$\mathcal{L}_2$-induced Gain Analysis of Switched Linear Systems via Finitely Parametrized Storage Functions

Kenji Hirata and João P. Hespanha

***Abstract*— This paper addresses the computation of the $\mathcal{L}_2$-induced gain for switched linear systems. The main contribution of the paper is to completely characterize the induced gain of a switched system though a differential inequalities on a *finitely parametrized* common storage function, one for each system being switched. The motivation for computing the induced gain of a switched system is the application of robust stability tools to the analysis of hybrid systems.**

I. Introduction

Hybrid dynamical systems whose behavior can be described using a mixture of event-based logic and differential or difference equations have been attracting significant interest. This is motivated by the observation that a wide variety of artificial/man-made and physical systems/processes are naturally modeled in a hybrid dynamical framework. Switched systems typically arise in the context of hybrid dynamical systems when it is possible to describe the behavior in each mode through a differential or difference equation and the event-based transitions as discontinuous switchings.

The stability of switched system has been extensively studied and several key results can be found in the survey papers [1], [2], [3] and references therein. Among these result, the ones in [4], [5] are especially relevant for the present paper. In these papers, it was proved that the existence of a *common* solution to a set of Lyapunov inequalities, one for each linear system being switched, is *equivalent* to the uniform asymptotic stability of the switched system for arbitrary switching signals. In this context, *uniformity* refers to the fact that asymptotic stability is guaranteed over an entire set of switching signals, which in the context of [4], [5] contains every switching signal with a finite number of discontinuities on every finite time interval. In [4], it was also proved that a common Lyapunov function actually admits a *finite parametrization*, and this fact was used to derive a finite dimensional algebraic criterion for uniform asymptotic stability of switched systems.

The input-output properties of dynamical systems, especially $\mathcal{L}_2$-induced gains are fundamental tools for robust control theories in linear [6], [7], and nonlinear [8] settings, particularly in $\mathcal{H}^\infty$ control problems [9]. In spite of their important roles, the progress on the study of input-output properties for switched systems has been difficult [10]. Non-conservative necessary and sufficient conditions that can be used to establish the value of the induced gain have been only available for special switched systems. A separation property between all the stabilizing and all the anti-stabilizing solutions to a set of algebraic Riccati equations of the systems being switched provides a complete solution to the induced gain analysis in the case of *slow-switching* signals, where slow-switching refers to the limit as the intervals between consecutive switchings grows to infinity [11]. The variational approaches in [12] also provide a complete and non-conservative characterization of the induced gain for single-input single-output first-order systems. In a recent paper [13], a complete characterization of the induced gain for general switched linear systems was proposed, where it was proved that the existence of a *common* storage function that is a solution to a set of Hamilton-Jacobi inequalities, one for each system being switched, is *equivalent* for the $\mathcal{L}_2$-induced gain of a switched linear system to lie below a prescribed value.

This paper further pursues non-conservative necessary and sufficient conditions for an induced gain estimate of a switched system. The main result shows that a common storage function that establishes both uniform asymptotic stability and a non-conservative induced gain estimate over the class of every piecewise constant switching signal, in fact, admits a *finite parametrization*. This result also allows us to derive finite dimensional algebraic criteria for an induced gain estimate.

The remainder of the paper is organized as follows: Section II describes the problem which will be considered here, and some required preliminary results are summarized. The main results are presented in Section III. The proofs of the main results are presented in Section IV. Section V contains some concluding remarks. Due to space limitations, the proofs of all lemmas are omitted, but they can be found in [14].

II. Problem Descriptions and Preliminary Results

This section formulates the problem that is considered here. Required preliminary results which derived in [13] are summarized in Section II-D.

A. Switched Systems

The switched systems under consideration are represented by equations of the form

$$\dot{x} = A_s x + B_s u \qquad y = C_s x + D_s u \tag{1a}$$

K. Hirata is with the Department of Mechanical Engineering, Nagaoka University of Technology, Nagaoka 940 2188, Japan.

J. P. Hespanha is with the Center for Control, Dynamical Systems, and Computation, and the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA.

where $s : [0,\infty) \to \mathcal{P}$ denotes a piecewise constant switching signal that selects appropriate quadruples (A_p, B_p, C_p, D_p) from a parametrized family

$$\{(A_p, B_p, C_p, D_p) |\ p \in \mathcal{P}\} \tag{1b}$$

of n-dimensional, m-input, k-output state space realizations, where $\mathcal{P}$ denotes an index set, and, throughout the paper, the set of matrices in (1b) is assumed to be compact.

The set of all piecewise constant switching signals is denoted by $\mathcal{S} = \{s : [0,\infty) \to \mathcal{P}\}$ and, by a piecewise constant signal, we mean a signal that exhibits a finite number of discontinuities on every bounded time interval and that is constant between consecutive discontinuities. By convention, each piecewise constant signal s is assumed to be continuous from above[1].

A function $x : [0,\infty) \to \mathbb{R}^n$ is said a solution to (1) if it is continuous and piecewise continuously differentiable and there exists a switching signal $s \in \mathcal{S}$ and an input signal[2] $u \in \mathcal{L}_2^m$ such that the time-varying differential equation $\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t)$ holds almost everywhere on $t \geq 0$.

B. Stability and Induced Gains

Our main interest is to determine the induced gain of the switched systems (1), but to proceed we need to introduce appropriate stability definitions.

