Communication Logic Design and Analysis for Networked Control Systems

Yonggang Xu and João P. Hespanha *

Dept. of Electrical and Computer Engineering
University of California,
Santa Barbara, CA 93106 USA
{yonggang,hespanha}@ece.ucsb.edu

Summary. This chapter addresses the control of spatially distributed processes via communication networks with a fixed delay. A distributed architecture is utilized in which multiple local controllers coordinate their efforts through a data network that allows information exchange. We focus our work on linear time invariant processes disturbed by Gaussian white noise and propose several logics to determine when the local controllers should communicate. Necessary conditions are given under which these logics guarantee boundedness and the trade-off is investigated between the amount of information exchanged and the performance achieved. The theoretical results are validated through Monte Carlo simulations. The resulting closed-loop systems evolve according to stochastic differential equations with resets triggered by stochastic counters. This type of stochastic hybrid system is interesting on its own.

1 Introduction

The architectures for feedback control of spatially distributed processes generally fall in one of the three classes, *centralized*, *decentralized* or *distributed*. Centralized architectures yield the best performance because they pose the least constraints on the structure of the controller, whereas decentralized architectures are the simplest to implement. We pursue here distributed architectures, as they provide a range of compromise solutions between the two extremes. The communication among local controllers is supported by a data network that allows information between local controllers to be exchanged at discrete time instants.

Our objective is to understand the trade-off between the amount of information exchanged and the performance achieved. Several results can be

*This research was supported by the National Science Foundation under the grant numbers CCR-0311084 and ECS-0242798.

found in the literature on how to reduce communication in networked control systems (NCS). The problem of stabilization with finite communication bandwidth is introduced by [21, 22], and further pursued by [14, 18, 11, 6, 12]. An estimation problem is investigated in [21] under the constraint that observations must be coded digitally and transmitted over a channel with finite capacity. The corresponding stabilization problem under similar limitations is addressed in [22]. [18, 14] and [6] determine the minimum bandwidth (measured in discrete symbols per second) needed to stabilize a linear process. In all these references a digital communication channel is assumed so that any information transmitted has to be quantized. [3] focuses on the quantization aspect and shows that the memoryless quantization scheme that minimizes the product of quantization density times sampling rate follows a logarithmic rule.

We depart from the work summarized above in that we only penalize the number of times that information is exchanged. This is motivated by the fact that in most widely used communication protocols there is a fixed overhead incurred by sending a packet over the network, which is not reduced by decreasing the number of data-bits. For example, a fixed-size ATM (Asynchronous Transfer Mode) cell consists of a 5-byte header and a 48-byte information field, whereas an Ethernet frame has a 14-byte or 22-byte header and a data field that must be at least 46 bytes long. In either case, one "pays" the same price for sending a single bit or 48/46 bytes of data.

Several practical issues motivate us to reduce packet rate in NCS. Higher data traffic may induce longer communication delay and more data dropouts, which are undesirable in real-time systems [10]. In sensor network applications, an important criterion in assessing communication protocols is energy efficiency and the primary source of energy consumption in the non-mobile wireless settings is the radio [1]. A smart communication scheduling method can extend the battery lifetime and therefore reduce the sensor network deployment costs.

The systems of interest are spatially distributed processes whose dynamics are decoupled but for which the control objectives are not, e.g., the control of a group of autonomous aircraft flying in a geometric formation far enough from each other so that their dynamics are decoupled. However, many of these ideas could be extended to coupled processes.

Each process with an associated local controller is viewed as a *node*. The overall control system consists of certain number of nodes connected via a communication network. Fig 1 depicts the internal structure of the ith node. It consists of a *local process*, a *local controller*, a *bank of estimators* and a *communication logic*. The *synchronized estimators* are used by the local controller to replace the state of remote processes that are not available locally. They are simply computational models of the remote processes and the reason to call them "synchronized" will become clear shortly. These estimators

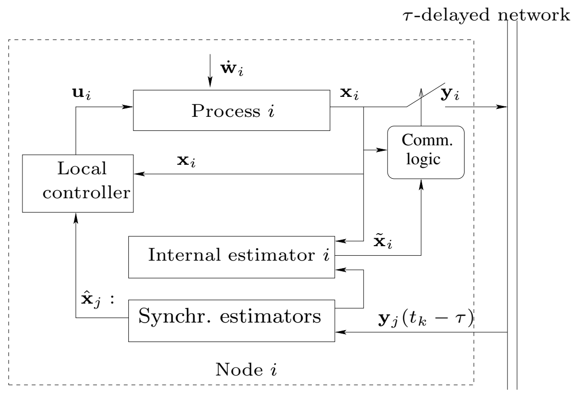

Fig. 1. One of the nodes in a networked control system.

run open-loop most of the time but are sometimes reset to "correct" values when state measurements are received through the network. These resets do not necessarily occur periodically and it is the responsibility of each node to decide when to broadcast to the network the state of its local process. The communication logic makes use of an *internal estimator* to determine how well other nodes can "predict" the state of its local process and decides when to broadcast it. In general, the communication network introduces delay and therefore this data only becomes available to the other nodes some time later. This type of architecture is proposed by Yook, Tilbury and Soparkar [24] for the control of discrete-time distributed systems over delay-free networks.

Several algorithms can be used by the communication logic to determine *when* the state of the local process should be broadcast. The quality of a communication logic should be judged in terms of the performance it can achieve for a given message broadcast rate. We measure performance in terms of the statistical moments of the estimation errors associated with the synchronized estimators, which provide a measure of the penalty introduce by the fact that the state measurements of the remote processes are not available locally.

One simple algorithm that can be used by the communication logic consists of broadcasting messages *periodically.* However, as we shall see, this is not optimal because data may be transmitted with little new information. In [24], it is proposed that a node should broadcast the true value of the state of its local process when it differs from the estimate known to the remaining nodes by more than a given threshold. For the linear discrete-time case, they showed that this scheme results in a system that is BIBO stable. The relation between the threshold level and the message exchange rate is investigated through simulation in the context of examples.

We proposed new communication logics that can be analyzed to determine stability as well as the trade-off between communication (in terms of average message exchange rates) and performance. We start by considering *stochastic communication logics* for which the probability of a node broadcasting a message is a function of the current estimation error. *Deterministic communication logics* similar to the ones proposed in [24] are also considered. We will see that the latter can be viewed as limiting cases of the former. We also simulate different communication logics, including periodic, stochastic and deterministic, and compare their performances.

The stochastic communication logics are based on doubly stochastic Poisson processes (DSPPs) [2]. In essence, the state of the local process is broadcast according to a Poisson process whose rate depends on the estimation error. This type of stochastic hybrid system seems to be interesting on its own. For stochastic communication logics, our stability analysis uses tools from jump diffusion processes. Deterministic logics are analyzed in context of first exit time problems.

In Section 2, the control-communication architecture is formally described for the case of two linear time-invariant processes. Stochastic communication logics are analyzed in Section 3, first for delay-free networks and later for networks that introduce a delay of τ time units. Deterministic logics are addressed in Section 4 for delay-free systems. Simulation results are presented in Section 5 for a second order leader-follower problem. We also provide trade-off curves showing the average communication rate versus the variance of the estimation error for an unstable process. Section 6 contains conclusions and directions for future work.

