k)R(\phi_k)^{-1}W(\phi_k)^\top & 0 \\ \star & R(\phi_k) \end{bmatrix} \begin{bmatrix} \bar{\xi}_k \\ \bar{u}_k \end{bmatrix}\right) \\ &= \sum_{k=0}^{\infty} E\left(\begin{bmatrix} \xi_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} Q(\phi_k) & W(\phi_k) \\ \star & R(\phi_k) \end{bmatrix} \begin{bmatrix} \xi_k \\ u_k \end{bmatrix}\right) = \mathcal{J}_{d1}(\widehat{\mu}, u). \end{aligned}$$

Hence u^{opt} is a solution to Problem 3.3 with the system (7) and the LQ cost (13) if and only if $\bar{u}^{\mathrm{opt}}$ with $\bar{u}_k^{\mathrm{opt}} = u_k^{\mathrm{opt}} + R(\phi_k)^{-1}W(\phi_k)^\top \xi_k$ is a solution to Problem 3.3 with the system (18) and the LQ cost (19). ∎

Finally, we can reduce the LQ problem for sampled-data systems with stochastic delays to the LQ problem for discrete-time Markov jump systems and LQ costs in the form (19).

THEOREM 3.6. *A control input* u^{opt} *is a solution to Problem* 2.3 *if and only if* $\bar{u}^{\mathrm{opt}}$ *with* $\bar{u}_k^{\mathrm{opt}} = u_k^{\mathrm{opt}} + R(\phi_k)^{-1}W(\phi_k)^\top \xi_k$ *is a solution to Problem* 3.3 *for the Markov jump system* (18) *and the LQ cost* (19)*, where the matrices* $\bar{A}$*,* C*, and* D *are obtained in* (15) *and* (16) *from the matrices* A*,* B*,* Q*,* W*, and* R *defined by* (8) *and* (11)*.*

Proof. This is an immediate consequence of Lemmas 3.4 and 3.5. ∎

Remark 3.7. If, in (17), we have strict positive definiteness

$$\begin{bmatrix} Q(\phi) & W(\phi) \\ \star & R(\phi) \end{bmatrix} > 0 \qquad \forall \phi \in \mathcal{M} \tag{20}$$

instead of the semidefiniteness, then

$$Q(\phi) - W(\phi)R(\phi)^{-1}W(\phi)^\top > 0 \qquad \forall \phi \in \mathcal{M}$$

by the Schur complement formula. In this case, $C(\phi)$ and $D(\phi)$, derived from the Cholesky decompositions (16), are unique in the following sense: For all $\phi \in \mathcal{M}$, there exist unique upper triangular matrices $C(\phi)$ and $D(\phi)$ with strictly positive diagonal entries such that (16) holds. Moreover, $C(\phi)$, $D(\phi)$ are continuous with respect to ϕ. See, e.g., Chapters 9 and 12 of [32]. Thus C and D satisfy $C \in \mathbb{H}_{\sup}^{n\times n}$ and $D \in \mathbb{H}_{\sup}^{m\times m}$.

4. LQ control for discrete-time Markov jump systems. In the previous section, we reduced the LQ problem for sampled-data systems with stochastic delays into the LQ problem for discrete-time Markov jump systems and LQ costs in the form (19). In this section, we recall results from [5] on such an LQ problem for Markov jump systems.

First we define stochastic stability for discrete-time Markov jump linear systems. On a probability space $(\Omega, \mathcal{F}, P)$, consider the following autonomous system:

$$\xi_{k+1} = A(\phi_k)\xi_k, \tag{21}$$

where $A \in \mathbb{H}_{\sup}^{n}$ and the sequence $\{\phi_k : k \in \mathbb{Z}_+\}$ is a time-homogeneous Markov chain in a Borel space $\mathcal{M}$. Throughout this section, we assume that the initial distribution $\widehat{\mu}$ of (ξ_0, ϕ_0), which is a probability measure on $\mathbb{R}^n \times \mathcal{M}$, satisfies the following conditions analogous to those in Assumption 2.2: (A1') $E(\|\xi_0\|^2) < \infty$, and (A2') $\widehat{\mu}_{\mathcal{M}}(\bullet) = \widehat{\mu}(\mathbb{R}^n \times \bullet)$ is absolutely continuous with respect to μ.

DEFINITION 4.1 (stochastic stability [4]). *The autonomous Markov jump linear system* (21) *is said to be stochastically stable if* $\sum_{k=0}^{\infty} E(\|\xi_k\|^2) < \infty$ *for any initial distribution* $\widehat{\mu}$ *satisfying* (A1') *and* (A2')*.*

Let $g(\bullet, \bullet)$ be the density function with respect to a σ-finite measure μ on $\mathcal{M}$ for the transition of the Markov chain $\{\phi_k : k \in \mathbb{Z}_+\}$ as in Assumption 2.1. For every $A \in \mathbb{H}_{\sup}^{n}$, define an operator $\mathcal{L}_A : \mathbb{H}_{1,\mathbb{C}}^{n} \to \mathbb{H}_{1,\mathbb{C}}^{n}$ by

$$\mathcal{L}_A(V)(\bullet) := \int_{\mathcal{M}} g(\ell, \bullet) A(\ell) V(\ell) A(\ell)^\top \mu(d\ell). \tag{22}$$

We recall a relationship among stochastic stability, the spectral radius $r_\sigma(\mathcal{L}_A)$, and a Lyapunov inequality condition.

THEOREM 4.2 (Theorem 5.3 of [4]). *The following assertions are equivalent:*

1. *The system* (21) *is stochastically stable.*

2. *The spectral radius $r_\sigma(\mathcal{L}_A) < 1$, where $\mathcal{L}_A$ is defined as in* (22).
3. *There exist $S \in \mathbb{H}^{n+}_{\sup,\mathbb{C}}$ and $\epsilon > 0$ such that the following Lyapunov inequality holds for μ-almost every $\phi \in \mathcal{M}$:*

$$S(\phi) - A(\phi)^\top \left(\int_{\mathcal{M}} g(\phi,\ell) S(\ell) \mu(d\ell) \right) A(\phi) \geq \epsilon I. \tag{23}$$

Note that the matrix S in statement 3 is a complex-valued function because the system matrix A is a complex-valued function in [4]. However, for a real-valued function A, it is enough to find a real-valued S in statement 3.

PROPOSITION 4.3. *For a real-valued function $A \in \mathbb{H}^n_{\sup}$, statement* 3 *in Theorem* 4.2 *is equivalent to the following:*

3'. *There exist $S \in \mathbb{H}^{n+}_{\sup}$ and $\epsilon > 0$ such that the Lyapunov inequality* (23) *holds for μ-almost every $\phi \in \mathcal{M}$.*

Proof. The statement 3' $\Rightarrow$ 3 is trivial because $\mathbb{H}^{n+}_{\sup} \subset \mathbb{H}^{n+}_{\sup,\mathbb{C}}$. To prove 3 $\Rightarrow$ 3', we let $S \in \mathbb{H}^{n+}_{\sup,\mathbb{C}}$ and $\epsilon > 0$ satisfy the Lyapunov inequality (23). Let $S_R(\phi) \in \mathbb{R}^{n\times n}$ and $S_I(\phi) \in \mathbb{R}^{n\times n}$ be the real and imaginary parts of $S(\phi)$, that is,

$$S(\phi) = S_R(\phi) + iS_I(\phi).$$

Since $S(\phi)$ is Hermitian, it follows that $S_R(\phi) = S_R(\phi)^\top$ and $S_I(\phi) = -S_I(\phi)^\top$. Therefore,

$$0 \leq x^\top S(\phi) x = x^\top (S_R(\phi) + iS_I(\phi))x = x^\top S_R(\phi) x \qquad \forall x \in \mathbb{R}^n.$$

Thus we obtain $S_R \in \mathbb{H}^{n+}_{\sup}$. Similarly, since $A(\phi)$ and $g(\phi,\ell)$ are real-valued, it follows that

$$\begin{aligned}\epsilon\|x\|^2 &\leq x^\top \left(S(\phi) - A(\phi)^\top \left(\int_{\mathcal{M}} g(\phi,\ell) S(\ell)\mu(d\ell) \right) A(\phi) \right) x \\ &= x^\top \left(S_R(\phi) - A(\phi)^\top \left(\int_{\mathcal{M}} g(\phi,\ell) S_R(\ell)\mu(d\ell) \right) A(\phi) \right) x \qquad \forall x \in \mathbb{R}^n.\end{aligned}$$

Hence S_R also satisfies the Lyapunov inequality (23) for μ-almost every $\phi \in \mathcal{M}$. This completes the proof. ∎

We next provide the definition of stochastic stabilizability and stochastic detectability.