Definition 1: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *uniformly asymptotically stable over* $\mathcal{S}'$ if there exists a function β of class $\mathcal{KL}$ such that[3] $\|x(t)\| \leq \beta(\|x(0)\|, t)$ for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $x(0) \in \mathbb{R}^n$, where $x(t)$ denotes the solution to (1) obtained for the switching signal s, zero input signal u, and initial condition $x(0)$.

Definition 2: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *(uniformly) exponentially stable over* $\mathcal{S}'$ if there exist constants $a > 0$ and $\lambda > 0$ such that the function β in Definition 1 can be chosen of the form $\beta(r,t) = ae^{-\lambda t}r$ for all $r \geq 0$ and all $t \geq 0$.

We now state the following definitions of induced gains.

Definition 3: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}'$* if

$$\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau \tag{2}$$

for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $u \in \mathcal{L}_2^m$, where $y(\tau)$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition.

Definition 4: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$* if the system has an $\mathcal{L}_2$-induced gain smaller than or equal to some $\gamma' < \gamma$ uniformly over $\mathcal{S}'$.

[1]For any $t \geq 0$, the limit from above of $s(\tau)$ as $\tau \downarrow t$ is equal to $s(t)$.

[2]We denote by $\mathcal{L}_2^m$ the set of square integrable functions with values on $\mathbb{R}^m$ defined on $[0,\infty)$.

[3]A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$, and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing and $\alpha(0) = 0$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$, and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Because of causality of the system dynamics [15], having an induced gain smaller than or equal to a given constant $\gamma > 0$ uniformly over $\mathcal{S}' \subset \mathcal{S}$ is equivalent to

$$\sup_{s \in \mathcal{S}'} \sup_{u \in \mathcal{L}_2^m} \lim_{t \to \infty} \int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \leq 0 \tag{3}$$

where $y(\tau)$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition. For a given $\mathcal{S}' \subset \mathcal{S}$, we denote by $\gamma_{\mathcal{S}'}$ the *exact* $\mathcal{L}_2$-induced gain of a switched system (1):

$$\begin{aligned}\gamma_{\mathcal{S}'} &= \inf\{\gamma > 0 |\ \text{(2) holds for all } t \geq 0, \text{ all } s \in \mathcal{S}' \\ &\qquad\qquad \text{and all } u \in \mathcal{L}_2^m\} \\ &= \inf\{\gamma > 0 |\ \text{(3) holds}\}\end{aligned}$$

Our goal is to provide non-conservative conditions on the realizations in (1b) that guarantee the system has an $\mathcal{L}_2$-induced gain smaller than (or equal to) a given constant $\gamma > 0$.

C. Switching Signal Classes

All classes of switching signals considered here are subsets of the class $\mathcal{S}$ of every piecewise constant switching signal. We state a few important regularity properties of switching signal classes. For a given subset $\mathcal{S}'$ of $\mathcal{S}$ and for constants $b > a \geq 0$, possibly with $b = \infty$, we denote by $\mathcal{S}'[a,b)$ the set of restrictions of all switching signals $s \in \mathcal{S}'$ to the interval $[a,b)$. Similarly, $\mathcal{L}_2^m[a,b)$ denotes a restriction of $\mathcal{L}_2^m$.

Let $\mathcal{S}' \subset \mathcal{S}$ and $t > 0$. Using the set of restricted switching signal signals $\mathcal{S}'[t,\infty)$, we can define a new set of switching signals defined on the whole time interval by ${}^t\mathcal{S}'[t,\infty) = \{\sigma |\ \sigma(\rho) = \mu(\rho + t),\ \mu \in \mathcal{S}'[t,\infty),\ \rho \geq 0\}$. The switching signal class $\mathcal{S}'$ is said to be *shift-invariant* if $\mathcal{S}' = {}^t\mathcal{S}'[t,\infty)$ for all $t > 0$.

Let $\mathcal{S}' \subset \mathcal{S}$ and $t > 0$. Suppose that $\sigma \in \mathcal{S}'[0,t)$ and $\mu \in \mathcal{S}'[t,\infty)$. We define a switching signal $\sigma \oplus \mu$ by $(\sigma \oplus \mu)(\tau) = \sigma(\tau)$ for $\tau \in [0,\tau)$ and $(\sigma \oplus \mu)(\tau) = \mu(\tau)$ for $t \in [\tau,\infty)$, and also define $\mathcal{S}'[0,t) \bigoplus \mathcal{S}'[t,\infty) = \{\sigma \oplus \mu |\ \sigma \in \mathcal{S}'[0,t),\ \mu \in \mathcal{S}'[t,\infty)\}$. The switching signal class $\mathcal{S}'$ is said to be *closed under concatenation* if $\mathcal{S}' = \mathcal{S}'[0,t) \bigoplus \mathcal{S}'[t,\infty)$ for all $t > 0$.

We now state the following definition.

Definition 5: Let $\mathcal{S}' \subset \mathcal{S}$. The switching signal class $\mathcal{S}'$ said to be *regular* if

1) For every $p \in \mathcal{P}$, the constant switching signal $s^p(t) = p(= \text{constant})$ belongs to $\mathcal{S}'$.
2) The switching signal class $\mathcal{S}'$ is shift-invariant.
3) The switching signal class $\mathcal{S}'$ is closed under concatenation.

The class $\mathcal{S}$ of every piecewise constant switching signal is regular, and examples of other regular and non-regular switching signal classes can be found in [14]. Our main results will be provided for regular switching signal classes.