2 Networked control system model

In this section we propose an estimator based architecture for distributed control. For simplicity of presentation we consider only two nodes like the ones in Fig 1.

2.1 An estimator based control architecture

The processes are assumed linear time-invariant with an exogenous disturbance input,

$$\begin{aligned} \dot{\mathbf{x}}_i &= A_i \mathbf{x}_i + B_i \mathbf{u}_i + \sigma_i \dot{\mathbf{w}}_i \\ \mathbf{y}_i &= \mathbf{x}_i + \boldsymbol{\zeta}_i \qquad\qquad \forall i \in \{1,2\}, \end{aligned}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ denotes the state, $\mathbf{u}_i \in \mathbb{R}^{m_i}$ the control input, $\mathbf{y}_i \in \mathbb{R}^{n_i}$ the state measurement, $\dot{\mathbf{w}}_i$ ℓ_i-dimensional standard Gaussian white noise, and $\boldsymbol{\zeta}_i$ zero-mean measurement noise and/or quantization errors. The two noise processes are assumed independent and all matrices are real and of appropriate dimensions. The process $\boldsymbol{\zeta}_i$ is also assumed stationary with known probability distribution $\mu(\cdot)$.

We assume given state-feedback control laws

$$\mathbf{u}_i = K_{i1}\mathbf{x}_1 + K_{i2}\mathbf{x}_2, \qquad \forall i \in \{1,2\} \tag{1}$$

that would provide adequate performance in a centralized configuration, i.e., if the states of both processes were available to both local controllers. In a centralized configuration, the closed-loop system would be

$$\begin{aligned} \dot{\mathbf{x}}_1 &= (A_1 + B_1K_{11})\mathbf{x}_1 + B_1K_{12}\mathbf{x}_2 + \sigma_1\dot{\mathbf{w}}_1 \\ \dot{\mathbf{x}}_2 &= (A_2 + B_2K_{22})\mathbf{x}_2 + B_2K_{21}\mathbf{x}_1 + \sigma_2\dot{\mathbf{w}}_2. \end{aligned} \tag{2}$$

Since the state of the ith process is not directly available at the jth node ($j \neq i$, $i, j \in \{1,2\}$), we build at the node j an estimate $\hat{\mathbf{x}}_i$ of the state $\mathbf{x}_i$. In the distributed architecture, the centralized laws (1) are replaced by

$$\begin{aligned} \mathbf{u}_1 &= K_{11}\mathbf{x}_1 + K_{12}\hat{\mathbf{x}}_2 \\ \mathbf{u}_2 &= K_{21}\hat{\mathbf{x}}_1 + K_{22}\mathbf{x}_2. \end{aligned} \tag{3}$$

The distributed control laws (3) result in a closed-loop dynamics given by

$$\begin{aligned} \dot{\mathbf{x}}_1 &= (A_1 + B_1K_{11})\mathbf{x}_1 + B_1K_{12}\hat{\mathbf{x}}_2 + \sigma_1\dot{\mathbf{w}}_1 \\ \dot{\mathbf{x}}_2 &= (A_2 + B_2K_{22})\mathbf{x}_2 + B_2K_{21}\hat{\mathbf{x}}_1 + \sigma_2\dot{\mathbf{w}}_2, \end{aligned} \tag{4}$$

to be contrasted with (2). To understand the effect of the distributed architecture on the performance of the closed-loop system, we write the closed-loop dynamics (4) in terms of the estimation errors,

$$\begin{aligned} \dot{\mathbf{x}}_1 &= (A_1 + B_1K_{11})\mathbf{x}_1 + B_1K_{12}\mathbf{x}_2 + \sigma_1\dot{\mathbf{w}}_1 + B_1K_{12}\mathbf{e}_2, \\ \dot{\mathbf{x}}_2 &= (A_2 + B_2K_{22})\mathbf{x}_2 + B_2K_{21}\mathbf{x}_1 + \sigma_2\dot{\mathbf{w}}_2 + B_2K_{21}\mathbf{e}_1, \end{aligned}$$

where the estimation error is defined as $\mathbf{e}_i := \hat{\mathbf{x}}_i - \mathbf{x}_i$. Comparing these equations with (2), we observe that the penalty paid for a distributed architecture is expressed by the additive "disturbance" terms $B_iK_{ij}\mathbf{e}_j$. The estimation error term $\mathbf{e}_i$ is the focus of our investigation.

2.2 Estimators

Since remote state information is not directly available, each node needs to construct *synchronized state-estimates* to be used in (3), based on the data received via the network, which is assumed to introduce a delay of τ time units. Moreover, each node also needs to send its own state to the remote nodes to allow them to construct their synchronized estimates. Each node's *communication logic* is responsible for determining when data transmission should take place and makes this decision based on an *internal estimate* of its own state. The difference between this internal state-estimate and the measured state provides a criterion to judge the quality of the synchronized estimates currently being used at the remote nodes.

To simplify the presentation, we only write the equations for the synchronized state-estimators inside node 2 and the internal estimator used by the communication logic inside node 1. These estimators are relevant to investigate the impact of the rate at which measurements are sent from node 1 to node 2. The flow of data in the opposite direction is determined by completely symmetric structures.

Node 2's synchronized state-estimates at some time t are based on all information received from node 1 up to time t:

$$\{\mathbf{y}_1(\mathbf{t}_j) : \mathbf{t}_j \le t - \tau\} \tag{5}$$

where $0 =: \mathbf{t}_0 < \mathbf{t}_1 < \mathbf{t}_2 < \cdots$ are the times at which node 1 sends its state measurement $\mathbf{y}_1(t_k)$ to node 2. The corresponding minimum-variance estimate is given by a Kalman filter for (4) with discrete measurements [13]. For simplicity we shall assume that the measurement noise $\boldsymbol{\zeta}_i(t)$ is negligible, in which case the filter takes a particularly simple form because one does not need to propagate the covariance matrix and the optimal estimator is given by the following open-loop "computational model"

$$\dot{\hat{\mathbf{x}}}_1 = (A_1 + B_1 K_{11})\hat{\mathbf{x}}_1 + B_1 K_{12}\hat{\mathbf{x}}_2, \tag{6}$$

which, upon receiving $\mathbf{y}_1(\mathbf{t}_k)$ at time $\mathbf{t}_k + \tau$, is updated according to

$$\begin{aligned}\hat{\mathbf{x}}_1(\mathbf{t}_k + \tau) = \mathbf{z}_1(\mathbf{t}_k + \tau) &:= \exp\{(A_1 + B_1 K_{11})\tau\}\mathbf{y}_1(\mathbf{t}_k) + \\ &+ \int_{\mathbf{t}_k}^{\mathbf{t}_k+\tau} \exp\{(A_1 + B_1 K_{11})(\mathbf{t}_k + \tau - r)\} B_1 K_{12}\hat{\mathbf{x}}_2(r)dr. \end{aligned}\tag{7}$$