DEFINITION 4.4 (stochastical stabilizability [5]). *Let $A \in \mathbb{H}^n_{\sup}$ and $B \in \mathbb{H}^{n\times m}_{\sup}$. We say that (A, B) is stochastically stabilizable if there exists $F \in \mathbb{H}^{m\times n}_{\sup}$ such that $r_\sigma(\mathcal{L}_{A+BF}) < 1$, where $\mathcal{L}_{A+BF}$ is defined as in* (22). *In this case, F is said to stochastically stabilize (A, B).*

DEFINITION 4.5 (stochastic detectability [5]). *Let $A \in \mathbb{H}^n_{\sup}$ and $C \in \mathbb{H}^{r\times n}_{\sup}$. We say that (C, A) is stochastically detectable if there exists $L \in \mathbb{H}^{n\times r}_{\sup}$ such that $r_\sigma(\mathcal{L}_{A+LC}) < 1$, where $\mathcal{L}_{A+LC}$ is defined as in* (22).

Consider the controlled system

$$\xi_{k+1} = A(\phi_k)\xi_k + B(\phi_k)u_k$$

and the LQ cost

$$\mathcal{J}_d(\widehat{\mu}, u) = \sum_{k=0}^{\infty} E\left(\|C(\phi_k)\xi_k\|^2 + \|D(\phi_k)u_k\|^2 \right),$$

where $A \in \mathbb{H}^n_{\sup}$, $B \in \mathbb{H}^{m\times n}_{\sup}$, $C \in \mathbb{H}^{n\times r}_{\sup}$, and $D \in \mathbb{H}^{m\times q}_{\sup}$. We assume that there exists $\epsilon_D > 0$ such that $D(\phi)^\top D(\phi) > \epsilon_D I$ for μ-almost every $\phi \in \mathcal{M}$. As in section 3, let $\{\mathcal{F}^d_k : k \in \mathbb{Z}_+\}$ denote a filtration, where $\mathcal{F}^d_k$ represents the σ-field generated by $\{\xi_0, \phi_0, \dots, \xi_k, \phi_k\}$, and set $\mathcal{U}_d$ as the class of control inputs $u = \{u_k : k \in \mathbb{Z}_+\}$ such that u_k is $\mathcal{F}^d_k$ measurable and $E(\|\xi_k\|^2) \to 0$ as $k \to \infty$ for every initial distribution $\widehat{\mu}$ satisfying (A1') and (A2').

Define operators $\mathcal{E} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{n+}_{\sup}$, $\mathcal{V} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{m+}_{\sup}$, and $\mathcal{R} : \mathbb{H}^{n+}_{\sup} \to \mathbb{H}^{n+}_{\sup}$ as follows:

$$\begin{aligned}\mathcal{E}(Z)(\bullet) &:= \int_{\mathcal{M}} Z(\ell) g(\bullet, \ell)\mu(d\ell),\\ \mathcal{V}(Z) &:= D^\top D + B^\top \mathcal{E}(Z) B,\\ \mathcal{R}(Z) &:= C^\top C + A^\top(\mathcal{E}(Z) - \mathcal{E}(Z)B\mathcal{V}(Z)^{-1}B^\top\mathcal{E}(Z))A.\end{aligned}$$

Using these operators, we can obtain a solution to the LQ problem for discrete-time Markov jump linear systems from the iterative computation of $\mathcal{R}(Z)$.

THEOREM 4.6 (Theorem 5.8 of [5]). *Consider the Markov jump system* (18) *with the LQ cost $\mathcal{J}_d$ in* (19)*. If (A, B) is stochastically stabilizable and (C, A) is stochastically detectable, then there exists a function $S \in \mathbb{H}^{n+}_{\sup}$ such that S is the unique solution in $\mathbb{H}^{n+}_{\sup}$ of the $\mathcal{M}$-coupled algebraic Riccati equation*

$$S(\phi) = \mathcal{R}(S)(\phi) \quad \mu\text{-almost every } \phi \in \mathcal{M} \tag{24}$$

and such that

$$K := -\mathcal{V}(S)^{-1} B^\top \mathcal{E}(S) A \in \mathbb{H}^{m\times n}_{\sup}$$

stochastically stabilizes (A, B). The control input $u^{\mathrm{opt}} \in \mathcal{U}_d$ defined by $u^{\mathrm{opt}}_k := K(\phi_k)\xi_k$ achieves

$$\mathcal{J}_d(\widehat{\mu}, u^{\mathrm{opt}}) = \inf_{u\in\mathcal{U}_d} \mathcal{J}_d(\widehat{\mu}, u) = E(\xi_0^\top S(\phi_0)\xi_0)$$

for every initial distribution $\widehat{\mu}$ satisfying (A1') *and* (A2')*. Moreover, we can compute the solution S of the Riccati equation* (24) *in the following way: For any $\Xi \in \mathbb{H}^{n+}_{\sup}$, the sequence $\{Y^\eta_k\}^\eta_{k=0}$ that is calculated by solving a (backward recursive) Riccati difference equation $Y^\eta_k = \mathcal{R}(Y^\eta_{k+1})$ with the initial value $Y^\eta_\eta = \Xi$ satisfies $Y^\eta_0(\phi) \to S(\phi)$ as $\eta \to \infty$ for μ-almost every $\phi \in \mathcal{M}$.*

Let us go back to the reduced LQ problem for the Markov jump system (18) and the LQ cost (19), where the matrices $\bar{A}$, C, and D are defined by (15) and (16). The Cholesky decompositions for all $\phi \in \mathcal{M}$ in (16) require heavy computational cost, but the weighting functions C and D appear only in $\mathcal{R}$ and $\mathcal{V}$ in the forms $C^\top C$ and $D^\top D$. Hence, to compute an optimal control input u^{opt}, we do not need the Cholesky decompositions in (16). Although we still need C to check the stochastic detectability of $(C, \bar{A})$, we see from Proposition 4.7 below that it is enough to test the stochastic detectability of $(C^\top C, \bar{A})$ if $C^\top C$ is positive definite.

PROPOSITION 4.7. *Define $\bar{A}$ and C as in* (15) *and* (16)*, respectively. The pair (A, B) is stochastically stabilizable if and only if $(\bar{A}, B)$ is stochastically stabilizable. Moreover, under the positive definiteness of the weighting matrix in* (20)*, $(C, \bar{A})$ is stochastically detectable if and only if $(C^\top C, \bar{A})$ is stochastically detectable.*

Proof. By the definition of $\bar{A}$, $F \in \mathbb{H}_{\sup}^{m\times n}$ stochastically stabilizes (A, B) if and only if $F + R^{-1}W^\top \in \mathbb{H}_{\sup}^{m\times n}$ stochastically stabilizes $(\bar{A}, B)$.

From the discussion in Remark 3.7, $C^{-1} \in \mathbb{H}_{\sup}^{n\times n}$ if (20) holds. Hence if $(C, \bar{A})$ is stochastically detectable with an observer gain $L \in \mathbb{H}_{\sup}^{n\times n}$, then $(C^\top C, \bar{A})$ is also stochastically detectable with an observer gain $L(C^\top)^{-1} \in \mathbb{H}_{\sup}^{n\times n}$, and vice versa. □

5. Sufficient conditions for stochastic stabilizability and detectability. From the results in sections 3 and 4, we can obtain an optimal controller under the assumption that (A, B) defined by (8) is stochastically stabilizable and $(C, \bar{A})$ defined by (15) and (16) is stochastically detectable. This assumption does not hold in general (and hence the solution of the Riccati difference equation may diverge) even if (A_c, B_c) is controllable and (Q_c, A_c) is observable. The major difficulty in this controller design is to check the stochastic stabilizability and detectability, namely, to show the existence of $F \in \mathbb{H}_{\sup}^{m\times n}$ and $L \in \mathbb{H}_{\sup}^{n\times r}$ such that the spectral radii of the operators $\mathcal{L}_{A+BF}$ and $\mathcal{L}_{\bar{A}+LC}$ are less than one. In this section, we provide sufficient conditions for these properties in terms of LMIs. To this end, we use the following technical result.

LEMMA 5.1 (section 2 of [14]). *For every square matrix U and every positive definite matrix S,*

$$US^{-1}U^\top \geq U + U^\top - S.$$

We here assume that $\{\phi_k : k \in \mathbb{Z}_+\}$ is a time-homogeneous Markov chain taking values in the box $\mathcal{M} = [\tau_{\min}, \tau_{\max}]^p$. As in Assumption 2.1, let $g(\bullet, \bullet)$ be the density function with respect to a σ-finite measure μ on $\mathcal{M}$ for the transition of the Markov chain $\{\phi_k : k \in \mathbb{Z}_+\}$.

5.1. Stochastic stabilizability. We first study the stochastic stabilizability of the pair $A \in \mathbb{H}_{\sup}^{n}$ and $B \in \mathbb{H}_{\sup}^{n\times m}$. In our LQ problem, we need to check the stochastic stabilizability of the pair (A, B) defined by (8).