D. Preliminary Results

Necessary and sufficient conditions for a switched system to be (uniformly) exponentially stable and admit an induced gain strictly smaller than a given constant $\gamma > 0$ uniformly over a regular switching signal class $\mathcal{S}' \subset \mathcal{S}$ were provided in [13], which can be summarized as in the following proposition:

Proposition 1: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that there exists a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. The following statements are equivalent:

1) The system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is strictly convex, zero at zero, and homogeneous of degree two, which satisfies
$$\begin{aligned}&\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x \\ &+ \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right)^{\mathrm{T}} \left(\gamma^2 I - D_p^{\mathrm{T}} D_p\right)^{-1} \\ &\quad \times \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right) \leq -\epsilon \|x\|^2 \qquad (4)\end{aligned}$$
for all $p \in \mathcal{P}$ and almost everywhere in x.
3) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$ and a compact set $K \subset \mathbb{R}^{n \times n}$ of positive definite matrices such that the piecewise quadratic function $V(x) = \max_{Q \in K} x^{\mathrm{T}} Q x$ satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x.

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree q if $f(kx) = k^q f(x)$ for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{R}$. Each function V in the statements 2) and 3) is differentiable almost everywhere in x, but not necessarily everywhere. Therefore, the quantification over x in (4) should be interpreted as for every $x \in \mathbb{R}^n$ except for the zero-measure set of points at which $x \mapsto V(x)$ is not differentiable.

Each statement 2) and 3) requires an existence of a *common* solution V that satisfies the inequalities in (4), one for every p in the index set $\mathcal{P}$. The function V can be regarded as a *common storage function* for all the systems being switched. It is probably not surprising that the existence of a common storage function suffices to guarantee an induced gain smaller than γ. It is perhaps more unexpected that this is actually a necessary condition.

The statements 2) and 3) do not depend on the specific class of switching signals $\mathcal{S}'$. We then conclude that every regular switching signal class exhibits the same induced gain. Since the class $\mathcal{S}$ of every piecewise constant switching signal is regular, the above theorem essentially shows that for every regular switching signal class $\mathcal{S}' \subset \mathcal{S}$, we have $\gamma_{\mathcal{S}} = \gamma_{\mathcal{S}'}$.

Since $s^q(\tau) = q(=$ constant$)$ is a possible switching signal [Definition 5, 1)], (uniform) exponential stability in the statement 1) implies that A_q is Hurwitz. We also notice that strict convexity, $V(0) = 0$ and homogeneity with degree two imply that the function V is positive definite, $V(x) > 0$, $x \neq 0$[4].

III. Main Results

This section summarizes main results of this paper. The following Theorem 1 shows that a common storage function that establishes both (uniform) exponential stability and a non-conservative induced gain estimate for any regular switching signal class, in fact, admits a *finite parametrization*.

Theorem 1: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that there exists a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. The following statements are equivalent:

1) The system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$, a constant integer M satisfying $M \geq n$ and a set of constant vectors $\ell_i \in \mathbb{R}^n$, $i = 1 \ldots, M$ such that the matrix $L \equiv [\, \ell_1 \ \cdots \ \ell_M \,]^{\mathrm{T}} \in \mathbb{R}^{M \times n}$ has full rank, $\mathrm{rank} L = n$, and the piecewise quadratic function $V(x) = \max_{i=1,\ldots,M} (\ell_i^{\mathrm{T}} x)^2$ satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x.

The finite parametrization property of a common storage function allows us to derive finite dimensional algebraic conditions for an induced gain estimate. We can state the following corollary.

Corollary 1: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that there exists a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. The following statements 1) and 2) are equivalent, and imply the statement 3):

1) There exists a constant integer M satisfying $M \geq n$, a set of constant vectors ℓ_i, $i = 1, \ldots, M$ such that the matrix $L \equiv [\, \ell_1 \ \cdots \ \ell_M \,]^{\mathrm{T}} \in \mathbb{R}^{M \times n}$ has full rank, $\mathrm{rank} L = n$, and a set of real numbers α_p^{ij}, $p \in \mathcal{P}$, $i, j = 1, \ldots, M$ which satisfy
$$\begin{bmatrix} \begin{pmatrix} \ell_i \ell_i^{\mathrm{T}} A_p + A_p^{\mathrm{T}} \ell_i \ell_i^{\mathrm{T}} \\ + \left((\alpha_p^{i1})^2 + \cdots + (\alpha_p^{iM})^2\right) \ell_i^{\mathrm{T}} \ell_i \\ -(\alpha_p^{i1} \ell_1)^{\mathrm{T}} \alpha_p^{i1} \ell_1 - \cdots \\ -(\alpha_p^{iM} \ell_M)^{\mathrm{T}} \alpha_p^{iM} \ell_M \end{pmatrix} & \ell_i \ell_i^{\mathrm{T}} B_p \\ B_p^{\mathrm{T}} \ell_i \ell_i^{\mathrm{T}} & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_p^{\mathrm{T}} \\ D_p^{\mathrm{T}} \end{bmatrix} [\, C_p \quad D_p \,] < 0$$
for all $p \in \mathcal{P}$.
2) There exists a constant integer M satisfying $M \geq n$, a set of positive semi-definite matrices with rank one P_i, $i = 1, \ldots, M$ satisfying $P_1 + \cdots + P_M > 0$ and a set of non-negative numbers β_p^{ij}, $p \in \mathcal{P}$, $i, j = 1, \ldots, M$

[4] Let $\theta \in (0, 1)$ and $x \neq 0$. We have $\theta^2 V(x) = V(\theta x) = V(\theta x + (1 - \theta) 0) < \theta V(x) + (1 - \theta) V(0) = \theta V(x)$, thus $\theta(1 - \theta) V(x) > 0$. Since $\theta \in (0, 1)$, this implies that $V(x) > 0$.

which satisfy

$$\begin{bmatrix} \begin{pmatrix} P_iA_p + A_p^{\mathrm{T}}P_i \\ +(\beta_p^{i1}+\cdots+\beta_p^{iM})P_i \\ -\beta_p^{i1}P_1-\cdots-\beta_p^{iM}P_M \end{pmatrix} & P_iB_p \\ B_p^{\mathrm{T}}P_i & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_p^{\mathrm{T}} \\ D_p^{\mathrm{T}} \end{bmatrix} [\, C_p \quad D_p \,] < 0$$

for all $p \in \mathcal{P}$.