To implement (6), node 2 also needs to compute $\hat{\mathbf{x}}_2$ based on the information that it has been sending to node 1. This is done using equations completely symmetric to (6)–(7):

$$\dot{\hat{\mathbf{x}}}_2 = (A_2 + B_2 K_{22})\hat{\mathbf{x}}_2 + B_2 K_{21}\hat{\mathbf{x}}_1 \tag{8}$$

$$\begin{aligned}\hat{\mathbf{x}}_2(\bar{\mathbf{t}}_k + \tau) = \mathbf{z}_2(\bar{\mathbf{t}}_k + \tau) &:= \exp\{(A_2 + B_2 K_{22})\tau\}\mathbf{y}_2(\bar{\mathbf{t}}_k) + \\ &+ \int_{\bar{\mathbf{t}}_k}^{\bar{\mathbf{t}}_k+\tau} \exp\{(A_2 + B_2 K_{22})(\bar{\mathbf{t}}_k + \tau - r)\} B_2 K_{21}\hat{\mathbf{x}}_1(r)dr, \end{aligned}\tag{9}$$

where each $\bar{\mathbf{t}}_k$ denotes a time at which node 2 sends the measurement $\mathbf{y}_2(\bar{\mathbf{t}}_k)$ to node 1, and $\bar{\mathbf{t}}_k + \tau$ the time at which this measurement is expected to arrive at its destination. Although node 2 has $\mathbf{x}_2$ always available, in building this estimate, it only does the discrete updates of $\hat{\mathbf{x}}_2$ τ time units after it sends the state measurement $\mathbf{y}_2$ to node 1, because only at that time will node 1 be able to update its estimate. By construction, the estimators inside node 2 defined by (6)–(9) will always remain equal to the corresponding estimators inside node 1. For this reason, we call them *synchronized state-estimators*.

2.3 Estimation error processes

The estimator equations (6)–(7) can be formally written as the following jump diffusion process

$$d\hat{\mathbf{x}}_1(t) = A\hat{\mathbf{x}}_1(t)dt + B_1K_{12}\hat{\mathbf{x}}_2(t)dt + \big(\mathbf{z}_1(t) - \hat{\mathbf{x}}_1(t)\big)d\mathbf{N}_1(t-\tau) \qquad (10)$$

where $A := A_1 + B_1K_{11}$ and $\mathbf{N}_1(t)$ is an integer counting process that is constant almost everywhere except at the times $\mathbf{t}_k$, $k \geq 0$ when it is increased by 1. Moreover, at any time t when the measurement $\mathbf{y}_1(t-\tau) = \mathbf{x}_1(t-\tau) + \boldsymbol{\zeta}_1(t-\tau)$ is received, we have that

$$\begin{aligned}\mathbf{z}_1(t) - \hat{\mathbf{x}}_1(t) &= \exp\{A\tau\}\boldsymbol{\zeta}_1(t-\tau) + \exp\{A\tau\}\mathbf{x}_1(t-\tau) \\ &\qquad + \int_{t-\tau}^{t} \exp\{A(t-r)\}B_1K_{12}\hat{\mathbf{x}}_2(r)dr - \mathbf{x}_1(t) - \mathbf{e}_1(t) \\ &= \boldsymbol{\eta}_1(t) - \mathbf{e}_1(t), \end{aligned} \qquad (11)$$

where

$$\boldsymbol{\eta}_1(t) = \exp\{A\tau\}\boldsymbol{\zeta}_1(t-\tau) - \int_{t-\tau}^{t} \exp\{A(t-r)\}\sigma_1 d\mathbf{w}_1(r), \qquad (12)$$

with the integral defined in the Itô sense [16, 9]. It is straightforward to show that the stochastic moments of $\boldsymbol{\eta}_1(t)$ are finite for any delay of τ.

¿From (4), (10) and (11) we conclude that the estimation error $\mathbf{e}_1$ satisfies

$$d\mathbf{e}_1(t) = A\mathbf{e}_1(t)dt - \sigma_1 d\mathbf{w}_1(t) + \big(\boldsymbol{\eta}_1(t) - \mathbf{e}_1(t)\big)d\mathbf{N}_1(t-\tau). \qquad (13)$$

Periodic communication is not optimal to reduce network utilization because a node does not need to send its measured state to the network if the other nodes have a good estimate of it. An optimal communication logic problem is solved in [23] for discrete-time systems, in which it is shown that the optimal communication decision for node 1 is a function of the estimation error associated with an additional estimate of its local state $\mathbf{x}_1$ that should be updated in a delay-free fashion right after data is transmitted [i.e., without waiting τ time units in the discrete update (7)], even though the network may exhibit significant delay. This *internal estimate* $\tilde{\mathbf{x}}_1$ is constructed inside node 1 as follows

$$d\tilde{\mathbf{x}}_1(t) = A\tilde{\mathbf{x}}_1(t)dt + B_1K_{12}\hat{\mathbf{x}}_2(t)dt + \big(\mathbf{y}_1(t) - \tilde{\mathbf{x}}_1(t)\big)d\mathbf{N}_1(t) \qquad (14)$$

in which $\mathbf{N}_1(t)$ is determined by node 1's communication logic. Inspired by the results in [23], we consider communication logics that base their decision on the internal estimation error $\tilde{\mathbf{e}}_i := \tilde{\mathbf{x}}_i - \mathbf{x}_i$, whose dynamics are given by

$$d\tilde{\mathbf{e}}_1(t) = A\tilde{\mathbf{e}}_1(t)dt - \sigma_1 d\mathbf{w}_1(t) + \big(\boldsymbol{\zeta}_1 - \tilde{\mathbf{e}}_1(t)\big)d\mathbf{N}_1(t) \tag{15}$$

For simplicity of notation, in the sequel we drop the subscript $_1$ in all the signals and rewrite (15) and (13) as follows

$$d\tilde{\mathbf{e}}(t) = A\tilde{\mathbf{e}}(t)dt - \sigma d\mathbf{w}(t) + \big(\boldsymbol{\zeta} - \tilde{\mathbf{e}}(t)\big)d\mathbf{N}(t) \tag{16}$$

$$d\mathbf{e}(t) = A\mathbf{e}(t)dt - \sigma d\mathbf{w}(t) + \big(\boldsymbol{\eta}(t) - \mathbf{e}(t)\big)d\mathbf{N}(t-\tau) \tag{17}$$

in which the integer process $\mathbf{N}(t)$ is determined by the communication logic based on $\tilde{\mathbf{e}}(t)$. For networks with negligible delay, (16) and (17) are identical and we can simply write

$$d\mathbf{e}(t) = A\mathbf{e}(t)dt - \sigma d\mathbf{w}(t) + (\boldsymbol{\zeta} - \mathbf{e}(t))d\mathbf{N}(t). \tag{18}$$

The reader is reminded that the equations above have analogous counterparts for all nodes in the network.