Divide $\mathcal{M} = [\tau_{\min}, \tau_{\max}]^p$ into N disjoint boxes $\{\mathcal{B}_i\}_{i=1}^N$ (whose union is $\mathcal{M}$), e.g., by splitting each interval $[\tau_{\min}, \tau_{\max}]$ into r intervals $[s_i, s_{i+1})$ $(i = 1, \dots, r-1)$ and $[s_r, s_{r+1}]$ such that

$$\tau_{\min} =: s_1 < s_2 < \cdots < s_{r+1} := \tau_{\max}.$$

For each $i = 1, \dots, N$, let $c_i \in \mathcal{B}_i$, e.g., the center of $\mathcal{B}_i$. Consider a piecewise-constant feedback gain $F \in \mathbb{H}_{\sup}^{m\times n}$ defined by

$$F(\phi) := F_i \in \mathbb{R}^{m\times n} \qquad \forall \phi \in \mathcal{B}_i. \tag{25}$$

We provide a sufficient condition for the feedback gain F in (25) to stochastically stabilize (A, B), inspired by the gridding approach developed, e.g., in [9, 10, 13, 18, 28].

THEOREM 5.2. *Let $A \in \mathbb{H}_{\sup}^{n}$ and $B \in \mathbb{H}_{\sup}^{n\times m}$. For each $i = 1, \dots, N$, define*

$$w_i(\phi) := \int_{\mathcal{B}_i} g(\phi, \ell)\mu(d\ell) \geq 0 \qquad \forall \phi \in \mathcal{M} \tag{26}$$

and

$$\begin{bmatrix} \Gamma_{A,i} & \Gamma_{B,i} \end{bmatrix} := \begin{bmatrix} \sqrt{w_1(c_i)}A(c_i) & \sqrt{w_1(c_i)}B(c_i) \\ \vdots & \vdots \\ \sqrt{w_N(c_i)}A(c_i) & \sqrt{w_N(c_i)}B(c_i) \end{bmatrix}.$$

Assume that, for every $i = 1, \ldots, N$, a scalar $\kappa_i > 0$ satisfies

$$\left\| \begin{bmatrix} \sqrt{w_1(\phi)}A(\phi) & \sqrt{w_1(\phi)}B(\phi) \\ \vdots & \vdots \\ \sqrt{w_N(\phi)}A(\phi) & \sqrt{w_N(\phi)}B(\phi) \end{bmatrix} - \begin{bmatrix} \Gamma_{A,i} & \Gamma_{B,i} \end{bmatrix} \right\| \leq \kappa_i \tag{27}$$

for μ-almost every $\phi \in \mathcal{B}_i$. If there exist positive definite matrices $R_i \in \mathbb{R}^{n\times n}$, (not necessarily symmetric) matrices $U_i \in \mathbb{R}^{n\times n}$, $\bar{F}_i \in \mathbb{R}^{m\times n}$, and scalars $\lambda_i > 0$ such that the following LMIs hold for all $i = 1, \ldots, N$,

$$\begin{bmatrix} U_i + U_i^\top - R_i & 0 & U_i^\top \Gamma_{A,i}^\top + \bar{F}_i^\top \Gamma_{B,i}^\top & \kappa_i \begin{bmatrix} U_i^\top & \bar{F}_i^\top \end{bmatrix} \\ \star & \lambda_i I & \lambda_i I & 0 \\ \star & \star & \boldsymbol{R} & 0 \\ \star & \star & \star & \lambda_i I \end{bmatrix} > 0, \tag{28}$$

where $\boldsymbol{R} := \operatorname{diag}(R_1, \ldots, R_N)$, then the pair (A, B) is stochastically stabilizable by the controller (25) *with $F_i := \bar{F}_i U_i^{-1}$.*

Proof. From Theorem 4.2 and Proposition 4.3, (A, B) is stochastically stabilizable if and only if there exist $S \in \mathbb{H}_{\sup}^{n+}$, $F \in \mathbb{H}_{\sup}^{m\times n}$, and $\epsilon > 0$ such that the following Lyapunov inequality holds for μ-almost every $\phi \in \mathcal{M}$:

$$S(\phi) - (A(\phi) + B(\phi)F(\phi))^\top \left(\int_{\mathcal{M}} g(\phi,\ell) S(\ell) \mu(d\ell) \right) (A(\phi) + B(\phi)F(\phi)) \geq \epsilon I. \tag{29}$$

We employ a piecewise-constant matrix function S for the Lyapunov inequality (29). Define

$$S(\phi) := S_i \qquad \forall \phi \in \mathcal{B}_i \tag{30}$$

with $S_i > 0$. In what follows, we prove that if the LMIs in (28) are feasible for all $i = 1, \ldots, N$, then the Lyapunov inequality (29) holds with $S \in \mathbb{H}_{\sup}^{n+}$ defined by (30).

By construction, we have that

$$\int_{\mathcal{M}} g(\phi,\ell) S(\ell) \mu(d\ell) = \sum_{i=1}^{N} w_i(\phi) S_i \qquad \forall \phi \in \mathcal{M}. \tag{31}$$

Note that

$$\sum_{i=1}^{N} w_i(\phi) S_i = \begin{bmatrix} \sqrt{w_1(\phi)} I \\ \vdots \\ \sqrt{w_N(\phi)} I \end{bmatrix}^\top \boldsymbol{S} \begin{bmatrix} \sqrt{w_1(\phi)} I \\ \vdots \\ \sqrt{w_N(\phi)} I \end{bmatrix}, \tag{32}$$

where $\boldsymbol{S} := \operatorname{diag}(S_1, \ldots, S_N)$. Substituting (31) and (32) into (29), we see that (29) can be transformed into the matrix inequality

$$S(\phi) - (A(\phi) + B(\phi)F(\phi))^\top \begin{bmatrix} \sqrt{w_1(\phi)} I \\ \vdots \\ \sqrt{w_N(\phi)} I \end{bmatrix}^\top \boldsymbol{S} \begin{bmatrix} \sqrt{w_1(\phi)} I \\ \vdots \\ \sqrt{w_N(\phi)} I \end{bmatrix} (A(\phi) + B(\phi)F(\phi)) \geq \epsilon I. \tag{33}$$

Moreover, by the Schur complement formula, the matrix inequality (33) is equivalent to

$$\begin{bmatrix} S(\phi)-\epsilon I & (A(\phi)+B(\phi)F(\phi))^\top \begin{bmatrix} \sqrt{w_1(\phi)}I \\ \vdots \\ \sqrt{w_N(\phi)}I \end{bmatrix}^\top \boldsymbol{S} \\ \star & \boldsymbol{S} \end{bmatrix} \geq 0. \tag{34}$$

Using the inequality (27), we next discretize (34); more specifically, we show that if the LMIs (28) are feasible for every $i = 1, \ldots, N$, then the matrix inequality (34) holds for μ-almost every $\phi \in \mathcal{M}$. In terms of the upper-right part of the matrix in (34), we obtain

$$\begin{aligned} &(A(\phi)+B(\phi)F(\phi))^\top \begin{bmatrix} \sqrt{w_1(\phi)}I \\ \vdots \\ \sqrt{w_N(\phi)}I \end{bmatrix}^\top \\ &\quad = \begin{bmatrix} I & F(\phi)^\top \end{bmatrix} \begin{bmatrix} \sqrt{w_1(\phi)}A(\phi)^\top & \cdots & \sqrt{w_N(\phi)}A(\phi)^\top \\ \sqrt{w_1(\phi)}B(\phi)^\top & \cdots & \sqrt{w_N(\phi)}B(\phi)^\top \end{bmatrix}. \end{aligned}$$

We also have from the inequality (27) that, for μ-almost every $\phi \in \mathcal{B}_i$, there exists

$$\begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix} \in \mathbb{R}^{nN\times(n+m)} \text{ with } \left\| \begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix} \right\| \leq 1$$

such that

$$\begin{bmatrix} \sqrt{w_1(\phi)}A(\phi)^\top & \cdots & \sqrt{w_N(\phi)}A(\phi)^\top \\ \sqrt{w_1(\phi)}B(\phi)^\top & \cdots & \sqrt{w_N(\phi)}B(\phi)^\top \end{bmatrix} = \begin{bmatrix} \Gamma_{A,i}^\top \\ \Gamma_{B,i}^\top \end{bmatrix} + \kappa_i \begin{bmatrix} \Phi_A^\top \\ \Phi_B^\top \end{bmatrix}.$$

Hence the matrix inequality (34) holds with some $\epsilon > 0$ for μ-almost every $\phi \in \mathcal{M}$ if

$$\begin{bmatrix} S_i & \begin{bmatrix} I & F_i^\top \end{bmatrix} \left(\begin{bmatrix} \Gamma_{A,i}^\top \\ \Gamma_{B,i}^\top \end{bmatrix} + \kappa_i \begin{bmatrix} \Phi_A^\top \\ \Phi_B^\top \end{bmatrix} \right) \boldsymbol{S} \\ \star & \boldsymbol{S} \end{bmatrix} > 0 \tag{35}$$

for every $i = 1 \ldots, N$ and for every $\begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix} \in \mathbb{R}^{nN\times(n+m)}$ with $\left\| \begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix} \right\| \leq 1$.