3) The system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.

The above corollary is derived by utilizing a so called S-procedure [16]. In general, it provides sufficient conditions for the statement 3), but the three statements in the corollary are equivalent if $M = 1$ or 2.

IV. Proofs of Main Results

This section proves the main results stated above. Due to space limitations, the proofs of Lemmas 1 and 2 are omitted, but they can be found in [14]

A. Proof of Theorem 1

1) Theorem 1, 2) ⇒ 1): It can be proved that the function V is convex and homogeneous of degree two. Thus, there exit constants α_1, α_2 which satisfy

$$\alpha_1\|x\|^2 \le V(x) \le \alpha_2\|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n \tag{5}$$

where both of α_1 and α_2 are positive, $\alpha_2, \alpha_1 > 0$, since the matrix L has full rank. Hence, the function V has quadratic shrink/growth rate bounds. Combining (4), we can prove exponential stability of the system (1). The remaining of the proof is identical to the proof of 2) ⇒ 1) of Proposition 1 in [14], showing that the system (1) has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.

2) Theorem 1, 1) ⇒ 2): The following two lemmas will be useful.

Lemma 1: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}'$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}}D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies

$$a^{\mathrm{T}}A_px + x^{\mathrm{T}}C_p^{\mathrm{T}}C_px + \left(\frac{1}{2}B_p^{\mathrm{T}}a + D_p^{\mathrm{T}}C_px\right)^{\mathrm{T}} \times \left(\gamma^2 I - D_p^{\mathrm{T}}D_p\right)^{-1}\left(\frac{1}{2}B_p^{\mathrm{T}}a + D_p^{\mathrm{T}}C_px\right) \le -\epsilon\|x\|^2 \tag{6}$$

for all $p \in \mathcal{P}$, all $a \in \partial V(x)$ and all $x \in \mathbb{R}^n$, where $\partial V(x)$ denotes the sub-differential of the convex function V at the point x.

Remark 1: Proposition 1 requires that a storage function should satisfy (4) almost everywhere in x. Lemma 1 basically says that the storage function actually satisfies the corresponding Hamilton-Jacobi inequality everywhere on $\mathbb{R}^n$, if one allows to consider sub-differentials for convex functions.

Lemma 2: Let $\mathcal{S}' \subset \mathcal{S}$ be regular, $\gamma > 0$, $0 < k < 1$ and $r > 0$. Suppose that there exists a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair, and the system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$, and therefore $\gamma^2 I - D_p^{\mathrm{T}}D_p > 0$ holds for each $p \in \mathcal{P}$ and there exit a constant $\epsilon > 0$ and a strictly convex function V for which the statement 2) (or 3)) of Proposition 1 holds. Let $\Pi(r) = \{x \in \mathcal{R}^n |\ V(x) \le r\}$ and pick any ξ from the boundary of $\Pi(r)$, i.e., $\xi \in \mathrm{bd}\Pi(r)$, and set $a_\xi \in \partial V(\xi)$. Then, for each $\xi \in \mathrm{bd}\Pi(r)$, there exists a constant $0 < \theta_\xi < 1$ such that

$$a_\xi^{\mathrm{T}}A_px + x^{\mathrm{T}}C_p^{\mathrm{T}}C_px + \left(\frac{1}{2}B_p^{\mathrm{T}}a_\xi + D_p^{\mathrm{T}}C_px\right)^{\mathrm{T}} \times \left(\gamma^2 I - D_p^{\mathrm{T}}D_p\right)^{-1}\left(\frac{1}{2}B_p^{\mathrm{T}}a_\xi + D_p^{\mathrm{T}}C_px\right) \le -k\epsilon\|\xi\|^2$$

holds for all $p \in \mathcal{P}$ and all $x \in R_\xi^{\ge}(\theta_\xi) \cap \Pi(r)$, where $R_\xi^{\ge}(\theta) = \{x \in \mathbb{R}^n |\ a_\xi^{\mathrm{T}}x \ge \theta a_\xi^{\mathrm{T}}\xi\}$ for some $0 \le \theta \le 1$.

Remark 2: Let ξ be a point in $\mathrm{bd}\Pi(r)$, and we have a gradient of the function V at this point. Let x be a point at which we would like to evaluate the inequality in (4). Lemma 2 says that, even we have a small 'miss-much' between ξ and x, the inequality (4) still holds, if the system admits an induced gain strictly smaller than γ, but it involves a price of a reduced decay rate specified by a parameter k.