2.4 Communication measure

The "communication cost" of a particular communication logic is measured in terms of the *communication rate*, defined to be the asymptotic rate that messages are sent, i.e.,

$$R := \lim_{k\to\infty} \mathbf{E}\Big[\frac{k}{\mathbf{t}_k}\Big].$$

Define $\mathbf{T}_k := \mathbf{t}_k - \mathbf{t}_{k-1}$ to be the *intercommunication time* between the $(k-1)$th and the kth messages. If all the $\mathbf{T}_k$ are *i.i.d.*, it is straightforward to show that

$$R = \lim_{k\to\infty} \mathbf{E}\Big[\frac{k}{\sum_{i=1}^{k}\mathbf{T}_k}\Big] = \frac{1}{\mathbf{E}[\mathbf{T}_k]}. \tag{19}$$

For several communication logics we proceed to investigate the relation between performance, measured in terms of the statistical moments of the estimation error $\mathbf{e}$, and communication cost, measured in terms of the communication rate R.

3 Stochastic communication logics

The idea behind stochastic communication logics is for each node to broadcast at an average rate that depends on the current value of the internal estimation error $\tilde{\mathbf{e}}$, as in (16). To this effect, we define $\mathbf{N}(t)$ to be a DSPP, whose increments are associated with message exchanges. The instantaneous rate at which increments occur is a function of the estimation error $\tilde{\mathbf{e}}(t)$. In particular we take $\mathbf{N}(t)$ to be a DSPP with *intensity* $\lambda\big(\tilde{\mathbf{e}}(t)\big)$, which has the property that

$$\mathbf{E}\big[\mathbf{N}(t)-\mathbf{N}(s)\big]=\mathbf{E}\Big[\int_s^t \lambda(\tilde{\mathbf{e}}(r))dr\Big], \qquad \forall t>s\geq 0,$$

where $\lambda:\mathbb{R}^n\to[0,\infty)$ is called the *intensity function*. For this type of communication logic, the communication rate R is given by

$$R=\lim_{t\to\infty}\frac{\mathbf{E}[\mathbf{N}(t)-\mathbf{N}(0)]}{t}=\lim_{t\to\infty}\frac{\int_0^t \mathbf{E}\big[\lambda(\tilde{\mathbf{e}}(r))\big]dr}{t}, \tag{20}$$

which shows that when $\mathbf{E}[\lambda(\tilde{\mathbf{e}}(t))]$ converges as $t\to\infty$, its limit is precisely the communication rate R.

We start by considering the delay-free case (18) and provide sufficient conditions for stochastic stability for both constant and state-dependent intensity functions. These results are later generalized for the case of a τ time units delay network expressed by (16)–(17).

3.1 Infinitesimal generators

For the stability analysis of (18), it is convenient to consider its *infinitesimal generator*: Given a twice continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}$, the generator $\mathcal{L}$ of a jump diffusion process $\mathbf{e}$ is defined by

$$(\mathcal{L}V)(e):=\lim_{t\to s}\frac{\mathbf{E}\big[V(\mathbf{e}(t)|\mathbf{e}(s)=e\big]-V(e)}{t-s}, \qquad \forall e\in\mathbb{R}^n,\ t>s\geq 0, \tag{21}$$

where $\mathbf{E}\big[V(\mathbf{e}(t)|\mathbf{e}(s)=e\big]$ denotes the expectation of $V(\mathbf{e}(t))$ given $\mathbf{e}(s)=e$ [15, 5]. The generator for the jump diffusion process described by (18) is given by

$$\mathcal{L}V(e)=\frac{\partial V(e)}{\partial e}\cdot Ae+\frac{1}{2}\mathrm{tr}\Big[\sigma'\frac{\partial^2 V(e)}{\partial e^2}\sigma\Big] + \lambda(e)\Big(\int V(\zeta)\,d\mu(\zeta)-V(e)\Big), \tag{22}$$

where $\frac{\partial V(e)}{\partial e}$ and $\frac{\partial^2 V(e)}{\partial e^2}$ denote the gradient vector and Hessian matrix of V, respectively [9]. Setting $e=\mathbf{e}(t)$ in (21) and taking expectation, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e}(t))]=\mathbf{E}[(\mathcal{L}V)(\mathbf{e}(t))], \tag{23}$$

from which the stochastic stability properties of the process $\mathbf{e}(t)$ can be deduced by appropriate choices of V.

3.2 Constant intensity

Consider a constant intensity function $\lambda(e) = \gamma$ for the DSPP. From (20), the corresponding communication rate is $R = \gamma$. The following statements hold.

Theorem 1. *Let* $\mathbf{e}$ *be the jump diffusion process defined by* (18) *with* $\lambda(e) = \gamma$, $\forall e$.

1. *If* $\gamma > \Re\{\lambda_i(A)\}$, *for every eigenvalue* $\lambda_i(A)$ *of* A, *then* $\mathbf{E}[\mathbf{e}(t)]$ *converges to zero exponentially fast.*
2. *If* $\gamma > 2\,m\,\Re\{\lambda_i(A)\}$, *for every eigenvalue* $\lambda_i(A)$ *of* A *and some* $m \geq 1$, *then* $\mathbf{E}[(\mathbf{e}(t)\cdot\mathbf{e}(t))^m]$ *is bounded.*
3. *If* $\gamma > 2\,\Re\{\lambda_i(A)\}$, *for every eigenvalue* $\lambda_i(A)$ *of* A, *and* P, Q *are* $n\times n$ *positive definite matrices and* c *a positive constant such that*

$$P\Big(A - \frac{\gamma}{2}I\Big) + \Big(A - \frac{\gamma}{2}I\Big)'P \leq -Q, \qquad\qquad Q \geq cP,$$

then $\mathbf{E}[\mathbf{e}(t)\cdot\mathbf{e}(t)]$ *is uniformly bounded and*

$$\lim_{t\to\infty} \mathbf{E}[\mathbf{e}(t)\cdot P\mathbf{e}(t)] \leq \frac{\gamma\rho^2 + \theta}{c}, \tag{24}$$

where $\rho^2 := \int \zeta\cdot P\zeta\, d\mu(\zeta)$, *and* $\theta := \mathrm{tr}(\sigma' P \sigma)$.

To prove this theorem, we need the following Lemma, which relates the expectations of different moments of a positive random variable.

Lemma 1. *Given a scalar random variable* $\mathbf{x}$ *that is nonnegative with probability one, a positive constant* δ, *and positive integers* $k > \ell > 0$, $\mathbf{E}[\mathbf{x}^k] \geq \delta^\ell \mathbf{E}[\mathbf{x}^{k-\ell}] - \delta^k$.