The resulting matrix inequality (35) has the product term of the variables F_i and $\boldsymbol{S}$. To remove this product term, we employ Lemma 5.1. Let $U_i \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Defining $R_i := S_i^{-1}$ and $\bar{F}_i := F_i U_i$, we have from Lemma 5.1 that

$$\begin{aligned} &\begin{bmatrix} U_i^\top & 0 \\ 0 & \boldsymbol{S}^{-1} \end{bmatrix} \begin{bmatrix} S_i & \begin{bmatrix} I & F_i^\top \end{bmatrix} \left(\begin{bmatrix} \Gamma_{A,i}^\top \\ \Gamma_{B,i}^\top \end{bmatrix} + \kappa_i \begin{bmatrix} \Phi_A^\top \\ \Phi_B^\top \end{bmatrix} \right) \boldsymbol{S} \\ \star & \boldsymbol{S} \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & \boldsymbol{S}^{-1} \end{bmatrix} \\ &\qquad \geq \begin{bmatrix} U_i + U_i^\top - R_i & G_i^\top + H_i^\top \Phi^\top \\ \star & \boldsymbol{R} \end{bmatrix}, \end{aligned} \tag{36}$$

where the matrices $\boldsymbol{R}$, Φ, G_i, and H_i are defined by

$$\begin{aligned} &\boldsymbol{R} := \operatorname{diag}(R_1, \ldots, R_N), \quad \Phi := \begin{bmatrix} \Phi_A & \Phi_B \end{bmatrix}, \\ &G_i := \begin{bmatrix} \Gamma_{A,i} & \Gamma_{B,i} \end{bmatrix} \begin{bmatrix} U_i \\ \bar{F}_i \end{bmatrix}, \quad H_i := \kappa_i \begin{bmatrix} U_i \\ \bar{F}_i \end{bmatrix}. \end{aligned}$$

Finally, we obtain the sufficient LMI condition (28) by removing Φ from the matrix in the right-hand side of the inequality (36). Since $\|\Phi\| \leq 1$, it follows that for every $\rho_i > 0$, we have $\rho_i H_i^\top (I - \Phi^\top \Phi) H_i \geq 0$. Moreover,

$$\begin{bmatrix} U_i + U_i^\top - R_i & G_i^\top + H_i^\top \Phi^\top \\ \star & \boldsymbol{R} \end{bmatrix} - \begin{bmatrix} \rho_i H_i^\top (I - \Phi^\top \Phi) H_i & 0 \\ \star & 0 \end{bmatrix} = V_i^\top \Omega_i V_i,$$

where

$$V_i := \begin{bmatrix} I & 0 \\ \Phi H_i & 0 \\ 0 & I \\ -H_i & 0 \end{bmatrix}, \qquad \Omega_i := \begin{bmatrix} U_i + U_i^\top - R_i & 0 & G_i^\top & \rho_i H_i^\top \\ \star & \rho_i I & I & 0 \\ \star & \star & \boldsymbol{R} & 0 \\ \star & \star & \star & \rho_i I \end{bmatrix}.$$

Since V_i is full column rank, it follows that if $\Omega_i > 0$, then $V_i^\top \Omega_i V_i > 0$, and hence the matrix inequality (35) holds. Note that $\rho_i H_i^\top$ in Ω_i has the product of the variables ρ_i and $\begin{bmatrix} U_i^\top & \bar{F}_i^\top \end{bmatrix}$. However, using the similarity transformation with $\operatorname{diag}(I, 1/\rho_i I, I, 1/\rho_i I)$, we see that Ω_i is similar to

$$\begin{bmatrix} U_i + U_i^\top - R_i & 0 & G_i^\top & H_i^\top \\ \star & \lambda_i I & \lambda_i I & 0 \\ \star & \star & \boldsymbol{R} & 0 \\ \star & \star & \star & \lambda_i I \end{bmatrix}, \tag{37}$$

in which $\lambda_i := 1/\rho_i$ and the variables appear in a linear form, and the matrix in (37) is that in the left-hand side of the LMIs (28). Thus if the LMIs (28) hold for all $i = 1, \ldots, N$, then the controller (25) with $F_i := \bar{F}_i U_i^{-1}$ stochastically stabilizes (A, B). ∎

The controller obtained in Theorem 5.2 is assumed to know to which box $\mathcal{B}_i$ the parameter ϕ_k belongs for each $k \in \mathbb{Z}_+$. The following result can be used to test stabilizability and to obtain a stabilizing controller when no information about the delays is available.

COROLLARY 5.3. *Under the same hypothesis as in Theorem* 5.2*, if there exist positive definite matrices $R_i \in \mathbb{R}^{n \times n}$, (not necessarily symmetric) matrices $U \in \mathbb{R}^{n \times n}$, $\bar{F} \in \mathbb{R}^{m \times n}$, and scalars $\lambda_i > 0$ such that the following LMIs hold for all $i = 1 \ldots, N$,*

$$\begin{bmatrix} U + U^\top - R_i & 0 & U^\top \Gamma_{A,i}^\top + \bar{F}^\top \Gamma_{B,i}^\top & \kappa_i \begin{bmatrix} U^\top & \bar{F}^\top \end{bmatrix} \\ \star & \lambda_i I & \lambda_i I & 0 \\ \star & \star & \boldsymbol{R} & 0 \\ \star & \star & \star & \lambda_i I \end{bmatrix} > 0,$$

where $\boldsymbol{R} := \operatorname{diag}(R_1, \ldots, R_N)$, then the delay-independent controller $F := \bar{F} U^{-1}$ stochastically stabilizes (A, B).

Proof. This is an immediate consequence of Theorem 5.2 with $U_1 = \cdots = U_N = U$ and $\bar{F}_1 = \cdots = \bar{F}_N = \bar{F}$. ∎

We next see how conservative the proposed gridding method is. We impose the following three assumptions on discrete-time Markov jump systems.

Assumption 5.4. For all $c \in \mathcal{M}$ and $\delta > 0$, the σ-finite measure μ on $\mathcal{M}$ satisfies

$$\mu\big(\{\phi \in \mathcal{M} : \|\phi - c\| < \delta\}\big) > 0.$$

Assumption 5.5. The functions $A \in \mathbb{H}^n_{\sup}$ and $B \in \mathbb{H}^{n\times m}_{\sup}$ are continuous.

Assumption 5.6. There exist $S \in \mathbb{H}^{n+}_{\sup}$, $F \in \mathbb{H}^{m\times n}_{\sup}$, and $\epsilon > 0$ such that the Lyapunov inequality (29) holds for μ-almost every $\phi \in \mathcal{M}$. Moreover, for every $\epsilon_a, \epsilon_b > 0$, there exist disjoint boxes $\{\mathcal{B}_i\}_{i=1}^N$ whose union is $\mathcal{M}$, points $c_i \in \mathcal{B}_i$ $(i = 1, \ldots, N)$, and piecewise constant functions $S_a \in \mathbb{H}^{n+}_{\sup}$ and $F_a \in \mathbb{H}^{m\times n}_{\sup}$ defined as in (30) and (25) such that the following three conditions hold:

1. $\|S - S_a\|_\infty < \epsilon_a$, $\|F - F_a\|_\infty < \epsilon_a$.
2. For all $i, j = 1, \ldots, N$, $w_j(\phi)$ defined by (26) is continuous at $\phi = c_i$.
3. For μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \ldots, N$,

$$\left\| \begin{bmatrix} A(\phi) & B(\phi) \end{bmatrix} - \begin{bmatrix} A(c_i) & B(c_i) \end{bmatrix} \right\| < \epsilon_b, \tag{38a}$$

$$\left\| \begin{bmatrix} \sqrt{w_1(\phi)} & \cdots & \sqrt{w_N(\phi)} \end{bmatrix} - \begin{bmatrix} \sqrt{w_1(c_i)} & \cdots & \sqrt{w_N(c_i)} \end{bmatrix} \right\| < \epsilon_b. \tag{38b}$$

Assumption 5.4 holds for the standard Borel measure. The functions A and B defined by (8) satisfy Assumption 5.5. The first sentence of Assumption 5.6, together with Theorem 4.2 and Proposition 4.3, implies that the pair (A, B) is stochastically stabilizable. Since we approximate a solution of the Lyapunov inequality (29) by piecewise-constant functions in Theorem 5.2, we need statement 1 of Assumption 5.6. We use statements 2 and 3 of Assumption 5.6 together with Assumptions 5.4 and 5.5 to show that a certain inequality holds at $\phi = c_i$ for every $i = 1, \ldots, N$. These assumptions on nonzero measure and continuity are required because the Lyapunov inequality (29) is assumed to be satisfied only at μ-almost everywhere.

The following proposition shows that if Assumptions 5.4–5.6 hold, then the presented gridding method will guarantee stochastic stabilizability given that approximation errors are sufficiently small.