From the statement 3) of Proposition 1, we have a positive constant ϵ and a compact set $K \subset \mathbb{R}^{n\times n}$ of positive definite matrices such that the piecewise quadratic function $V(x) = \max_{Q\in K} x^{\mathrm{T}}Qx$ satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x. We also have constants $\alpha_1, \alpha_2 > 0$ which satisfy (5), and this concludes that, for any given $r > 0$, the set $\Pi(r) = \{x \in \mathcal{R}^n |\ V(x) \le r\}$ is compact. We next construct a polyhedral approximation $\mathrm{approx}(\Pi(r))$ of the set $\Pi(r)$.

Let $0 < k < 1$ be any given constant and pick any $\xi \in \mathrm{bd}\Pi(r)$. Suppose that θ_ξ is a constant which satisfies the conditions in Lemma 2. Since $\theta_\xi < 1$, the set $R_\xi^{\ge}(\theta_\xi)$ contains ξ in its interior, i.e., $\xi \in \mathrm{int}R_\xi^{\ge}(\theta_\xi) \equiv R_\xi^{>}(\theta_\xi) = \{x \in \mathbb{R}^n |\ a_\xi^{\mathrm{T}}x > \theta_\xi a_\xi\xi\}$. Hence, the family of open sets $R_\xi^{>}(\xi)$, $\xi \in \mathrm{bd}\Pi(r)$ defines an open covering of the set $\mathrm{bd}\Pi(r)$, i.e., $\mathrm{bd}\Pi(r) \subset \bigcup_{\xi\in\mathrm{bd}\Pi(r)} R_\xi^{>}(\theta_\xi)$.

Since the set $\mathrm{bd}\Pi(r)$ is compact, we can pick finitely many points $\xi_i \in \mathrm{bd}\Pi(r)$, $i = 1, \ldots, 2M$ which also define an open covering of the set $\mathrm{bd}\Pi(r)$. In addition, since $V(-x) = V(x)$ for all $x \in \mathbb{R}^n$, we can assume that $\xi_{M+i} = -\xi_i$, $i = 1, \ldots, M$ without loss of generality. Thus

$$\mathrm{bd}\Pi(r) \subset \bigcup_{i=1,\ldots,2M} R_{\xi_i}^{>}(\theta_{\xi_i}) \tag{7}$$

defines a centrally symmetrical finite open covering of the set $\mathrm{bd}\Pi(r)$. For notational simplicity, let us denote $a_i \equiv a_{\xi_i}$ and $\theta_i \equiv \theta_{\xi_i}$ for each $i = 1, \ldots, 2M$. We note that our

assumption $\xi_{M+1} = -\xi_i$ implies $a_{M+i} = -a_i$ and $\theta_{M+i} = \theta_i$ for all $i = 1, \ldots, M$.

Let us define

$$\mathrm{approx}(\Pi(r)) = \bigcap_{i=1,\ldots,2M} R^{\leq}_{\xi_i}(\theta_i)$$

where $R^{\leq}_{\xi_i}(\theta_i) = \{x \in \mathbb{R}^n |\ a_i^{\mathrm{T}} x \leq \theta_i a_i^{\mathrm{T}} \xi_i\}$. Because of the definition, $\theta_i > 0$ and $a_i^{\mathrm{T}} \xi_i > 0$, the set $\mathrm{approx}(\Pi(r))$ is closed, convex and contains the origin in its interior. We next see that $\mathrm{approx}(\Pi(r)) \subset \Pi(r)$, and hence it is compact.

We assume that there exits a point x such that $x \in \mathrm{approx}(\Pi(r))$ and $x \notin \Pi(r)$. Let us set $y = \sqrt{r/V(x)}x$. Since $x \notin \Pi(r)$ implies $V(x) > r$ and $\sqrt{r/V(x)} < 1$, the point y is localed on the line segment connecting x and the origin. Convexity of the set $\mathrm{approx}(\Pi(r))$ concludes that $y \in \mathrm{approx}(\Pi(r))$, and we have $a_i^{\mathrm{T}} y \leq \theta_i a_i^{\mathrm{T}} \xi_i$ for all $i = 1, \ldots, 2M$. On the other hand, because of $V(y) = (\sqrt{r/V(x)})^2 V(x) = r$, we have $y \in \mathrm{bd}\Pi(r)$. Since the family $R^{>}_{\xi_i}(\theta_i)$, $i = 1, \ldots, 2M$ defines the open covering of the set $\mathrm{bd}\Pi(r)$ in (7), we have $a_j^{\mathrm{T}} y > \theta_j a_j^{\mathrm{T}} \xi_j$ for some $j = 1, \ldots, 2M$, which leads a contradiction, and this proves compactness of $\mathrm{approx}(\Pi(r))$. We notice here that compactness of $\mathrm{approx}(\Pi(r))$ concludes that the matrix $[\, a_1 \ \cdots \ a_M \,]^{\mathrm{T}} \in \mathbb{R}^{M \times n}$ has full rank.

Let us set $\ell_i = a_i/(2\sqrt{\theta_i r})$ for each $i = 1, \ldots, 2M$. We notice that the vectors satisfy $\ell_{M+i} = -\ell_i$, since $a_{M+i} = -a_i$ for all $i = 1, \ldots, M$, and the matrix L has full rank, since the rank of the matrix $[\, a_1 \ \cdots \ a_M \,]^{\mathrm{T}}$ is equal to n. Let us define the piecewise quadratic function $v(x) = \max_{i=1,\ldots,M} (\ell_i^{\mathrm{T}} x)^2$. We see next that there exists a constant $\epsilon' > 0$ such that the function v satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x.