Proof (Lemma 1). Suppose $\mathbf{x}$ has distribution $\mu(\mathbf{x})$. For every $\delta > 0$, the following inequalities hold

$$\begin{aligned}\mathbf{E}[\mathbf{x}^k] &\geq \int_{x\geq\delta} x^k\, d\mu(x) \geq \delta^\ell \int_{x\geq\delta} x^{k-\ell}\, d\mu(x)\\ &= \delta^\ell\Big(\int_{x\geq 0} x^{k-\ell}\, d\mu(x) - \int_{x<\delta} x^{k-\ell}\, d\mu(x)\Big) \geq \delta^\ell(\mathbf{E}[\mathbf{x}^{k-\ell}] - \delta^{k-\ell}). \qquad \square\end{aligned}$$

Proof (Theorem 1). To prove 1, put $V(e) = e$. From (22),

$$\mathcal{L}V(e) = Ae - \gamma e$$

because $\int V(\zeta) d\mu(\zeta) = 0$. Therefore (23) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[\mathbf{e}(t)] = (A - \gamma I)\mathbf{E}[\mathbf{e}(t)],$$

and statement 1 follows since $A-\gamma I$ is stable. To prove 2, let P and Q be $n\times n$ positive definite matrices and c a positive constant such that

$$P\Big(A-\frac{\gamma}{2m}I\Big)+\Big(A-\frac{\gamma}{2m}I\Big)'P\leq -Q, \qquad\qquad Q\geq cP.$$

Such matrices exist as long as $\gamma>2\,m\max\{\Re[eig(A)]\}$. For $m\geq 1$, define

$$V(e):=(e\cdot Pe)^m. \tag{25}$$

We conclude from (22) that

$$\begin{aligned}\mathcal{L}V(e)&=m(e\cdot Pe)^{m-1}\,e\cdot(PA+A'P)e+\lambda(e)\rho^{2m}-\lambda(e)V(e)\\&\qquad+2m(m-1)(e\cdot Pe)^{m-2}\,e\cdot P\sigma\sigma' Pe+m(e\cdot Pe)^{m-1}\theta\\&=m(e\cdot Pe)^{m-1}\,e\cdot[P(A-\frac{\gamma}{2m}I)+(A-\frac{\gamma}{2m}I)'P]e+\gamma\,\rho^{2m}\\&\qquad+2m(m-1)(e\cdot Pe)^{m-2}\,e\cdot P\sigma\sigma' Pe+m(e\cdot Pe)^{m-1}\theta\\&\leq -m(e\cdot Pe)^{m-1}\,e\cdot Qe+\gamma\,\rho^{2m}\\&\qquad+2m(m-1)(e\cdot Pe)^{m-2}\,e\cdot P\sigma\sigma' Pe+m(e\cdot Pe)^{m-1}\theta\\&\leq -cmV(e)+\gamma\,\rho^{2m}+m\big(2c_2(m-1)+\theta\big)(e\cdot Pe)^{m-1},\end{aligned}$$

where $\rho^{2m}:=\int(\zeta\cdot P\zeta)^m d\mu(\zeta)$ and $c_2>0$ is such that $P\sigma\sigma' P\leq c_2P$. From this and (23), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e})]\leq -cm\mathbf{E}[V(\mathbf{e})]+\gamma\,\rho^{2m}+m\big(2c_2(m-1)+\theta\big)\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m-1}].$$

Given some $\delta>0$, from Lemma 1,

$$\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m-1}]\leq\frac{1}{\delta}\mathbf{E}[V(\mathbf{e})]+\delta^{m-1},$$

and therefore

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e})] \qquad (26)\\&\leq -cm\mathbf{E}[V(\mathbf{e})]+\gamma\rho^{2m}+m\big(2c_2(m-1)+\theta\big)\Big(\frac{1}{\delta}\mathbf{E}[V(\mathbf{e})]+\delta^{m-1}\Big)\\&=-m\Big(c-\frac{2c_2(m-1)+\theta}{\delta}\Big)\mathbf{E}[V(\mathbf{e})]+\gamma\rho^{2m}+m\delta^{m-1}\big(2c_2(m-1)+\theta\big).\end{aligned}$$

For sufficiently large δ, $c-\frac{2c_2(m-1)+\theta}{\delta}>0$ and the boundedness of $\mathbf{E}[V(\mathbf{e})]$ and consequently that of $\mathbf{E}[(\mathbf{e}\cdot\mathbf{e})^m]$ follows.

To prove 3, we re-write (26) for $m=1$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e})]\leq -\Big(c-\frac{\theta}{\delta}\Big)\mathbf{E}[V(\mathbf{e})]+\gamma\rho^2+\theta.$$

Applying the Comparison Lemma [7], we conclude that

$$\lim_{t\to\infty} \mathbf{E}[V(\mathbf{e})] \le \frac{\gamma\rho^2+\theta}{c-\frac{\theta}{\delta}},$$

from which (24) follows as we make $\delta \to \infty$. □

3.3 Error-dependent intensity

We now consider an intensity for the DSPP that depends on the current estimation error. The rationale is that a larger estimation error should more rapidly lead to a message exchange. We consider intensities of the form

$$\lambda(e) = (e \cdot Pe)^k, \qquad \forall e \in \mathbb{R}^n, \tag{27}$$

where P is some positive definite matrix and k a positive integer.

Theorem 2. *Let* $\mathbf{e}$ *be the jump diffusion process defined by* (18) *with intensity* (27)*. For every* $k > 0$*, the communication rate and all finite moments of* $\mathbf{e}(t)$ *are bounded.*

Proof (Theorem 2). Choose c_1 sufficiently large so that $A - \frac{c_1}{2}I$ is asymptotically stable. Then there exists a matrix $P > 0$ such that

$$P\Big(A - \frac{c_1}{2}I\Big) + \big(A - \frac{c_1}{2}I\big)P < 0,$$

i.e., $PA + A'P < c_1 P$. Moreover $P\sigma\sigma' P \le c_2 P$ for sufficiently large $c_2 > 0$.

We start by proving that the mth moment of $\mathbf{e}(t)$ is bounded for $m > k$. Let V be as in (25) and $\rho^{2m} := \int (\zeta \cdot P\zeta)^m d\mu(\zeta)$. From, (22) we obtain

$$\begin{aligned}
\mathcal{L}V(e) &= m(e\cdot Pe)^{m-1}\, e\cdot(PA+A'P)e + \lambda(e)\rho^{2m} - \lambda(e)V(e)\\
&\quad + 2m(m-1)(e\cdot Pe)^{m-2}\, e\cdot P\sigma\sigma' Pe + m(e\cdot Pe)^{m-1}\theta\\
&= m(e\cdot Pe)^{m-1}\, e\cdot(PA+A'P)e + \rho^{2m}(e\cdot Pe)^k - (e\cdot Pe)^{m+k}\\
&\quad + 2m(m-1)(e\cdot Pe)^{m-2}\, e\cdot P\sigma\sigma' Pe + m(e\cdot Pe)^{m-1}\theta\\
&\le c_1 m(e\cdot Pe)^m + \rho^{2m}(e\cdot Pe)^k - (e\cdot Pe)^{m+k}\\
&\quad + m\big(2c_2(m-1)+\theta\big)(e\cdot Pe)^{m-1}.
\end{aligned}$$

From this and (23), we conclude that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e})] &\le c_1 m\mathbf{E}[V(\mathbf{e})] + \rho^{2m}\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^k] - \mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m+k}]\\
&\quad + m\big(2c_2(m-1)+\theta\big)\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m-1}].
\end{aligned}$$

