Finite-time Convergence Policies in State-dependent Social Networks

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Abstract— **This paper addresses the problem of finite-time convergence in a social network for a political party or an association, modeled as a distributed iterative system with a graph dynamics chosen to mimic how people interact. It is firstly shown that, in this setting, finite-time convergence is achieved only when nodes form a complete network, and that contacting with agents with distinct opinions reduces to a half the required interconnections. Two novel strategies are presented that enable finite-time convergence, even for the case where each node only contacts the two closest neighbors. These strategies are of prime importance, for instance, in a company environment where agents can be motivated to reach faster conclusions. The performance of the proposed policies is assessed through simulation, illustrating, in particular the finite-time convergence property.**

I. Introduction

The study of social networks relates to understanding how a group of agents perform a decision-making process. We are interested in the problem of showing convergence for a state-dependent social network and what are the implications that small changes on the way the nodes interact can cause in the convergence time. Such observations are interesting in practical terms in the sense that marketing campaigns and proper information dissemination in a network can significantly reduce the convergence time.

In [1], the authors study a classic model of influence networks and opinion formation processes found in sociology, which considers the evolution of power of each agent based on previous opinion formation process outcomes. The focus is on finding how the weights assigned to each agent evolve if they are constructed using previous relevance of a specific node and respective previous weight.

The problem of proving convergence in the presence of leaders is addressed in [2], assuming that the network is state-dependent in the sense that nodes communication is based on the difference in state being below a certain threshold of their own. Given that the initial graph has a directed spanning tree, the authors of [2] provide a potential function so as to get the system to consensus while maintaining network connectivity. In this paper, we turn our focus on how ti define the state-dependent rule to achieve faster convergence, by using an alternative approach, based on recent advances in consensus methods available in the literature.

In [3], it is consider the problem of selecting leaders which are more determinant both to the steady and transient states. Different metric measures for social influence are presented, allowing the construction of a non-convex optimization for the optimal leader selection problem. Convex relaxation techniques are employed and a distributed solution is found. Note that [3] is related to our work since both focus on how fast the agents reach a solution. However, the dynamics assumed in [3] are time-invariant whereas we consider a more general framework.

In this paper, we focus on analyzing a social network within a political party or an association where agents are rational when evaluating each argument of other nodes, and influence occurs among agents whose opinion is closer. The main contributions are as follows:

- We model a social network as a distributed algorithm where the network is state-dependent with a fixed parameter of maximum number of connections;
- Considering only nodes with distinct opinion is shown to reduced the number of required neighbors to half the nodes in the network to obtain finite-time convergence;
- Finally, two strategies are investigated — one where nodes with extreme opinions contact with each other and another where agents require a fixed number of neighbors — and proved to converge in finite time, even when only communicating with 2 other nodes.

The topic of studying convergence of social networks is closely related to that of consensus (see, e.g., [4],[5]). The dynamic system generated has similarities and most tools used in the convergence proofs are common to both fields (see [6], [7], [8], [4] and [5]). The work of [9] addresses the problem of consensus with state-dependent dynamics and the tools to obtain the proofs are similar to those adopted in this paper. Nevertheless, our results focus on how to select the network dynamics to increase the rate of convergence.

The main observation in this paper is that people in social networks such as a political party or a sports association, tend to contact other agents with similar opinions. The work of [10] and [11] points to the same conclusion. Different models are studied in [10], resulting in nodes converging, polarizing or fragmenting into various opinion clusters. In [11], it investigates when randomly selected pairs of nodes average their beliefs, as long as their opinions are close, converge to a single or multiple opinion clusters. In this paper, the main difference is our assumption that each node has a fixed number of influence connections, which is motivated by people having a limited number of acquaintances and considering a small number of agents in their decision.

The remainder of this paper is organized as follows. In Section II, we describe the standard social network with state-dependent dynamics. The proposed policies are given in Section III and the main properties of the obtained results are stated in Section IV and illustrated in simulation in Section V. Concluding remarks are provided in Section VI.

Notation : The transpose of a matrix A is denoted by $A^\intercal$. We let $1_n := [1 \dots 1]^\intercal$ indicate an n-dimensional vector of ones and I_n denotes the identity matrix of dimension n. The notation $||.||$ refers to $\|v\| := \sup_i |v_i|$ for a vector v. Moreover, card(S) represents the cardinality of set S.