To this effect, we first need to confirm that the function v has quadratic shrink and growth rate bounds. It can be proved that the function v is convex, homogeneous of degree two. Thus, there exist constants β_1, β_2 such that

$$\beta_1 \|x\|^2 \leq v(x) \leq \beta_2 \|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n \tag{8}$$

where both of β_2 and β_1 are positive, $\beta_2, \beta_1 > 0$, since the matrix L has full rank.

We next consider a polyhedral partition of $\mathbb{R}^n$. Let us define a set of cones in $\mathbb{R}^n$ such as

$$K_j = \{x \in \mathbb{R}^n |\ \ell_j^{\mathrm{T}} x \geq \ell_i^{\mathrm{T}} x \quad \text{for all } i = 1, \ldots, 2M\} \tag{9}$$

for $j = 1, \ldots, 2M$. The union of the sets K_j, $j = 1, \ldots, 2M$ covers whole $\mathbb{R}^n$, and the zero measure set

$$\Omega = \{x \in \mathbb{R}^n |\ \ell_j^{\mathrm{T}} x = \ell_i^{\mathrm{T}} x \quad \text{for some } j \neq i \text{ where } i, j = 1, \ldots, 2M\} \tag{10}$$

includes all non-differentiable points of the function v. Thus, we see that (4) holds for all $p \in \mathcal{P}$ and all $x \in \mathbb{R}^n \setminus \Omega = \bigcup_{j=1,\ldots 2M} \mathrm{int} K_j$.

We now in position to evaluate the inequality (4) for the function v, and we start our investigation for the points in $\mathrm{bd}(\mathrm{approx}(\Pi(r)))$. Let us pick any $j = 1, \ldots, 2M$ and consider any point x in $\mathrm{bd}(\mathrm{approx}(\Pi(r))) \cap \mathrm{int} K_j$, where $x \in \mathrm{int} K_j$ implies the function $x \mapsto v(x)$ is differentiable. We have that

$$v(x) = \max_{i=1,\ldots,M} (\ell_i^{\mathrm{T}} x)^2 = (\ell_j x)^2 = \frac{1}{4\theta_j r} (a_j^{\mathrm{T}} x)^2 \tag{11a}$$

$$\frac{\partial v}{\partial x}(x) = 2x^{\mathrm{T}} \ell_j \ell_j^{\mathrm{T}} = \frac{1}{2\theta_j r} x^{\mathrm{T}} a_j a_j^{\mathrm{T}} \tag{11b}$$

Because of $x \in \mathrm{bd}(\mathrm{approx}(\Pi(r)))$, we also have

$$a_j^{\mathrm{T}} x = \theta_j a_j^{\mathrm{T}} \xi_j \tag{12}$$

We now suppose that the piecewise quadratic function V is differentiable at the point ξ_j, i.e., $a_j = (\partial V/\partial x)(\xi_j)$, then we have

$$a_j^{\mathrm{T}} = 2\xi_j^{\mathrm{T}} Q_j \tag{13}$$

for a unique matrix $Q_j \in K$ which satisfies $V(\xi_j) = \max_{Q \in K} \xi_j^{\mathrm{T}} Q \xi_j = \xi_j^{\mathrm{T}} Q_j \xi_j = r$. By substituting (13) into the right hand side of (12), we conclude

$$a_j^{\mathrm{T}} x = 2\theta_j \xi_j^{\mathrm{T}} Q_j \xi_j = 2\theta_j r \tag{14}$$

On the other hand, if the function $\xi_j \mapsto V(\xi_j)$ is not differentiable, then a_j is an element of $\partial V(\xi_j) = \mathrm{cl}(\mathrm{conv} S(\xi_j))$, where $\mathrm{conv} S(\xi)$ denotes the convex hull of the set $S(\xi)$ of all limits of the sequences of $a_{x^j} = (\partial V/\partial x)^{\mathrm{T}}(x^j)$, $j = 1, 2, \ldots$ such that the function V is differentiable at each x^j, and x^j tends to ξ (see, e.g., [17]). Because of Carathéodory's Theorem, we can have $a_i \in S(\xi_j)$ and $\lambda_i \geq 0$, $i = 1, \ldots, n+1$ which satisfy $a_j = \lambda_1 a_i + \cdots + \lambda_{n+1} a_{n+1}$ and $\lambda_1 + \cdots + \lambda_{n+1} = 1$. Let x_i^k, $i = 1, \ldots, n+1$, $k = 1, 2, \ldots$ be the sequences such that the function V is differentiable at each x_i^k, in addition x_i^k and $a_i^k = (\partial V/\partial x)^{\mathrm{T}}(x_i^k)$ tend to ξ_j and a_i as k goes to infinity, respectively. Since the function V differentiable at each x_i^k, we have $(a_i^k)^{\mathrm{T}} = (\partial V/\partial x)(x_i^k) = 2x_i^k Q_{x_i^k}$ where $Q_{x_i^k} \in K$ is a unique matrix satisfying $V(x_i^k) = \max_{Q \in K} (x_i^k)^{\mathrm{T}} Q x_i^k = (x_i^k)^{\mathrm{T}} Q_{x_i^k} x_i^k$. By making a convex combination of a_i's with coefficients λ_i, we have

$$\begin{aligned} a_j &= \lambda_1 a_1 + \cdots + \lambda_{n+1} a_{n+1} \\ &= \lim_{k \to \infty} (\lambda_1 a_1^k + \cdots + \lambda_{n+1} a_{n+1}^k) \\ &= 2 \lim_{k \to \infty} (\lambda_1 Q_{x_1^k} x_1^k + \cdots + \lambda_{n+1} Q_{x_{n+1}^k} x_{n+1}^k) \end{aligned} \tag{15}$$

By substituting (15) into the right hand side of (12), we obtain

$$\begin{aligned} a_j^{\mathrm{T}} x = 2\theta_j \lim_{k \to \infty} (&\lambda_1 (x_1^k)^{\mathrm{T}} Q_{x_1^k} + \cdots \\ &+ \lambda_{n+1} (x_{n+1}^k)^{\mathrm{T}} Q_{x_{n+1}^k}) \xi_j = 2\theta_j r \end{aligned} \tag{16}$$

where we use the fact that all x_i^k's tend to ξ_j as k goes to infinity, and $V(\xi_j) = r$.