Given some $\delta_1, \delta_2, \delta_3 > 0$, we conclude from Lemma 1 that

$$\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^k] \le \frac{\mathbf{E}[V(\mathbf{e})]}{\delta_1^{m-k}} + \delta_1^k,$$
$$\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m+k}] \ge \delta_2^k \mathbf{E}[V(\mathbf{e})] - \delta_2^{m+k},$$
$$\mathbf{E}[(\mathbf{e}\cdot P\mathbf{e})^{m-1}] \le \frac{1}{\delta_3}\mathbf{E}[V(\mathbf{e})] + \delta_3^{m-1},$$

therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[V(\mathbf{e})] &\le c_1 m\mathbf{E}[V(\mathbf{e})] + \rho^{2m}\Big(\frac{\mathbf{E}[V(\mathbf{e})]}{\delta_1^{m-k}} + \delta_1^k\Big) - (\delta_2^k\mathbf{E}[V(\mathbf{e})] - \delta_2^{m+k}) \\ &\quad + m\big(2c_2(m-1)+\theta\big)\Big(\frac{\mathbf{E}[V(\mathbf{e})]}{\delta_3} + \delta_3^{m-1}\Big) \\ &\le \big(c_1 m + \frac{\rho^{2m}}{\delta_1^{m-k}} - \delta_2^k + m\frac{2c_2(m-1)+\theta}{\delta_3}\big)\mathbf{E}[V(\mathbf{e})] \\ &\quad + \rho^{2m}\delta_1^k + \delta_2^{m+k} + m\big(2c_2(m-1)+\theta\big)\delta_3^{m-1}.\end{aligned}$$

For sufficiently large δ_2,

$$c_1 m + \frac{\rho^{2m}}{\delta_1^{m-k}} - \delta_2^k + \frac{m\big(2c_2(m-1)+\theta\big)}{\delta_3} < 0,$$

and the boundedness of $\mathbf{E}[V(\mathbf{e})]$ and consequently of $\mathbf{E}[(\mathbf{e}\cdot\mathbf{e})^m]$ follows.

To prove the boundedness of the mth moment of $\mathbf{e}(t)$ for $m \le k$, we use Lemma 1 to bound

$$\mathbf{E}[(\mathbf{e}\cdot\mathbf{e})^m] \le \frac{\mathbf{E}[(\mathbf{e}\cdot\mathbf{e})^{k+1}]}{\delta_4^{k+1-m}} + \delta_4^m \tag{28}$$

where $\delta_4 > 0$. Since the boundedness of the $(k+1)$th moment has already been established, we conclude that the mth moment is also bounded for $m \le k$. □

3.4 τ-delayed network

We now extend the stochastic stability results to networks with τ time units delay. For constant intensity functions, the estimation error process in (17) is driven by a constant intensity Poisson process $\mathbf{N}(t-\tau)$ and Theorem 1 still holds. It turns out that, for intensity functions like (27), a result analogous to Theorem 2 but for delayed networks can also be proved.

Theorem 3. *Let* $\tilde{\mathbf{e}}$ *and* $\mathbf{e}$ *be the jump diffusion processes defined by* (16) *and* (17) *with the intensity of the DSPP* $\mathbf{N}(t)$ *given by* $\lambda\big(\tilde{\mathbf{e}}(t)\big)$*, with*

$$\lambda(\tilde{e}) = (\tilde{e}\cdot P\tilde{e})^k, \qquad \forall \tilde{e}\in\mathbb{R}^n,$$

for some $k > 0$*. The communication rate and all the finite moments of both* $\tilde{\mathbf{e}}(t)$ *and* $\mathbf{e}(t)$ *are bounded.*

For a given time $t \geq 0$, let $\mathbf{s}(t)$ be the time at which the communication logic sends the last measurement before or at t, i.e.,

$$\mathbf{s}(t) := \max\big\{r \leq t : d\mathbf{N}(r) > 0\big\}.$$

The random variable $\mathbf{s}(t)$ is a *stopping time* [15], which is independent of any event after time t. Since no data is sent during the time interval $(\mathbf{s}(t), t]$, the processes $\mathbf{N}$ and $\tilde{\mathbf{e}}$ in (16) have no jumps on this interval. Consequently, because of the network delay of τ, the remote node does not receive any data on $(\mathbf{s}(t)+\tau, t+\tau]$ and during this interval the process $\mathbf{e}$ in (17) has no jumps.

Proof (Theorem 3). Since the process $\tilde{\mathbf{e}}$ is not affected by the delay, the boundedness of all moments of this process as well as the communication rate follows directly from the proof of Theorem 2. It remains to prove boundedness of all moments of the estimation error $\mathbf{e}$ associated with the synchronized state-estimators. To this effect, we consider the function $V : \mathbb{R}^n \to \mathbb{R}$ defined in (25) and show that for an arbitrary time t, $\mathbf{E}\big[V\big(\mathbf{e}(t+\tau)\big)\big]$ is bounded. Note that once we prove boundedness of $\mathbf{E}[V(\mathbf{e})]$, the boundedness of $\mathbf{E}[(\mathbf{e}\cdot\mathbf{e})^m]$ follows. At time $\mathbf{s}(t)+\tau$, $\mathbf{e}$ is reset to $\boldsymbol{\eta}(\mathbf{s}(t)+\tau)$ defined by (12), and therefore

$$\mathbf{e}(\mathbf{s}(t)+\tau) = \exp\{A\tau\}\boldsymbol{\zeta} - \int_{\mathbf{s}(t)}^{\mathbf{s}(t)+\tau} \exp\{A(\mathbf{s}(t)+\tau-r)\}\sigma\, d\mathbf{w}(r). \tag{29}$$

On the other hand, since the process $\tilde{\mathbf{e}}$ is reset to $\boldsymbol{\zeta}$ at time $\mathbf{s}(t)$ and this process has no jumps on $(\mathbf{s}(t), t]$, we conclude from (16) that

$$\tilde{\mathbf{e}}(t) = \exp\{A(t-\mathbf{s}(t))\}\boldsymbol{\zeta} - \int_{\mathbf{s}(t)}^{t} \exp\{A(t-r)\}\sigma\, d\mathbf{w}(r),$$

which is equivalent to

$$\boldsymbol{\zeta} = \exp\{A(-t+\mathbf{s}(t))\}\tilde{\mathbf{e}}(t) + \int_{\mathbf{s}(t)}^{t} \exp\{A(\mathbf{s}(t)-r)\}\sigma\, d\mathbf{w}(r). \tag{30}$$

Using (30) to eliminate $\boldsymbol{\zeta}$ in (29), we conclude that

$$\begin{aligned}\mathbf{e}(\mathbf{s}(t)+\tau) = {} & \exp\{A(\tau+\mathbf{s}(t)-t)\}\tilde{\mathbf{e}}(t) \\ & - \int_{t}^{\mathbf{s}(t)+\tau} \exp\{A(\mathbf{s}(t)+\tau-r)\}\sigma\, d\mathbf{w}(r).\end{aligned} \tag{31}$$

Moreover, since the process $\mathbf{e}$ has no jumps on $(\mathbf{s}(t)+\tau, t+\tau]$, we conclude from (17) that

$$\mathbf{e}(t+\tau) = \exp\{A(t-\mathbf{s}(t))\}\mathbf{e}(\mathbf{s}(t)+\tau) - \int_{\mathbf{s}(t)+\tau}^{t+\tau} \exp\{A(t+\tau-r)\}\sigma\, d\mathbf{w}(r).$$