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II. Problem Statement

We have introduced the motivation for the social network in a political party where people contact a subset of the group which has similar opinions on a subject. These opinions translate objective arguments and people are rational and will regard the interval that they receive rather than the individual opinions. The same problem can be found in different scenarios such as: a group of people are discussing a location to rendezvous and have communication devices with that have a variable power to transmit. To avoid the cost of transmitting to other people that are further away from their location, the subjects can only contact a small nearby subset. The discussed scenarios motivate the following problem.

We consider a social network where a set of n agents, also called nodes, interact and influence each other about a personal belief or opinion regarding some subject or discussion. The belief of agent i is denoted by a scalar $x_i(k), 1 \leq i \leq n$, where we consider the time as a discrete variable k which is incremented whenever agents communicate among themselves and their beliefs are updated.

The objective is to determine the final belief of the social network, x_∞, defined as

$$x_\infty := \lim_{k \to \infty} x_i(k)$$

The network of interconnections representing the influence that each agent has on another agent is modeled by a time-varying directed graph $G(k) = (\mathcal{V}, E(k))$, where $\mathcal{V}$ represents the set of n agents, also denoted by nodes, and $E(k) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of influence links that change over time. Node i influences the opinion of node j, at time k, if $(i,j) \in E(k)$. In general, the graphs $G(k)$ have self-loops, motivated by the fact that an agent will regard its previous opinion in the formation of his new belief, which means that $(i,i) \in E(k)$. $N_i(k)$ represents the set of neighbors of agent i, i.e., $N_i(k) = \{j : (i,j) \in E(k)\}$.

Definition 1 (standard network): For each node $i \in \mathcal{V}$, there is a link entering i from each node in the set $N_i(k) := N_i^-(k) \cup N_i^+(k)$, where

$$N_i^-(k) = \big\{\eta \text{ closest nodes to } x_i \text{ among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}$$

and

$$N_i^+(k) = \big\{\eta \text{ closest nodes to } x_i \text{ among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}$$

where $\eta \in \mathbb{Z}^+$. In the definitions of $N_i^-(k)$ and $N_i^+(k)$, the qualifier "closest" refers to the nodes whose belief x_j is closest in value to the belief x_i.Notice that the cardinality of $N_i^-(k)$ or $N_i^+(k)$ can be zero, so no assumption is made on the degree of the nodes in $G(k)$.

In this paper, we envisage a social network where the opinion translates a set of arguments regarding a subject and that agents are objective, meaning that, at time k, all nodes would reach the same conclusion if they had access to all the remaining opinions. Notice that the way nodes evaluate the arguments can change over time. These observations translate into the following dynamics for agent i

$$x_i(k+1) = \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \alpha_k) \max_{j \in N_i(k)} x_j(k) \quad (1)$$

where the parameter $\alpha_k \in [0,1]$ models how the agents grade the conclusions with respect to the extreme (minimum and maximum) opinions of their neighbors.

The parameter α_k allow us to consider asymmetric agents that may favor optimistic or pessimistic views: if we associate a positive stance to high values of the belief, then $\alpha_k = 1$ would correspond to optimist agents that favor only take into account beliefs more positive than their own, whereas $\alpha_k = 0$ would correspond to pessimistic agents.

The problem can be summarized as that of determining whether the opinions of the agents will converge. Moreover, for practical applications it is often useful to know if the desired convergence is met in finite-time, i.e.,

$$\exists k_f : \forall i, j, k \geq k_f, |x_i(k) - x_j(k)| < \epsilon.$$

for some appropriate tolerance $\epsilon \geq 0$. When the previous condition is met, one would like to determine k_f as a condition on the number of nodes n and neighbors η.

III. Proposed Solution

We study different network graph dynamics that may yield faster convergence than when nodes exchange information with all their closest neighbors. However, the focus of this work is not on optimizing the convergence speed based on choosing the parameter values α_k in equation (1), but rather based on the selection of the graph dynamics, i.e., the sets that define the graph neighborgood relations.

The neighbor selection policy outlined in the previous section may lead to slow convergence because nodes with equal beliefs can be chosen and nodes near the minimum or maximum values have fewer links, as either the set $N_i^-(k)$ or $N_i^+(k)$ has cardinality smaller than η. While in real social networks, people with extreme opinions may indeed interact with less neighbors precisely because of their extreme views, it is still interesting to study how deviations from the policy outlined in the previous section may lead to faster convergence.

In the following definitions by "distinct neighbors" we are referring to neighbors i and j such that for a given $\epsilon > 0$ $|x_i - x_j| > \epsilon$.