PROPOSITION 5.7. *If Assumptions* 5.4–5.6 *hold, then there exist disjoint boxes* $\{\mathcal{B}_i\}_{i=1}^N$ *whose union is* $\mathcal{M}$ *and points* $c_i \in \mathcal{B}_i$ *(*$i = 1, \ldots, N$*) such that the LMIs in* (28) *are feasible.*

The proof of Proposition 5.7 can be found in Appendix A.

5.2. Stochastic detectability. Next we study the stochastic detectability of the pair $A \in \mathbb{H}^n_{\sup}$ and $C \in \mathbb{H}^{r\times n}_{\sup}$. In our LQ problem, we need to check the stochastic detectability of $(C, \bar{A})$ or $(Q - WR^{-1}W^\top, \bar{A})$ in (15) and (16).

Define an observer gain $L \in \mathbb{H}^{n\times r}_{\sup}$ as the piecewise-constant function

$$L(\phi) := L_i \in \mathbb{R}^{n\times r} \qquad \forall \phi \in \mathcal{B}_i, \tag{39}$$

where the disjoint boxes $\{\mathcal{B}_i\}_{i=1}^N$ are chosen as in the previous subsection. Note that the positions of the variables F, L are different between $A + BF$ (stabilizability) and $A + LC$ (detectability). Moreover, unlike the case of countable-state Markov chains (see, e.g., [6]), the duality of stochastic stabilizability and stochastic detectability is not proved yet for the case of continuous-state Markov chains. Hence we cannot use Theorem 5.2 directly, but the gridding method still provides a sufficient condition for stochastic detectability in terms of LMIs.

THEOREM 5.8. *Let* $A \in \mathbb{H}^n_{\sup}$ *and* $C \in \mathbb{H}^{r\times n}_{\sup}$*. For each* $i = 1, \ldots, N$*, define* w_i *as in Theorem* 5.2 *and*

$$\begin{bmatrix} \Upsilon_{A,i} \\ \Upsilon_{C,i} \end{bmatrix} := \begin{bmatrix} A(c_i) \\ C(c_i) \end{bmatrix}, \quad \Upsilon_{w,i} := \begin{bmatrix} \sqrt{w_1(c_i)} I & \cdots & \sqrt{w_N(c_i)} I \end{bmatrix}.$$

Assume that, for all $i = 1, \dots, N$, scalars $\kappa_{A,i}, \kappa_{w,i} > 0$ satisfy

$$\left\| \begin{bmatrix} A(\phi) \\ C(\phi) \end{bmatrix} - \begin{bmatrix} \Upsilon_{A,i} \\ \Upsilon_{C,i} \end{bmatrix} \right\| \leq \kappa_{A,i}, \tag{40a}$$

$$\left\| \begin{bmatrix} \sqrt{w_1(\phi)} I & \cdots & \sqrt{w_N(\phi)} I \end{bmatrix} - \Upsilon_{w,i} \right\| \leq \kappa_{w,i} \tag{40b}$$

for μ-almost every $\phi \in \mathcal{B}_i$. If there exist positive definite matrices $S_i \in \mathbb{R}^{n \times n}$, (not necessarily symmetric) matrices $U_i \in \mathbb{R}^{n \times n}$, $\bar{L}_i \in \mathbb{R}^{n \times r}$, and scalars $\lambda_i, \rho_i > 0$ such that the following LMIs hold for all $i = 1, \dots, N$,

$$\begin{bmatrix} U_i + U_i^\top & 0 & U_i \Upsilon_{A,i} + \bar{L}_i \Upsilon_{C,i} & \kappa_{A,i} \begin{bmatrix} U_i & \bar{L}_i \end{bmatrix} & 0 & \Upsilon_{w,i} \boldsymbol{S} & \rho_i I \\ \star & \lambda_i I & \lambda_i I & 0 & 0 & 0 & 0 \\ \star & \star & S_i & 0 & 0 & 0 & 0 \\ \star & \star & \star & \lambda_i I & 0 & 0 & 0 \\ \star & \star & \star & \star & \rho_i I & \kappa_{w,i} \boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \star & \rho_i I \end{bmatrix} > 0, \tag{41}$$

where $\boldsymbol{S} := \operatorname{diag}(S_1, \dots, S_N)$, then the pair (C, A) is stochastically detectable by the observer gain L in (39) *with $L_i := U_i^{-1} \bar{L}_i$.*

Proof. As in the proof of Theorem 5.2, we see from Theorem 4.2 and Proposition 4.3 that (C, A) is stochastically detectable if and only if there exist $S \in \mathbb{H}^{n+}_{\text{sup}}$, $L \in \mathbb{H}^{n \times r}_{\text{sup}}$, and $\epsilon > 0$ such that the following Lyapunov inequality holds for μ-almost every $\phi \in \mathcal{M}$:

$$S(\phi) - (A(\phi) + L(\phi)C(\phi))^\top \left(\int_{\mathcal{M}} g(\phi, \ell) S(\ell) \mu(d\ell) \right) (A(\phi) + L(\phi)C(\phi)) \geq \epsilon I. \tag{42}$$

We prove that if the LMIs in (41) are feasible for all $i = 1, \dots, N$, then the Lyapunov inequality (42) holds with $S \in \mathbb{H}^{n+}_{\text{sup}}$ defined as a piecewise-constant matrix function S in (30).

By the definitions of L and S, we have from the Schur complement formula and (31) that the Lyapunov inequity (42) can be transformed into

$$\begin{bmatrix} \left(\sum_{j=1}^N w_j(\phi) S_j \right)^{-1} & A(\phi) + L_i C(\phi) \\ \star & S_i - \epsilon I \end{bmatrix} \geq 0 \tag{43}$$

for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \dots, N$. First we remove the nonlinear term $\left(\sum_{j=1}^N w_j(\phi) S_j \right)^{-1}$ in (43) by using Lemma 5.1. If $U_i \in \mathbb{R}^{n \times n}$ is nonsingular, then the matrix in the left-hand side of this inequality is similar to

$$\begin{bmatrix} U_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \left(\sum_{j=1}^N w_j(\phi) S_j \right)^{-1} & A(\phi) + L_i C(\phi) \\ \star & S_i - \epsilon I \end{bmatrix} \begin{bmatrix} U_i^\top & 0 \\ 0 & I \end{bmatrix}.$$

Using Lemma 5.1, we see that

$$\begin{aligned}
&\begin{bmatrix} U_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \left(\sum_{j=1}^N w_j(\phi) S_j\right)^{-1} & A(\phi) + L_i C(\phi) \\ \star & S_i - \epsilon I \end{bmatrix} \begin{bmatrix} U_i^\top & 0 \\ 0 & I \end{bmatrix} \\
&\quad \geq \begin{bmatrix} U_i + U_i^\top - \sum_{j=1}^N w_j(\phi) S_j & U_i(A(\phi) + L_i C(\phi)) \\ \star & S_i - \epsilon I \end{bmatrix} \\
&\quad = \begin{bmatrix} U_i + U_i^\top & U_i(A(\phi) + L_i C(\phi)) \\ \star & S_i - \epsilon I \end{bmatrix} - \begin{bmatrix} \sqrt{w_1(\phi)} I & \\ \vdots & 0 \\ \sqrt{w_N(\phi)} I & \end{bmatrix}^\top \boldsymbol{S} \begin{bmatrix} \sqrt{w_1(\phi)} I & 0 \\ \vdots & 0 \\ \sqrt{w_N(\phi)} I & 0 \end{bmatrix},
\end{aligned}$$

where $\boldsymbol{S} := \operatorname{diag}(S_1, \ldots, S_N)$. Therefore, from the Schur complement formula, the matrix inequality (43) holds if

$$\begin{bmatrix} U_i + U_i^\top & U_i(A(\phi) + L_i C(\phi)) & \boldsymbol{S}_w(\phi) \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} > 0, \tag{44}$$

where $\boldsymbol{S}_w$ is defined by

$$\boldsymbol{S}_w(\phi) := \begin{bmatrix} \sqrt{w_1(\phi)} I & \cdots & \sqrt{w_N(\phi)} I \end{bmatrix} \boldsymbol{S} \qquad \forall \phi \in \mathcal{M}. \tag{45}$$

Let us next discretize the matrix inequality (44) by using the inequalities (40). In other words, we show that if the LMIs (41) are feasible for every $i = 1, \ldots, N$, then the matrix inequality (44) holds for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \ldots, N$. From the inequalities (40), we see that for μ-almost every $\phi \in \mathcal{B}_i$, there exist

$$\begin{bmatrix} \Phi_A \\ \Phi_C \end{bmatrix} \in \mathbb{R}^{(n+r) \times n}, \ \Phi_w \in \mathbb{R}^{n \times nN} \text{ with } \left\| \begin{bmatrix} \Phi_A \\ \Phi_C \end{bmatrix} \right\| \leq 1, \ \|\Phi_w\| \leq 1$$

such that

$$\begin{gathered}
\begin{bmatrix} A(\phi) \\ C(\phi) \end{bmatrix} = \begin{bmatrix} \Upsilon_{A,i} \\ \Upsilon_{C,i} \end{bmatrix} + \kappa_{A,i} \begin{bmatrix} \Phi_A \\ \Phi_C \end{bmatrix}, \\
\begin{bmatrix} \sqrt{w_1(\phi)} I & \cdots & \sqrt{w_N(\phi)} I \end{bmatrix} = \Upsilon_{w,i} + \kappa_{w,i} \Phi_w.
\end{gathered}$$