From this and (31), we obtain

$$\mathbf{e}(t+\tau) = \exp\{A\tau\}\tilde{\mathbf{e}}(t) - \int_t^{t+\tau} \exp\{A(t+\tau-r)\}\sigma\, d\mathbf{w}(r).$$

Since $V(a+b) \leq 2^{2m}V(a) + 2^{2m}V(b)$, $\forall a, b \in \mathbb{R}^n$, we conclude that

$$V(\mathbf{e}(t+\tau)) \leq 2^{2m} V\big(\exp\{A\tau\}\tilde{\mathbf{e}}(t)\big) + 2^{2m} V\Big(\int_t^{t+\tau} \exp\{A(t+\tau-r)\}\sigma\, d\mathbf{w}(r)\Big).$$

Since the process that appears in the integral is a Gaussian white noise, independent of $\mathbf{s}(t)$, we conclude that there exist finite constants c_5, c_6 such that

$$\mathbf{E}\big[V(\mathbf{e}(t+\tau))\big] \leq c_5 \mathbf{E}\big[V\big(\tilde{\mathbf{e}}(t)\big)\big] + c_6.$$

The boundedness of $\mathbf{E}\big[V\big(\mathbf{e}(t+\tau)\big)\big]$ then follows from the already established boundedness of $\mathbf{E}[V(\tilde{\mathbf{e}}(t))]$. □

4 Deterministic communication logics

We now consider communication logics that utilize deterministic rules but restrict our attention to delay-free networks, whose estimation error satisfies (18). The communication logic monitors a continuous positive and radially unbounded *communication index* $S : \mathbb{R}^n \to \mathbb{R}^+$ and force a node to broadcast its state when $S(\mathbf{e}) \geq 1$. In particular, a message broadcast occurs at time $\mathbf{t}_k$ when $\lim_{t \uparrow t_k} S\big(\mathbf{e}(t)\big) \geq 1$. To avoid chattering, the post-reset value $\boldsymbol{\zeta}(t_k)$ should satisfy $S\big(\boldsymbol{\zeta}(t_k)\big) < 1$ with probability one. This type of resetting guarantees that $\mathbf{e}(t)$ is bounded, since

$$\mathbf{e}(t) \in \mathcal{D} := \{e \in \mathbb{R}^n | S(e) \leq 1\}, \qquad \forall t \geq 0, \tag{32}$$

with probability one.

To determine the communication rate, suppose that a message exchange occurred at time $\mathbf{t}_{k-1}$ and $\mathbf{e}(\mathbf{t}_{k-1})$ was reset to $\boldsymbol{\zeta}(t_{k-1})$. From $\mathbf{t}_{k-1}$ to the next reset time $\mathbf{t}_k$, $\mathbf{e}(t)$ is a pure diffusion process

$$\dot{\mathbf{e}} = A\mathbf{e} - \sigma\dot{\mathbf{w}}. \tag{33}$$

Given $\boldsymbol{\zeta}(t_{k-1})$, define $\mathbf{T}_k(\boldsymbol{\zeta})$ to be the inter-communication time, i.e.,

$$\mathbf{T}_k(\boldsymbol{\zeta}) = \inf\{t - \mathbf{t}_{k-1} \geq 0 : \mathbf{e}(t) \in \partial\mathcal{D}, \mathbf{e}(\mathbf{t}_{k-1}) = \boldsymbol{\zeta}\},$$

where $\mathbf{e}(t)$ is governed by (33) for $t \geq \mathbf{t}_{k-1}$ and $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$. The random variable $\mathbf{T}_k(\boldsymbol{\zeta})$ is called the first exit time of $\mathbf{e}(t)$ from $\mathcal{D}$. It is in general not easy to obtain the distribution of $\mathbf{T}_k(\boldsymbol{\zeta})$ in closed form, but its expected value can be obtained from Dynkin's equation. In particular, defining $g(\zeta) := \mathbf{E}[\mathbf{T}_k(\zeta)]$, it is known that $g(\zeta)$ is a solution to the following boundary value problem:

$$\frac{\partial g(\zeta)}{\partial \zeta} \cdot A\zeta + \frac{1}{2}\mathrm{tr}\Big[\sigma' \frac{\partial^2 g(\zeta)}{\partial \zeta^2}\sigma\Big] = -1, \quad (34)$$
$$\forall \zeta \in \mathcal{D}, \quad g(\zeta) = 0, \quad \forall \zeta \in \partial\mathcal{D},$$

where $\frac{\partial g(\zeta)}{\partial \zeta}$ and $\frac{\partial^2 g(\zeta)}{\partial \zeta^2}$ denote the gradient vector and Hessian matrix of g respectively [17]. Once $g(\zeta)$ is known, the expected intercommunication time $\mathbf{T}_k$ can be obtained from

$$\mathbf{E}[\mathbf{T}_k] = \mathbf{E}[g(\zeta_{k-1})] = \int g(\zeta) d\mu(\zeta),$$

and the communication rate follows from (19)

$$R = \frac{1}{\int g(\zeta) d\mu(\zeta)}.$$

In practice, (34) needs to be solved numerically. Since $\mathcal{D}$ is compact, (32) provides an upper bound on $\mathbf{e}(t)$ and consequently on its statistical moments. To obtain tighter bounds one can use Kolmogorov's forward equation with appropriate boundary conditions to compute the probability density function of the error $\mathbf{e}(t)$. However, this method is computationally intensive for higher-order systems.

5 Simulation results

In this section we validate the theoretical results through Monte Carlo simulations. All the simulations are done in Matlab/Simulink. The DSPP $\mathbf{N}(t)$ is realized by a single binomial test. Specifically, for a fixed time step h, a message exchange is triggered at time $t := k\,h$, $k \in \mathbb{N}$, if a binomial test characterized by a probability of success $p = 1 - e^{-h\lambda(\mathbf{e}(t))}$ succeeds. Convergence results for similar procedures can be found in [4] and references therein.

5.1 Leader-follower

A leader-follower problem is used to illustrate the distributed control architecture with different communication logics. The two processes have identical dynamics and are disturbed by uncorrelated white Gaussian noise processes. The dynamics of the leading and following vehicles are given by

$$\text{leader:} \qquad \dot{\mathbf{x}}_1(t) = A\mathbf{x}_1(t) + Br(t) + \sigma\dot{\mathbf{w}}_1(t) \quad (35)$$
$$\text{follower:} \qquad \dot{\mathbf{x}}_2(t) = A\mathbf{x}_2(t) + Bu_2(t) + \sigma\dot{\mathbf{w}}_2(t), \quad (36)$$

where each state $\mathbf{x}_i$ contains the position and velocity of one of the vehicles, u_i are the controls, r is an external reference, each $\dot{\mathbf{w}}_i(t)$ is standard Gaussian white noise, and $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & -0.5 \end{smallmatrix}\right]$, $\sigma = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}'$. The follower's control objective is to follow the leader's position. The reference r is also known by the follower. The open-loop state estimator for the leader's state is given by

$$\dot{\hat{\mathbf{x}}}_1 = A\hat{\mathbf{x}}_1 + Br, \qquad \hat{\mathbf{x}}_1(\mathbf{t}_k) = \mathbf{y}_1(\mathbf{t}_k) := \mathbf{x}_1(\mathbf{t}_k) + \boldsymbol{\zeta}_k,$$

where the $\mathbf{t}_k$ denote the times at which the leader broadcasts its state $\mathbf{x}(\mathbf{t}_k)$ to the follower, and the $\boldsymbol{\zeta}_k$ are zero-mean uniformly distributed random vectors over an interval of about 5% of the maximum estimation error. The follower uses the following controller

$$u_2 = -K(\mathbf{x}_2 - \hat{\mathbf{x}}_1).$$

where $K = [32.6 \quad 8.07]$ is obtained from an LQR design.