Definition 2 (nearest distinct value): For each node $i \in \mathcal{V}$, there is a link entering i from each node in the set $N_i(k) := N_i^-(k) \cup N_i^+(k) \cup \{i\}$, where

$$N_i^-(k) = \big\{\eta \text{ closest distinct nodes to } x_i \text{ among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}$$

and

$$N_i^+(k) = \big\{\eta \text{ closest distinct nodes to } x_i \text{ among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}$$

By only counting distinct neighbors (i.e., nodes with distinct beliefs) we focus our attention on policies where nodes seek to be informed by a diversified set of opinions in their decision processes.

A second network structure (or policy), referred to as *nearest distinct neighbors*, is defined as follows:

Definition 3 (nearest distinct neighbors): For each node $i \in \mathcal{V}$, there is a link entering i from each node in the set $N_i(k) := N_i^-(k) \cup N_i^+(k) \cup \{i\}$, where

$$N_i^-(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}$$

and

$$N_i^+(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}.$$

Moreover, if $\kappa_i = \text{card}(N_i^-(k)) < \eta$ then,

$$N_i^+(k) = \big\{2\eta - \kappa_i \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}$$

Finally, if $\kappa_i = \text{card}(N_i^+(k)) < \eta$ then,

$$N_i^-(k) = \big\{2\eta - \kappa_i \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}.$$

In this definition, nodes correct their lower degrees by contacting with other nearest neighbors. Even though the behavior of this strategy is completely different, it establishes that convergence rate is governed by the ability to form clusters, i.e., a group of nodes sharing a common opinion. This relationship is made clearer by the next network dynamics, referred to as *nearest circular value*.

Definition 4 (nearest circular value): For each node $i \in \mathcal{V}$, there is a link entering i from each node in the set $N_i(k) := N_i^-(k) \cup N_i^+(k) \cup \{i\}$, where

$$N_i^-(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}$$

and

$$N_i^+(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}.$$

Moreover, if $\kappa_i = \text{card}(N_i^-(k)) < \eta$ then,

$$N_i^+(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\} \bigcup \big\{\eta - \kappa_i \text{ nodes farthest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) > x_i(k)\}\big\}.$$

Finally, if $\kappa_i = \text{card}(N_i^+(k)) < \eta$ then,

$$N_i^-(k) = \big\{\eta \text{ nodes closest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\} \bigcup \big\{\eta - \kappa_i \text{ nodes farthest to } x_i \text{ but distinct among those in } \{j \in \mathcal{V} : x_j(k) < x_i(k)\}\big\}.$$

By *farthest* we mean the opposite of *closest* in the sense that node j is the farthest of i if there is no node ℓ such that $|x_i - x_j| \geq |x_i - x_\ell|$. The nearest circular value enforces all nodes to establish 2η links. In a social context, this definition amounts to a node with a strong opinion complementing it with some nodes with the opposite opinion as an attempt to increase the convergence rate. Notice that this unlikely to happen naturally in a social network, but could be enforced by policy or in scenarios where agents are given incentives to cooperate. This type of rule is often used in public debates where people with a wide range of opinions are asked to share their views on a topic of interest.

IV. Main Properties

In this section, we show the convergence properties of the standard social network dynamics with particular focus on the conditions for achieving finite-time convergence. The same analysis is also performed for the three policies introduced in this paper as "rules" to get a faster convergence in a social network about a given topic.

The two following Lemmas are very straightforward to deduce and we present them here to simplify the proofs of convergence for social network with the four graph dynamics.

Lemma 1 (order preservation): Take any two nodes $i, j \in \mathcal{V}$ and a graph dynamics where the neighbor selection is based on the states being close such as in Definition 1, Definition 2 and Definition 3. If $x_i(k) \leq x_j(k)$, then $x_i(k+1) \leq x_j(k+1)$.

Proof: The lemma results from the relationship that if $x_i(k) \leq x_j(k)$, then

$$\min_{\ell \in N_i(k)} x_\ell(k) \leq \min_{m \in N_j(k)} x_\ell(k)$$

and also

$$\max_{\ell \in N_i(k)} x_\ell(k) \leq \max_{m \in N_j(k)} x_\ell(k)$$

and by (1), we get the conclusion. ■

Notice that Lemma 1 is not valid for Definition 4 as nodes interact with neighbors that are "farthest". However, Lemma 1 is only used to prove asymptotic convergence whereas a different technique is used when addressing finite-time convergence, which is the case of the result for Definition 4. Lemma 1 states that the relative order of the opinions is preserved along time. The result will be helpful since, in the analysis, we can assume the numbering of each node to correspond to the sorting of their beliefs.