Then we obtain

$$\begin{aligned}
&\begin{bmatrix} U_i + U_i^\top & U_i(A(\phi) + L_i C(\phi)) & \boldsymbol{S}_w(\phi) \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} \\
&\quad = \begin{bmatrix} U_i + U_i^\top & G_{A,i} + H_{A,i} \Phi & G_{w,i} + \Phi_w H_{w,i} \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix},
\end{aligned}$$

where, using $\bar{L}_i := U_i L_i$, we define the matrices Φ, $G_{A,i}$, $H_{A,i}$, $G_{w,i}$, and $H_{w,i}$ by

$$\begin{gathered}
\Phi := \begin{bmatrix} \Phi_A \\ \Phi_C \end{bmatrix}, \quad G_{A,i} := \begin{bmatrix} U_i & \bar{L}_i \end{bmatrix} \begin{bmatrix} \Upsilon_{A,i} \\ \Upsilon_{C,i} \end{bmatrix}, \\
H_{A,i} := \kappa_{A,i} \begin{bmatrix} U_i & \bar{L}_i \end{bmatrix}, \quad G_{w,i} := \Upsilon_{w,i} \boldsymbol{S}, \quad H_{w,i} := \kappa_{w,i} \boldsymbol{S}.
\end{gathered}$$

Since $\|\Phi\|, \|\Phi_w\| \leq 1$, it follows that for all $\rho_i, \rho_{w,i} > 0$, we have

$$\rho_i H_{A,i}(I - \Phi\Phi^\top)H_{A,i}^\top + \rho_{w,i}(I - \Phi_w\Phi_w^\top) \geq 0.$$

Moreover,

$$\begin{bmatrix} U_i + U_i^\top & G_{A,i} + H_{A,i}\Phi & G_{w,i} + \Phi_w H_{w,i} \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} - \begin{bmatrix} \rho_i H_{A,i}(I - \Phi\Phi^\top)H_{A,i}^\top + \rho_{w,i}(I - \Phi_w\Phi_w^\top) & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix} = V_i^\top \Omega_i V_i,$$

where

$$V_i := \begin{bmatrix} I & 0 & 0 \\ \Phi^\top H_{A,i}^\top & 0 & 0 \\ 0 & I & 0 \\ -H_{A,i}^\top & 0 & 0 \\ \Phi_w^\top & 0 & 0 \\ 0 & 0 & I \\ -I & 0 & 0 \end{bmatrix}, \quad \Omega_i := \begin{bmatrix} U_i + U_i^\top & 0 & G_{A,i} & \rho_i H_{A,i} & 0 & G_{w,i} & \rho_{w,i} I \\ \star & \rho_i I & I & 0 & 0 & 0 & 0 \\ \star & \star & S_i & 0 & 0 & 0 & 0 \\ \star & \star & \star & \rho_i I & 0 & 0 & 0 \\ \star & \star & \star & \star & \rho_{w,i} I & H_{w,i} & 0 \\ \star & \star & \star & \star & \star & \boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \star & \rho_{w,i} I \end{bmatrix}.$$

Since V_i is full column rank, $\Omega_i > 0$ leads to $V_i \Omega_i V_i > 0$, which implies that the matrix inequality (44) is satisfied for μ-almost every $\phi \in \mathcal{B}_i$. Note that $\rho_i H_{A,i}$ has the product of the variables ρ_i and $\begin{bmatrix} U_i & \bar{L}_i \end{bmatrix}$. However, applying the similarity transformation with $\mathrm{diag}(I, 1/\rho_i I, I, 1/\rho_i I, I, I, I)$, we see that Ω_i is similar to the following matrix:

$$\begin{bmatrix} U_i + U_i^\top & 0 & G_{A,i} & H_{A,i} & 0 & G_{w,i} & \rho_{w,i} I \\ \star & \lambda_i I & \lambda_i I & 0 & 0 & 0 & 0 \\ \star & \star & S_i & 0 & 0 & 0 & 0 \\ \star & \star & \star & \lambda_i I & 0 & 0 & 0 \\ \star & \star & \star & \star & \rho_{w,i} I & H_{w,i} & 0 \\ \star & \star & \star & \star & \star & \boldsymbol{S} & 0 \\ \star & \star & \star & \star & \star & \star & \rho_{w,i} I \end{bmatrix},$$

where $\lambda_i := 1/\rho_i$, and this matrix is that in the left-hand side of (41). Thus, if the LMIs (41) are feasible, then (C, A) is stochastically detectable. ∎

As in the case of stochastic stabilizability, we see that the proposed gridding method does not introduce conservatism if approximation errors are sufficiently small.

Assumption 5.9. The functions $A \in \mathbb{H}_{\sup}^n$ and $C \in \mathbb{H}_{\sup}^{r \times n}$ are continuous.

Assumption 5.10. There exist $S \in \mathbb{H}_{\sup}^{n+}$, $L \in \mathbb{H}_{\sup}^{n \times r}$, and $\epsilon > 0$ such that the Lyapunov inequality (42) holds for μ-almost every $\phi \in \mathcal{M}$. Moreover, for every $\epsilon_a, \epsilon_b > 0$, there exist disjoint boxes $\{\mathcal{B}_i\}_{i=1}^N$ whose union is $\mathcal{M}$, points $c_i \in \mathcal{B}_i$ $(i = 1, \dots, N)$, and piecewise constant functions $S_a \in \mathbb{H}_{\sup}^{n+}$ and $L_a \in \mathbb{H}_{\sup}^{n \times r}$ defined by (30) and (39) such that the following three conditions hold:

1. $\|S - S_a\|_\infty < \epsilon_a$, $\|L - L_a\|_\infty < \epsilon_a$.
2. For all $i, j = 1, \dots, N$, $w_j(\phi)$ defined by (26) is continuous at $\phi = c_i$.
3. For μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \dots, N$, the following inequality and (38b) are satisfied:

$$\left\| \begin{bmatrix} A(\phi) \\ C(\phi) \end{bmatrix} - \begin{bmatrix} A(c_i) \\ C(c_i) \end{bmatrix} \right\| < \epsilon_b.$$

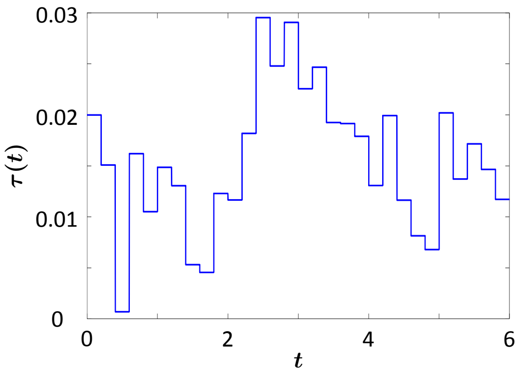

FIG. 2. *Sample path of* $\tau(t)$.

PROPOSITION 5.11. *If Assumptions* 5.4, 5.9, *and* 5.10 *hold, then there exist disjoint boxes* $\{\mathcal{B}_i\}_{i=1}^N$ *whose union is* $\mathcal{M}$ *and points* $c_i \in \mathcal{B}_i$ $(i = 1, \ldots, N)$ *such that the LMIs in* (41) *are feasible.*

The proof of Proposition 5.11 can be found in Appendix B.

6. Numerical example. Consider the unstable batch reactor studied in [30], where the system matrices A_c and B_c in the continuous-time plant (1) are given by

$$A_c := \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \qquad B_c := \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

This model is widely used as a benchmark example. We take the sampling period $h = 0.2$ and the delay interval $[\tau_{\min}, \tau_{\max}] = [0, 0.03]$.

We consider that the latest delay τ_k is stochastically determined by the average of the last two delays τ_{k-1} and τ_{k-2}. More precisely, the sequence $\{\phi_k : k \in \mathbb{Z}_+\}$, where $\phi_k := \left[\begin{smallmatrix} \tau_k \\ \tau_{k-1} \end{smallmatrix}\right]$, is a Markov chain, and its transition probability kernel $\mathcal{G}$ is given in the following way: For every box $\mathcal{B} = [b_{11}, b_{12}] \times [b_{21}, b_{22}]$,

$$\mathcal{G}\left(\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \mathcal{B}\right) = \begin{cases} \frac{\Phi_{d_{\text{ave}}}(b_{12}) - \Phi_{d_{\text{ave}}}(b_{11})}{\Phi_{d_{\text{ave}}}(\tau_{\max}) - \Phi_{d_{\text{ave}}}(\tau_{\min})} & \text{if } d_1 \in [b_{21}, b_{22}], \\ 0 & \text{otherwise,} \end{cases} \tag{46}$$

where $d_{\text{ave}} := (d_1 + d_2)/2$ and $\Phi_d(x)$ is the probability distribution function of the normal distribution with mean d and standard deviation σ. Figure 2 illustrates a sample path of the delay $\tau(t)$ with the initial data $\tau_0 = \tau_{-1} = 0.02$ and the standard derivation $\sigma = 1/100$, where $\tau(t)$ is defined by

$$\tau(t) := \tau_k \qquad \forall t \in [kh + \tau_k, (k+1)h + \tau_{k+1}).$$

The weighting matrices Q_c, R_c for the state and the input in (5) are the identity matrices with compatible dimensions. Using Theorem 5.2, we can confirm that (A, B) in (8) is stochastically stabilizable. Additionally, $(Q - WR^{-1}W^\top, \bar{A})$ in Lemma 3.2 is stochastically detectable by Theorem 5.8. Hence, by Theorems 3.6 and 4.6, we can derive an optimal controller u^{opt} from the iteration of a Riccati difference equation.