By substituting (14) or (16) into (11), we conclude that

$$v(x) = \theta_j r \quad \text{and} \quad \frac{\partial v}{\partial x}(x) = a_j^{\mathrm{T}} \tag{17}$$

for each $j = 1, \ldots, 2M$ and each $x \in \mathrm{bd}(\mathrm{approx}(\Pi(r))) \cap \mathrm{int} K_j$.

From (17), we have

$$\frac{\partial v}{\partial x}(x)A_p x + x^{\mathrm{T}}C_p^{\mathrm{T}}C_p x + \left(\frac{1}{2}B_p^{\mathrm{T}}\frac{\partial v^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}}C_p x\right)^{\mathrm{T}} \times \left(\gamma^2 I - D_p^{\mathrm{T}}D_p\right)^{-1}\left(\frac{1}{2}B_p^{\mathrm{T}}\frac{\partial v^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}}C_p x\right)$$
$$= a_j^{\mathrm{T}}A_p x + x^{\mathrm{T}}C_p^{\mathrm{T}}C_p x + \left(\frac{1}{2}B_p^{\mathrm{T}}a_j + D_p^{\mathrm{T}}C_p x\right)^{\mathrm{T}} \times \left(\gamma^2 I - D_p^{\mathrm{T}}D_p\right)^{-1}\left(\frac{1}{2}B_p^{\mathrm{T}}a_j + D_p^{\mathrm{T}}C_p x\right)$$

For a given constant $0 < k < 1$, applying Lemma 2 to the function V, the fact $x \in R^{\geq}_{\xi_j}(\theta_j) \cap \Pi(r)$ concludes that the quantity in the above equation is smaller than or equals to $-\epsilon k\|\xi_j\|^2$ for all $p \in \mathcal{P}$.

Since j is arbitrary, by using (5), (8) and (17), we further have

$$-k\epsilon\|\xi_j\|^2 \leq -k\epsilon\frac{r}{\alpha_1} = -\frac{k\epsilon}{\alpha_1}\frac{1}{\theta_j}\theta_j r = -\frac{k\epsilon}{\alpha_1\theta_j}v(x) \leq -\frac{k\epsilon}{\alpha_1\theta_j}\beta_2\|x\|^2 \leq -\epsilon'\|x\|^2$$

for all $p \in \mathcal{P}$ and all $x \in \bigcup_{j=1,\ldots,2M}[\mathrm{bd}(\mathrm{approx}(\Pi(r))) \cap \mathrm{int}K_j] = \mathrm{bd}(\mathrm{approx}(\Pi(r))) \setminus \Omega$, where we set $\epsilon' \equiv (k\epsilon\beta_2)/(\alpha_1\theta) > 0$ and $0 < \theta \equiv \max_{i=1,\ldots,M}\theta_i = \max_{i=1,\ldots,2M}\theta_i < 1$.

We now consider any point y in $\mathbb{R}^n \setminus \Omega$. For each y, we have unique $j = 1,\ldots,2M$, $x \in \mathrm{bd}(\mathrm{approx}(\Pi(r)))\setminus\Omega$, and $t > 0$ such that $y = tx$. Because of the definition of v, we have $v(y) = t^2 v(x)$ and $(\partial v/\partial x)(y) = t(\partial v/\partial x)(x)$. These facts lead to

$$\frac{\partial v}{\partial x}(y)A_p y + y^{\mathrm{T}}C_p^{\mathrm{T}}C_p y + \left(\frac{1}{2}B_p^{\mathrm{T}}\frac{\partial v^{\mathrm{T}}}{\partial x}(y) + D_p^{\mathrm{T}}C_p y\right)^{\mathrm{T}} \times \left(\gamma^2 I - D_p^{\mathrm{T}}D_p\right)^{-1}\left(\frac{1}{2}B_p^{\mathrm{T}}\frac{\partial v^{\mathrm{T}}}{\partial x}(y) + D_p^{\mathrm{T}}C_p y\right)$$
$$\leq -t^2\epsilon'\|x\|^2 = -\epsilon'\|y\|^2$$

for all $p \in \mathcal{P}$. Since y is arbitrary, this inequality holds almost everywhere on $\mathbb{R}^n$.

B. Proof of Corollary 1

We see that the condition in the statement 1) or 2) implies the statement 2) in Theorem 1, hence it concludes the corollary.