Lemma 2 (convergence for higher connectivity): Define

$$V^\eta(k) := \max_{i \in \mathcal{V}} x_i^\eta(k) - \min_{i \in \mathcal{V}} x_i^\eta(k)$$

where $x_i^\eta(k)$ represents the state at time instant k evolving according to (1) when the number of larger or smaller neighbors is η.

Take any of the network dynamics in Section II and Section III and two integers $1 \leq \eta_1 \leq \eta_2$. Then, for any initial conditions $x(0)$, $V^{\eta_1}(k) \geq V^{\eta_2}(k)$.

Proof: Select a node i such that

$$x_i(k_0) = \min_{\ell \in \mathcal{V}} x_\ell(k_0)$$

and j such that

$$x_j(k_0) = \max_{\ell \in \mathcal{V}} x_\ell(k_0).$$

Observe that

$$\max_{m \in N_i^{\eta_1}(k_0)} x_m(k_0) \leq \max_{m \in N_i^{\eta_2}(k_0)} x_m(k_0)$$

and

$$\min_{m \in N_j^{\eta_1}(k_0)} x_m(k_0) \geq \min_{m \in N_j^{\eta_2}(k_0)} x_m(k_0)$$

where $N_i^\eta(k)$ represents the set of neighbors of node i considering a value of η for the number of nodes to include. Applying recursively, we get $x_i^{\eta_1}(k_0+s) \leq x_i^{\eta_2}(k_0+s)$ and $x_j^{\eta_1}(k_0+s) \geq x_j^{\eta_2}(k_0+s)$, for all integers s. Thus, the conclusion follows. ■

A. Standard Network

The next theorem, which can be seen as a generalization of [9], presents convergence results for the standard social network.

Theorem 1: Consider a social network as in Definition 1 with updating rule (1) and any sequence $\{\alpha_k\}$. Then,

(i) If $\eta \geq n-1$, the network is guaranteed to have finite-time convergence;

(ii) If $\eta < n-1$, only asymptotic convergence can be guaranteed.

Proof: (i) The proof is straightforward by noticing that for $\eta = n-1$, we get a complete graph and finite-time convergence is achieved in one time instant for any sequence $\{\alpha_k\}$.

(ii) We start by considering the case of $\eta = 1$. Take the node i and j to be respectively the node with the smallest and largest state. Then, $V^1(0) = x_j^1(0) - x_i^1(0)$, unless $x_i^1(k) = x_j^1(k)$ (in which case convergence has already been achieved), we get by definition

$$x_i^1(k) \leq x_i^1(k+1)$$
$$x_j^1(k) \geq x_j^1(k+1)$$

The important step here is to notice that at least one of the conditions must be a strict inequality. Equality only happens when $\alpha_k = 0$ or $\alpha_k = 1$. In the first case, $x_i^1(k) < x_i^1(k+1)$ since the smallest state is subject to a maximization with a greater value. The converse is also true for the case of $\alpha_k = 1$. Thus, $V^1(k+1) < V^1(k)$ which means that the sequence $V^1(k)$ is monotonically decreasing. In addition with the fact that $V^1(k) > 0$ except when $x^1(k) = c1_n$ for some constant c since by definition the neighbor set $N_i(k) = \{i\}$ and by equation (1) we get $x_i^1(k+1) = x_i^1(k)$ and $V^1(k+1) = V^1(k)$. By the discrete-time version of the La Salle Invariance Principle, the conclusion follows. Due to Lemma 2, since $V^1(k)$ converges so does $V^\eta(k)$, which concludes the proof ∎

Remark 1 (Distinct state values): In any of the graph dynamics considered in this paper, the neighbors and the node itself must have different state values, therefore any two nodes i and j with $x_i = x_j$ have $N_i(k) = N_j(k), \forall k \geq 0$. Thus, the cardinality of the set of (distinct) node values

$$\Phi(k) = \text{card}(\{x_1(k), \cdots, x_n(k)\})$$

is a non-increasing function. Moreover, if the initial states are not distinct, then the conclusions of all theorems hold, but instead of n in the corresponding statements, one can have $n - \Phi(0)$. Also notice that, in the previous theorem, if $\alpha_k = 0$ or $\alpha_k = 1$, then $\Phi(k+1) = \Phi(k) - 1$, which means that after n time instants convergence is achieved.

The following proposition gives the asymptotic convergence rate in Theorem 1 when the sequence of α_k is constant.