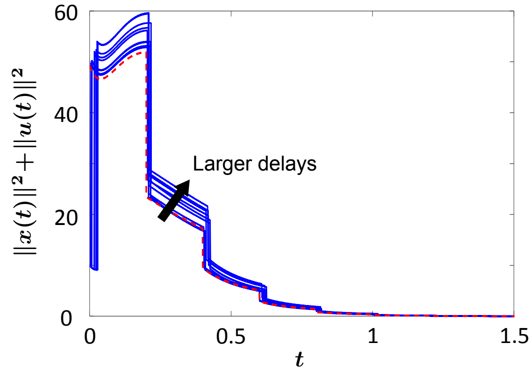

FIG. 3. *Ten sample paths of* $\|x(t)\|^2 + \|u(t)\|^2$*: The solid lines are the time responses with stochastic delays. The dotted line is the time response with no delays, for which we used the (conventional) discrete-time LQ regulator computed with the same weighting matrices.*

Time responses are computed for a deterministic initial state

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \quad u_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Figure 3 depicts 10 sample paths of the performance function $\|x(t)\|^2 + \|u(t)\|^2$, where initial delays τ_0, τ_{-1} are uniformly distributed in the interval $[\tau_{\min}, \tau_{\max}]$ and the standard deviation σ of the probability distribution function Φ_d in (46) is given by $\sigma = 1/100$.

We observe that the time responses with small initial delays are similar to the response with no delays by the conventional discrete-time LQ regulator with the same weighting matrices. Although larger delays degrade the control performance, the optimal controller achieves almost the same decrease rate for every initial delay.

7. Concluding remarks. We provided the design of delay-dependent optimal controllers for sampled-data systems whose sensor-to-controller delays are stochastically determined by the last few delays. Our optimal control problem was reduced to the LQ problem of discrete-time Markov jump systems. We can efficiently compute an optimal controller by iteratively solving a Riccati difference equation. Moreover, we derived the sufficient conditions for stochastic stabilizability and detectability in terms of LMIs via the gridding approach. From these conditions, we can also construct stabilizing controllers and state observers. Future work will focus on addressing more general systems by incorporating packet losses and output feedback.

To solve the stabilization problem of discrete-time systems with continuous-valued Markovian jumping parameters, we approximated the function S in the Lyapunov inequality (23) by a piecewise constant function. Another interesting study is to approximate the system

$$\{(A(\phi), B(\phi)) : \phi \in \mathcal{M}\}$$

by

$$\{(A(c_i), B(c_i)) : c_i \text{ is the center of } \mathcal{B}_i, \quad i = 1, \ldots, N\}$$

and then to consider the stabilization problem of systems having discrete-valued Markovian jumping parameters but time-varying uncertainty in the coefficient matrices.

Appendix A. Proof of Proposition 5.7. First we prove that if $\epsilon_a > 0$ in Assumption 5.6 is sufficiently small, then there exists $\epsilon_1 > 0$ such that, for μ-almost every $\phi \in \mathcal{M}$,

$$\begin{bmatrix} S_a(\phi) & (A(\phi)+B(\phi)F_a(\phi))^\top \begin{bmatrix} \sqrt{w_1(\phi)}I \\ \vdots \\ \sqrt{w_N(\phi)}I \end{bmatrix}^\top \\ \star & \boldsymbol{S}^{-1} \end{bmatrix} > \epsilon_1 I, \tag{47}$$

where $\boldsymbol{S} := \operatorname{diag}(S_1, \dots, S_N)$.

Since

$$\int_{\mathcal{M}} g(\phi,\ell)\mu(d\ell) = 1 \tag{48}$$

for all $\phi \in \mathcal{M}$, it follows from statement 1 of Assumption 5.6 that

$$\left\| \int_{\mathcal{M}} g(\phi,\ell)(S(\ell)-S_a(\ell))\mu(d\ell) \right\| \le \int_{\mathcal{M}} g(\phi,\ell)\|S(\ell)-S_a(\ell)\|\mu(d\ell) \le \epsilon_a$$

for μ-almost every $\phi \in \mathcal{M}$. We obtain

$$\begin{aligned} &\Bigg\| \left(S(\phi) - (A(\phi)+B(\phi)F(\phi))^\top \left(\int_{\mathcal{M}} g(\phi,\ell)S(\ell)\mu(d\ell) \right) (A(\phi)+B(\phi)F(\phi)) \right) \\ &\quad - \left(S_a(\phi) - (A(\phi)+B(\phi)F(\phi))^\top \left(\int_{\mathcal{M}} g(\phi,\ell)S_a(\ell)\mu(d\ell) \right) (A(\phi)+B(\phi)F(\phi)) \right) \Bigg\| \\ &\quad \le \left(1 + \|A+BF\|_\infty^2\right)\epsilon_a \end{aligned}$$

for μ-almost every $\phi \in \mathcal{M}$. Therefore, if we set $\bar{\epsilon}_a > 0$ to be a value with

$$\bar{\epsilon}_a < \frac{\epsilon}{1+\|A+BF\|_\infty^2},$$

then $\epsilon_2 := \epsilon - \left(1+\|A+BF\|_\infty^2\right)\bar{\epsilon}_a > 0$ satisfies

$$S_a(\phi) - (A(\phi)+B(\phi)F(\phi))^\top \left(\int_{\mathcal{M}} g(\phi,\ell)S_a(\ell)\mu(d\ell) \right) (A(\phi)+B(\phi)F(\phi)) > \epsilon_2 I \tag{49}$$

for μ-almost every $\phi \in \mathcal{M}$ and for every $\epsilon_a \in (0, \bar{\epsilon}_a)$. Substituting (31) and (32) into (49) and applying the Schur complement formula, we have that there exists $\epsilon_3 > 0$ such that

$$\begin{bmatrix} S_a(\phi) & (A(\phi)+B(\phi)F(\phi))^\top \boldsymbol{w}(\phi) \\ \star & \boldsymbol{S}^{-1} \end{bmatrix} > \epsilon_3 I$$

for μ-almost every $\phi \in \mathcal{M}$, where

$$\boldsymbol{w}(\phi) := \begin{bmatrix} \sqrt{w_1(\phi)}I & \cdots & \sqrt{w_N(\phi)}I \end{bmatrix}.$$

We obtain

$$\begin{bmatrix} S_a(\phi) & (A(\phi)+B(\phi)F_a(\phi))^\top \boldsymbol{w}(\phi) \\ \star & \boldsymbol{S}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} S_a(\phi) & (A(\phi)+B(\phi)F(\phi))^\top \boldsymbol{w}(\phi) \\ \star & \boldsymbol{S}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & (F(\phi)-F_a(\phi))^\top B(\phi)^\top \boldsymbol{w}(\phi) \\ \star & 0 \end{bmatrix}.$$

By (48), $\{w_i\}_{i=1}^N$ satisfies

$$\sum_{i=1}^N w_i(\phi) = \sum_{i=1}^N \int_{\mathcal{B}_i} g(\phi,\ell)\mu(d\ell) = \int_{\mathcal{M}} g(\phi,\ell)\mu(d\ell) = 1 \qquad \forall \phi \in \mathcal{M}. \tag{50}$$

It follows that $\|\boldsymbol{w}(\phi)\| = 1$ for every choice of disjoint boxes $\{\mathcal{B}_i\}_{i=1}^N$ and for every $\phi \in \mathcal{M}$. Hence, if $\|F - F_a\|_\infty < \epsilon_a$, then

$$\left\| \begin{bmatrix} 0 & (F(\phi)-F_a(\phi))^\top B(\phi)^\top \boldsymbol{w}(\phi) \\ \star & 0 \end{bmatrix} \right\| \le \|B\|_\infty \epsilon_a$$

for μ-almost every $\phi \in \mathcal{M}$. If $\epsilon_a \in (0, \bar{\epsilon}_a)$ satisfies $\epsilon_a < \epsilon_3/\|B\|_\infty$, then the desired inequality (47) holds with $\epsilon_1 := \epsilon_3 - \|B\|_\infty \epsilon_a$.