Let us consider the partition of $\mathbb{R}^n$ in (9). Since $\ell_{M+i} = -\ell_i$ for each $i = 1,\ldots,M$, we have

$$K_j = \{x \in \mathbb{R}^n |\ \ell_j^{\mathrm{T}}x \geq \ell_i^{\mathrm{T}}x \text{ and } \ell_j^{\mathrm{T}}x \geq -\ell_i^{\mathrm{T}}x \quad \text{for all } i = 1,\ldots,M\}$$
$$= \{x \in \mathbb{R}^n |\ \ell_j^{\mathrm{T}}x \leq |\ell_i^{\mathrm{T}}x| \quad \text{for all } i = 1,\ldots,M\}$$

for $j = 1,\ldots,2M$. We also have

$$K_j \cup K_{M+j} = K_j \cup (-K_j) = \{x \in \mathbb{R}^n |\ (\ell_j^{\mathrm{T}}x)^2 \geq (\ell_i^{\mathrm{T}}x)^2 \quad \text{for all } i = 1,\ldots,M\}$$

where we use the fact that $K_{M+j} = -K_j$ and $K_j \cap (-K_j) = \{0\}$ for each $i = 1,\ldots,M$. The union of the sets K_j, $j = 1,\ldots,2M$ covers whole $\mathbb{R}^n$, and the measure zero set (10) includes all non-differentiable points of the function $V(x) = \max_{i=i,\ldots,M}(\ell_i^{\mathrm{T}}x)^2$ in Theorem 1.

We notice that the condition in the statement 2) in Theorem 1 is confirmed if the inequality (4) holds for all $p \in \mathcal{P}$, and for all $x \in \mathbb{R}^n \setminus \Omega$ or equivalently for all $x \in \mathrm{int}K_j$ and all $j = 1,\ldots,2M$. In addition, if the inequality (4) holds for a point $x \in K_j$, then it is automatically holds for $-x \in K_{M+j}$. Therefore, we have an equivalent condition to the statement 2) in Theorem 1 as

$$x^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}}A_p x + x^{\mathrm{T}}A_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}}x + x^{\mathrm{T}}C_p^{\mathrm{T}}C_p x + x^{\mathrm{T}}(B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p)^{\mathrm{T}}(\gamma^2 I - D_p^{\mathrm{T}}D_p)^{-1} \times (B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p)x^{\mathrm{T}} \leq -\epsilon\|x\|^2$$
$$\text{subject to} \quad (\ell_j^{\mathrm{T}}x)^2 > (\ell_i^{\mathrm{T}}x)^2 \qquad \text{for all } i = 1,\ldots,M$$

holds for all $p \in \mathcal{P}$ and all $j = 1,\ldots M$. Where the vectors ℓ_i, $i = 1,\ldots,M$ should satisfy $\mathrm{rank}[\,\ell_1\ \cdots\ \ell_M\,]^{\mathrm{T}} = n$.

Let us pick any $p \in \mathcal{P}$ and $j = 1,\ldots,M$, and consider real numbers α_p^{ji}, $i = 1,\ldots,M$. We have a sufficient condition to the above constrained feasibility problem as

$$x^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}}A_p x + x^{\mathrm{T}}A_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}}x + x^{\mathrm{T}}C_p^{\mathrm{T}}C_p x + x^{\mathrm{T}}(B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p)^{\mathrm{T}}(\gamma^2 I - D_p^{\mathrm{T}}D_p)^{-1} \times (B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p)x$$
$$\leq -\epsilon\|x\|^2 - (\alpha_p^{j1})^2 x^{\mathrm{T}}(\ell_j^{\mathrm{T}}\ell_j - \ell_1^{\mathrm{T}}\ell_1)x - \cdots - (\alpha_p^{jM})^2 x^{\mathrm{T}}(\ell_j^{\mathrm{T}}\ell_j - \ell_M^{\mathrm{T}}\ell_M)x$$

for all $x \in \mathbb{R}^n$. Since $\epsilon > 0$, it is equivalent to the following strict matrix inequality condition

$$\ell_j\ell_j^{\mathrm{T}}A_p + A_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + \left((\alpha_p^{j1})^2 + \cdots + (\alpha_p^{jM})^2\right)\ell_j^{\mathrm{T}}\ell_j - (\alpha_p^{j1}\ell_1)^{\mathrm{T}}\alpha_p^{j1}\ell_1 - \cdots - (\alpha_p^{jM}\ell_M)^{\mathrm{T}}\alpha_p^{jM}\ell_M + C_p^{\mathrm{T}}C_p + (B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p)^{\mathrm{T}}(\gamma^2 I - D_p^{\mathrm{T}}D_p)^{-1} \times (B_p^{\mathrm{T}}\ell_j\ell_j^{\mathrm{T}} + D_p^{\mathrm{T}}C_p) < 0$$

The positive definiteness of the matrix $\gamma^2 I - D_p^{\mathrm{T}}D_p$ concludes that the statement 1) implies the statement 2) in Theorem 1.

We set rank one matrices $P_i = \ell_i^{\mathrm{T}}\ell_i \geq 0$ and non-negative numbers $\beta_p^{ji} = (\alpha_p^{ji})^2 \geq 0$ for $i = 1,\ldots,M$, respectively. Then, this and the rank constraint, $\mathrm{rank}(P_1 + \cdots + P_M) = n$, conclude that the statement 2) implies 3).

V. Conclusions

This paper provided non-conservative necessary and sufficient conditions to compute the induced gain of a switched system in terms of a common solution to a system of Hamilton-Jacobi inequalities. The main contribution of this paper is to prove that a common storage function that establishes both uniform asymptotic stability and a non-conservative induced gain estimate over any regular switching signal class admits a finite parametrization. This fact also allows us to derive finite dimensional algebraic criteria for an induced gain estimate. Future work is needed to propose a specific numerical procedure to solve (perhaps approximately) the Hamilton-Jacobi inequalities.

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