Proposition 1: Consider the social network as in Definition 1 with updating rule (1) and distinct initial condition $x(0)$, a constant sequence $\{\alpha\}$ and $\eta < n-1$. Then, the following holds true

$$||x(k) - x_\infty|| \leq \lambda_2^k ||x(0) - x_\infty||$$

where λ_2 is the second largest eigenvalue of the matrix $A \in \mathbb{R}^{n \times n}$ defined by

$$A_{ij} := \begin{cases} \alpha, \text{ if } j = \max(1, i-\eta) \\ 1-\alpha, \text{ if } j = \min(n, i+\eta) \\ 0, \text{ otherwise} \end{cases}.$$

Proof: Since α is constant and by Lemma 1 the ordering of the nodes does not change, we can write the state-dependent network dynamics as a state-independent iteration. Matrix A, representing one iteration of the social interaction network, can be used to define a linear time-invariant dynamic system written as

$$x(k+1) = Ax(k)$$

Given that A is row stochastic it has an eigenvector 1_n corresponding to the eigenvalue 1 and $x_\infty = c1_n$ for some constant c defining the final social opinion. Therefore, $x_\infty = Ax_\infty$ and we can rewrite

$$\begin{aligned} x(k+1) - x_\infty &= A(x(k) - x_\infty) \\ &= A^k(x(0) - x_\infty) \end{aligned}$$

The network is strongly connected by definition which leads to all eigenvalues being within the unit circle and a single eigenvalue equal to 1. The convergence speed is governed by the magnitude of the second largest eigenvalue. ∎

B. Nearest Distinct Values

In a realistic scenario, Theorem 1 for the standard network states that finite-time convergence of all the agents cannot be guaranteed unless $\eta = n-1$, which corresponds to the complete network. The next theorem shows the convergence results when the graph dynamics is described as in Definition 2.

Theorem 2: Consider the social network with graph dynamics as in Definition 2, updating rule (1), and any sequence $\{\alpha_k\}$. Then,

(i) If $\eta \geq \frac{n}{2}$, the network is guaranteed to have finite-time convergence in no more than $\lceil \log_2 n \rceil$;

(ii) If $\eta < \frac{n}{2}$, the network can only be guaranteed to achieve asymptotic convergence.

Proof: (i) Without loss of generality, we assume $n = 2\eta$, the initial states are all distinct as in Remark 1 and the nodes numbering corresponds to their sorting according to their state, namely identifying the maximum and minimum value nodes by the subscript x_1 and x_n, respectively. Since $n = 2\eta$, there exist at least two nodes reaching the minimum and maximum nodes, i.e., there are i, j :

$$\min_{\ell \in N_i(0)} x_\ell = \min_{\ell \in N_j(0)} x_\ell = x_1(0)$$
$$\max_{\ell \in N_i(0)} x_\ell = \max_{\ell \in N_j(0)} x_\ell = x_n(0)$$

Thus, $\Phi(1) = \Phi(0) - 1$. In the following iterations the cardinality reduces by $2, 4, \cdots$ by nodes fulfilling the previous conditions, which leads to $\Phi_k = n - (2^k - 1)$. Thus, $\Phi(k) \leq 1 \Leftrightarrow k \geq \log_2 n$, thus leading to the conclusion.

(ii) Using the previous argument, one determines that if $\eta < \frac{n}{2}$, it is not possible to find at least a pair of nodes communicating with the whole network and we obtain $\Phi(k) = \Phi(0), \forall k$. Asymptotic convergence is achieved by the same argument as in the proof of Theorem 1. ∎

C. Nearest Distinct Neighbors

The following result shows convergence for the case when the network graph dynamics is as in Definition 3.

Theorem 3: Consider the social network with the graph dynamics as in Definition 3, updating rule (1) and any sequence $\{\alpha_k\}$. Then, for any $\eta \geq 1$, the network has finite-time convergence in no more than $\lceil \frac{n-(2\eta+1)}{2\eta} \rceil + 1$ time steps.

Proof: Without loss of generality, we assume distinct initial states as in Remark 1 and that the nodes labels are sorted according to their state ordering. Similarly to the previous theorem, if $\Phi(k) \leq 2\eta + 1$ then the network is complete between all the nodes with distinct values and finite-time consensus is achieved in a single time instant.