Let us next derive the feasibility of the LMIs in (28) from the inequality (47). Using the similarity transformation $\operatorname{diag}(S_a^{-1}, I)$, we find that there exists $\epsilon_4 > 0$ such that

$$\begin{bmatrix} R_i & \begin{bmatrix} R_i & \bar{F}_i^\top \end{bmatrix} \begin{bmatrix} A(\phi)^\top \\ B(\phi)^\top \end{bmatrix} \boldsymbol{w}(\phi) \\ \star & \boldsymbol{R} \end{bmatrix} > \epsilon_4 I$$

for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \dots, N$, where $R_i := S_i^{-1}$, $\boldsymbol{R} := \boldsymbol{S}^{-1}$, and $\bar{F}_i := F_i R_i$. Assumption 5.4 on the nonzero property of μ and Assumptions 5.5 and 5.6 on the continuity of $A, B, \boldsymbol{w}$ at $\phi = c_i$ imply that for every $\epsilon_5 \in (0, \epsilon_4)$,

$$\begin{bmatrix} R_i & R_i^\top \Gamma_{A,i}^\top + \bar{F}_i^\top \Gamma_{B,i}^\top \\ \star & \boldsymbol{R} \end{bmatrix} > \epsilon_5 I \qquad \forall i = 1, \dots, N, \tag{51}$$

where $\Gamma_{A,i}$ and $\Gamma_{B,i}$ are defined as in Theorem 5.2.

By statement 3 of Assumption 5.6, we see that for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \dots, N$,

$$\left\| \boldsymbol{w}(\phi)^\top \begin{bmatrix} A(\phi) & B(\phi) \end{bmatrix} - \begin{bmatrix} \Gamma_{A,i} & \Gamma_{B,i} \end{bmatrix} \right\| \le \left(1 + \left\| \begin{bmatrix} A & B \end{bmatrix} \right\|_\infty \right) \epsilon_b =: \epsilon_\kappa. \tag{52}$$

On the other hand, the LMI in (28) with $U_i = R_i$ and $\kappa_i = \epsilon_\kappa$ is equivalent to

$$\begin{bmatrix} R_i & R_i^\top \Gamma_{A,i}^\top + \bar{F}_i^\top \Gamma_{B,i}^\top & 0 & \epsilon_\kappa \begin{bmatrix} R_i^\top & \bar{F}_i^\top \end{bmatrix} \\ \star & \boldsymbol{R} & \lambda_i I & 0 \\ \star & \star & \lambda_i I & 0 \\ \star & \star & \star & \lambda_i I \end{bmatrix} > 0.$$

By the Schur complement formula, if $\bar{F}_i = F_i R_i$, then the above inequality is equivalent to

$$\begin{bmatrix} R_i & R_i^\top \Gamma_{A,i}^\top + \bar{F}_i^\top \Gamma_{B,i}^\top \\ \star & \boldsymbol{R} \end{bmatrix} - \begin{bmatrix} (\epsilon_\kappa^2/\lambda_i) \cdot R_i^\top \left(I + F_i^\top F_i\right) R_i & 0 \\ \star & \lambda_i I \end{bmatrix} > 0. \tag{53}$$

From (49), we know that

$$\|R_i\| = \|S_i^{-1}\| < \frac{1}{\epsilon_2} \qquad \forall i = 1, \ldots, N.$$

Therefore, if $\lambda_i > 0$ and $\epsilon_\kappa > 0$ satisfy

$$\lambda_i < \epsilon_5, \qquad \epsilon_\kappa < \epsilon_2 \sqrt{\frac{\lambda_i \epsilon_5}{1 + (\|F\|_\infty + \epsilon_a)^2}} \qquad \forall i = 1, \ldots, N,$$

then the inequality (51) leads to the desired conclusion (53). If we choose sufficiently small $\epsilon_b > 0$, then $\epsilon_\kappa > 0$ satisfies the above inequality. This completes the proof.

Appendix B. Proof of Proposition 5.11. By the same discussion as in the proof of Proposition 5.7, there exists $\epsilon_1 > 0$ such that

$$\begin{bmatrix} \left(\sum_{j=1}^N w_j(\phi) S_j\right)^{-1} & A(\phi) + L_i C(\phi) \\ \star & S_i \end{bmatrix} > \epsilon_1 I$$

for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \ldots, N$. Using the similarity transformation $\operatorname{diag}\left(\sum_{j=1}^N w_j(\phi) S_j, I\right)$ and applying the Schur complement formula, we find that there exists $\epsilon_2 > 0$ such that

$$\begin{bmatrix} 2U(\phi) & U(\phi)(A(\phi) + L_i C(\phi)) & \boldsymbol{S}_w(\phi) \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} > \epsilon_2 I$$

for μ-almost every $\phi \in \mathcal{B}_i$ and for every $i = 1, \ldots, N$, where $U(\phi) := \sum_{j=1}^N w_j(\phi) S_j$, $\boldsymbol{S} := \operatorname{diag}(S_1, \ldots, S_N)$, and $\boldsymbol{S}_w$ is defined as in (45). Assumption 5.4 on the nonzero property of μ and Assumptions 5.9 and 5.10 on the continuity of $A, C, \boldsymbol{w}$ at $\phi = c_i$ lead to

$$\begin{bmatrix} 2U_i & U_i \Upsilon_{A,i} + \bar{L}_i \Upsilon_{C,i} & \Upsilon_{w,i} \boldsymbol{S} \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} > \epsilon_3 I \qquad \forall \epsilon_3 \in (0, \epsilon_2),\ \forall i = 1, \ldots, N, \tag{54}$$

where $\Upsilon_{A,i}$, $\Upsilon_{C,i}$, $\Upsilon_{w,i}$ are defined as in Theorem 5.8 and

$$U_i := U(c_i) = \sum_{j=1}^N w_j(c_i) S_j, \quad \bar{L}_i := U_i L_i. \tag{55}$$

The LMIs (41) with $\kappa_{A,i} = \epsilon_b = \kappa_{w,i}$ are equivalent to

$$\begin{bmatrix} U_i + U_i^\top & U_i \Upsilon_{A,i} + \bar{L}_i \Upsilon_{C,i} & \Upsilon_{w,i} \boldsymbol{S} & \epsilon_b \begin{bmatrix} U_i & \bar{L}_i \end{bmatrix} & 0 & 0 & \rho_i I \\ \star & S_i & 0 & 0 & 0 & \lambda_i I & 0 \\ \star & \star & \boldsymbol{S} & 0 & \epsilon_b \boldsymbol{S} & 0 & 0 \\ \star & \star & \star & \lambda_i I & 0 & 0 & 0 \\ \star & \star & \star & \star & \rho_i I & 0 & 0 \\ \star & \star & \star & \star & \star & \lambda_i I & 0 \\ \star & \star & \star & \star & \star & \star & \rho_i I \end{bmatrix} > 0$$

and hence to

$$\begin{bmatrix} U_i+U_i^\top & U_i\Upsilon_{A,i}+\bar{L}_i\Upsilon_{C,i} & \Upsilon_{w,i}\boldsymbol{S} \\ \star & S_i & 0 \\ \star & \star & \boldsymbol{S} \end{bmatrix} - \begin{bmatrix} \frac{\epsilon_b^2}{\lambda_i}(U_iU_i^\top+\bar{L}_i\bar{L}_i^\top)+\rho_i I & 0 & 0 \\ 0 & \lambda_i I & 0 \\ 0 & 0 & \frac{\epsilon_b^2}{\rho_i}\boldsymbol{S}^2 \end{bmatrix} > 0$$

by the Schur complement formula. From (50), we know that U_i and $\bar{L}_i$ defined by (55) satisfy

$$\|U_i\| \le \|U\|_\infty \le \sum_{j=1}^N w_j(\phi)\|S_j\| \le \|S_a\|_\infty \le \|S\|_\infty + \epsilon_a,$$
$$\|\bar{L}_i\| \le \|U_i\| \cdot \|L_i\| \le (\|S\|_\infty + \epsilon_a)\cdot(\|L\|_\infty + \epsilon_a)$$

for every $i = 1, \dots, N$. Moreover, $\boldsymbol{S} = \operatorname{diag}(S_1, \dots, S_N)$ satisfies $\|\boldsymbol{S}\| \le \|S\|_\infty + \epsilon_a$. Hence, for the matrices U_i and $\bar{L}_i$ in (55) and $\boldsymbol{S} = \operatorname{diag}(S_1, \dots, S_N)$, there exist $\lambda_i > 0$, $\rho_i > 0$, $\epsilon_b > 0$ such that

$$\begin{bmatrix} \frac{\epsilon_b^2}{\lambda_i}(U_iU_i^\top+\bar{L}_i\bar{L}_i^\top)+\rho_i I & 0 & 0 \\ 0 & \lambda_i I & 0 \\ 0 & 0 & \frac{\epsilon_b^2}{\rho_i}\boldsymbol{S}^2 \end{bmatrix} < \frac{\epsilon_3}{2}I.$$

By (54), the desired LMIs in (41) are satisfied.

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