At each time k, there are $\eta + 1$ nodes that have access to $x_1(k)$ and $x_{1+\eta}(k)$, and $\eta + 1$ nodes receive the information $x_{n-\eta}(k)$ and $x_n(k)$. Thus, $\Phi(k) = n - 2\eta k$ and we need to have $\Phi(k) \leq 2\eta + 1 \Leftrightarrow k \geq \frac{n-(2\eta+1)}{2\eta}$ to get to a configuration where finite-time convergence is achieved in a single instant, which concludes the proof. ■

D. Nearest Circular Value

The next theorem shows the convergence results when the graph dynamics are as in Definition 4.

Theorem 4: Consider the social network with graph dynamics as in Definition 4, updating rule (1) and any sequence $\{\alpha_k\}$. Then, for any $\eta \geq 1$, the network has finite-time convergence in no more than $\lceil \frac{n-(2\eta+1)}{2\eta-1} \rceil + 1$ time steps.

Proof: Without loss of generality, we assume distinct initial states as in Remark 1 and that the nodes labels are sorted according to their state ordering. If $\Phi(k) \leq 2\eta + 1$, then we have the complete network and finite-time consensus is achieved in a single time instant.

At each time instant k, there are 2η nodes that have access both to $x_1(k)$ and $x_n(k)$. Thus, $\Phi(k) = n - (2\eta - 1)k$ and we need to have $\Phi(k) \leq 2\eta + 1 \Leftrightarrow k \geq \frac{n-(2\eta+1)}{2\eta-1}$ to get to a configuration where finite-time convergence is achieved in a single time instant, which concludes the proof. ■

Remark 2: In a first analysis, the convergence time provided in Theorem 2, i.e., $\log_2 n$, could appear significantly faster when compared to $\lceil \frac{n-(2\eta+1)}{2\eta-1} \rceil + 1$ from Theorem 4. However, we stress that, in Theorem 2, such a rate is achieved when $n = 2\eta$, which would lead to convergence in a single instant in the conditions of Theorem 4.

V. Simulation Results

In the previous section, we showed convergence results for four different settings: what we devise as a standard social network according to the observations and assumptions; a version where people contact only other agents with distinct opinions; a first strategy where people with strong beliefs search for agents with opposite arguments; and a last setting where nodes contact with exactly 2η nodes.

In order to compare these four policies, we consider a social network with $n = 20$ agents and set their initial states to $x_i(0) = i^2, i = 1, \cdots, n$.

Fig. 1 depicts the evolution of function $V(k)$ in each iteration of the standard social network. Recall that $V(k)$ denotes, as defined in the statement of Lemma 2, the distance between the largest and smallest nodes of the network.The case where $\eta = 19$is overlapped by the case of $\eta = 20$ as in both cases we are dealing with the complete network

The simulation for the case of a social network where nodes follow the distinct value policy is presented in Fig. 2. The cases of finite-time convergence (depicted in thick lines for $\eta \geq \frac{n}{2}$) and the maximum number of iterations both correspond to the values provided by Theorem 2. Whereas in the standard network finite-time convergence is only guaranteed for the complete network, in this case only two nodes must receive information from the whole network.

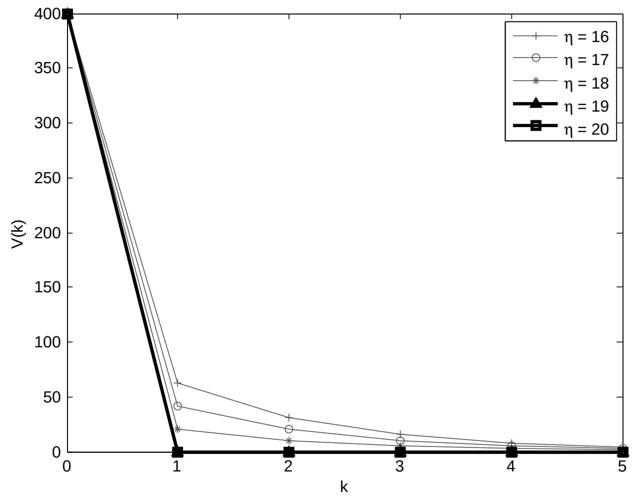

Fig. 1. Evolution of $V(k)$ for the case of a standard social network for values of $\eta = 16, \cdots, 20$.

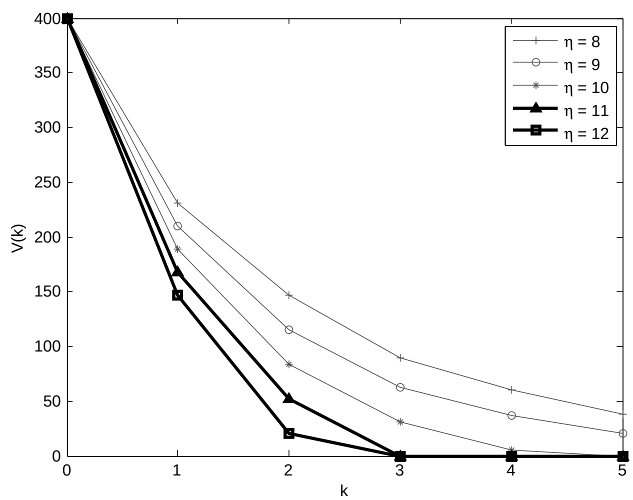

Fig. 2. Evolution of $V(k)$ for the case of distinct value network for values of $\eta = 8, \cdots, 12$.

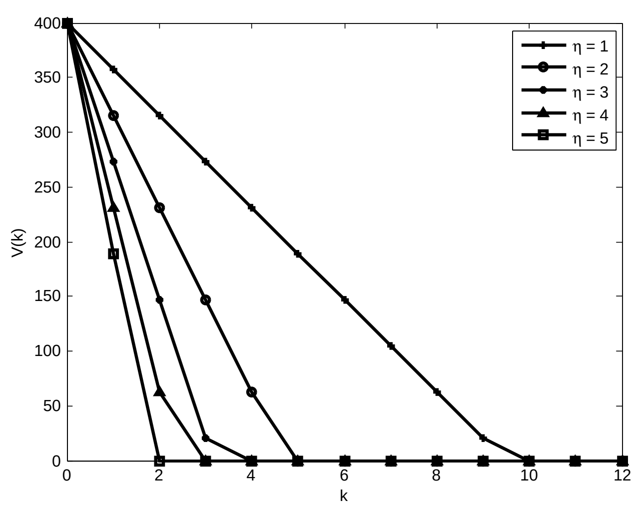

Fig. 3. Evolution of $V(k)$ for the case of a circular value for values of $\eta = 1, \cdots, 5$.

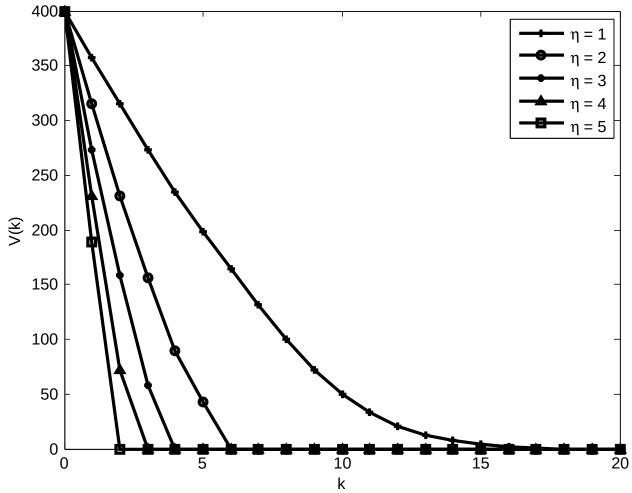

Fig. 4. Evolution of $V(k)$ for the case of distinct neighbors for values of $\eta = 1, \cdots, 5$.

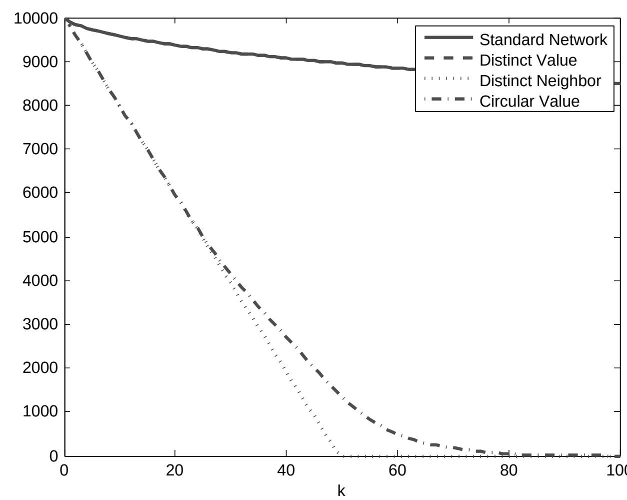

Fig. 5. Comparison of the evolution of $V(k)$ for the four cases with $\eta = 1$.

Figures 3 and 4 show the simulation results for the circular graph dynamics and the closest distinct neighbor policy, respectively. We draw attention to the fact that both rules lead to finite-time convergence regardless of the choice of η, but that the closest distinct policy has a faster rate. In the circular policy, a cluster of nodes contacting the two nodes with the strongest opinions is formed in each iteration. In contrast, for the closest neighbor strategy, two clusters of nodes with the same opinion are formed in the first iteration and in each subsequent step new nodes are added.

The previous simulations are useful to illustrate the results presented in this paper. However, it is not straightforward to compare the convergence of all four scenarios. In a different simulation, we increase the number of nodes to $n = 100$ and set $\eta = 1$ as to make the results comparable, since the first two scenarios have a number of links equal to $\eta(2n - \eta - 1)$ and the following two have $2n\eta$ links.

Figure 5 depicts the range of the state, as measured by the function $V(k)$ for the different networks. Both the circular and distinct neighbor achieve finite-time convergence. The main conclusion is that both the graph dynamics corresponding to Definition 1 and Definition 2, are restrictions, leading to slow convergence rates and for the case of $\eta = 1$ they are the same since the lines are overlayed. We also point out the behavior of the circular and distinct neighbor policies to enforce fast convergence. This indicates that forcing the establishment of clusters of opinions leads to finite-time convergence and that the rate is governed both by the number of clusters and how fast other nodes join those clusters.

VI. Conclusions

In this paper, the problem of opinion convergence in social networks present in a political party or an association is addressed using a distributed iterative algorithm with different types of graph dynamics that express how agents interact. It is assumed that each agent is rational and regards the opinion of others, and will only contact at most η close neighbors in terms of opinion. Such definition arises from the observation that people tend to interact with others that share similar values and beliefs. We show convergence results for the standard social network and show it can be improved by considering only nodes with distinct opinions. By doing so, finite-convergence is achieve only requiring half of the links when compared to the standard case.

Two policies are introduced to reduce the parameter η, namely a strategy where nodes with extreme opinions seek the influence of others with opposite argument; and a policy where agents ensure at all time 2η links by just considering the closest agents in belief without concerns whether it is greater or lower. Convergence results are provided that establish finite-time convergence. Both graph dynamics have different ways of creating clusters of opinions which influences the transient behavior. These results are useful in a company or organization environment, where agents can be motivated to cooperate according to one of these rules, which will attain a faster social convergence.

References

[1] P. Jia, A. Mirtabatabaei, N. E. Friedkin, and F. Bullo, "On the dynamics of influence networks via reflected appraisal," in *American Control Conference (ACC), 2013*, 2013, pp. 1249–1254.

[2] Z. Kan, J. Klotz, E. Pasiliao, and W. Dixon, "Containment control for a directed social network with state-dependent connectivity," in *American Control Conference (ACC), 2013*, 2013, pp. 1950–1955.

[3] M. Fardad, F. Lin, X. Zhang, and M. Jovanovic, "On new characterizations of social influence in social networks," in *American Control Conference (ACC), 2013*, June 2013, pp. 4777–4782.

[4] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *Automatic Control, IEEE Transactions on*, vol. 49, no. 9, pp. 1520 – 1533, sept. 2004.

[5] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009, electronically available at http://coordinationbook.info.

[6] S. Patterson, B. Bamieh, and A. El Abbadi, "Convergence rates of distributed average consensus with stochastic link failures," *Automatic Control, IEEE Transactions on*, vol. 55, no. 4, pp. 880 –892, april 2010.

[7] F. Fagnani and S. Zampieri, "Average consensus with packet drop communication," *SIAM J. Control Optim*, no. 48, pp. 102–133, 2009.

[8] R. Carli, F. Bullo, and S. Zampieri, "Quantized average consensus via dynamic coding/decoding schemes," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 2, pp. 156–175, 2010. [Online]. Available: http://dx.doi.org/10.1002/rnc.1463

[9] G. Shi and K. H. Johansson, "Convergence of distributed averaging and maximizing algorithms part ii: State-dependent graphs," in *American Control Conference (ACC), 2013*, 2013, pp. 6875–6880.

[10] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis and simulation," *Journal of Artificial Societies and Social Simulation*, vol. 5, no. 3, p. 2, 2002. [Online]. Available: http://ideas.repec.org/a/jas/jasssj/2002-5-2.html

[11] G. Weisbuch, G. Deffuant, F. Amblard, and J.-P. Nadal, "Meet, discuss, and segregate!" *Complexity*, vol. 7, no. 3, pp. 55–63, 2002. [Online]. Available: http://dx.doi.org/10.1002/cplx.10031