Tab. 1 summarizes the communication rates and the variances of both the estimation and the tracking errors for three communication logics: periodic, DSPP with quadratic intensity $\lambda(e)$ and deterministic with a quadratic communication index $S(e) := e \cdot Pe$, where $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$. For fair comparison, the parameters are selected to achieve communication rates approximately equal to 0.2 for all logics. We see that both the deterministic logic outperforms both the DSPP logic and the periodic communication.

Table 1. Communication rate versus variance of the estimation and tracking errors

Logics	Parameters	Comm. rate	Est. err. var.	Trck. err. var.
Determ.	$S(e) \leq 0.070$	0.19	0.011	0.017
DSPP	$\lambda(e) := 0.5\frac{e \cdot Pe}{0.070}$	0.22	0.029	0.037
Period.	$period = 5$	0.20	0.037	0.042

Fig. 2 shows sample trajectories of the position tracking and estimation errors for a 20-second period, in which $r(t)$ is a sinusoid. The communication instants are indicated by markings in the horizontal lines at the bottom of the left figure. Under similar communication rates, both the deterministic and the DSPP logics show advantage over that of periodic communication, as they both exhibit lower error variances. Aperiodic transmission in the stochastic updating rules requires data to be time-stamped.

5.2 Rate-variance curves

To study the trade-off between communication rate and estimation error variance, we consider the remote state estimator of a first order unstable process $d\mathbf{x} = \mathbf{x}dt + d\mathbf{w}$. This corresponds to a jump diffusion process defined by (18)

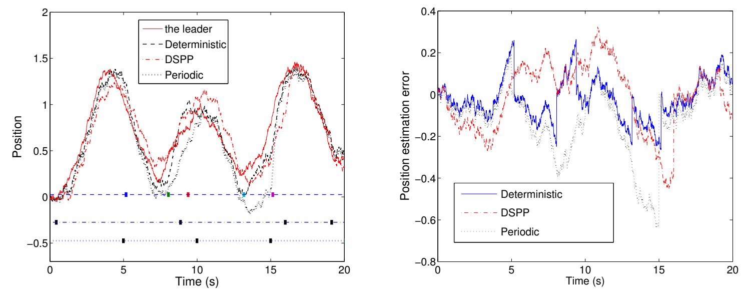

Fig. 2. Leader and follower positions (left) and leader estimation error (right) obtained with the different logics: deterministic, DSPP, and Periodic. The message exchange time instants are indicated with bars in the left plot.

with $A = 1$, $\sigma = 1$. The results presented refer to a simulation time of 1000 seconds. The system's instability presents an added challenge to a distributed architecture.

Fig. 3 (left) depicts the trade-off between the communication rate and the variance of the estimation error for four different communication logics: periodic, DSPP with constant intensity, DSPP with quadratic intensity, and deterministic with quadratic communication index. The curves are obtained by varying the parameters that define these logics. For a given communication rate, the DSPP logic with constant intensity results in the largest error, whereas the deterministic logic results in the smallest. The communication rate obtained with the DSPP logic for the quadratic $\lambda(e)$ is significantly smaller than the upper bound provided by (28), which for this example numerically equals 1.

Fig. 3 (right) provides a comparison between deterministic and DSPP logics. The deterministic logics have a communication index of form $S(e) := \frac{e^2}{\Delta} \leq 1$, and the different points on the curve are generated by changing Δ. The DSPP logics have intensities of the form $\lambda(e) = (\frac{e^2}{\Delta})^k$, where Δ is a positive parameter and $k \in \{1, 2, 3, 4, 5\}$. For large k, $\lambda(e)$ essentially provides a barrier at $e^2 = \Delta$, which acts as the bound in the deterministic logics. It is therefore not surprising to see that as k increases, the DSPP logics converge to the deterministic logics. As proved for discrete systems, the deterministic curve provides optimal trade-off between communication cost and control performance [23].

6 Conclusion and future work

Deterministic and stochastic communication logics are proposed to determine when local controllers should communicate in a distributed control architecture. Using tools from jump diffusion processes and the Dynkin's equation,

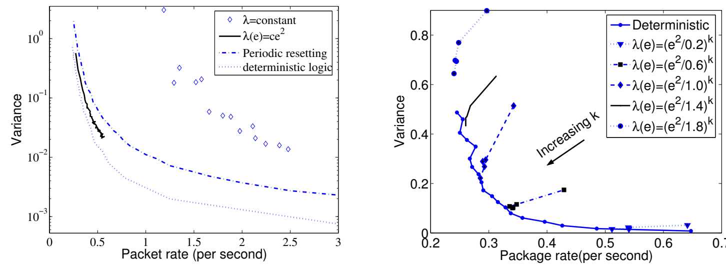

Fig. 3. Communication rate versus variance of the estimation error: for different communication logics (left), and for deterministic and polynomial-intensity DSPP logics (right).

we investigated conditions under which these logics guarantee boundedness as well as the trade-off between the amount of information exchanged and the performance achieved. Monte Carlo simulations confirm that these communication logics can save communication resources over periodic schemes.

In this work, a linear certainty equivalence controller structure (3) is assumed, which may not achieve optimal control performance in the distributed settings. In fact, the counter example in [19] shows that optimal LQG controllers for distributed linear processes are in general nonlinear. Our problem falls under the class of delayed sharing information patterns for which a general separation theorem for controller and estimator design does not appear to exist [8, 20]. We are currently investigating if some form of separation can hold for this specific problem under consideration.

Other future work includes studying the impact of modeling errors on the system's performance as well as the impact of a non-ideal networks that drop packets.

References

1. Coleri S, Puri A, Varaiya P (2003) Power efficient system for sensor networks p. 837–842. In: Eighth IEEE International Symposium on Computers and Communication
2. Cox D. R (1955) Some statistical methods connected with series of events. Journal of the Royal Statistical Society, 17(2):129–164
3. Elia N, Mitter S. K (2001) Stabilization of linear systems with limited information. IEEE Trans. on Automat. Contr., 46(9):1384–1400
4. Glasserman P, Merener. N (2003) Numerical solution of jump-diffusion LIBOR market models. Finance and Stochastics, 7:1–27
5. Hespanha J. P (2004) Stochastic hybrid systems: Applications to communication networks In: Alur R, Pappas G. J (eds), Hybrid Systems: Computation

and Control, , number 2993 in Lect. Notes in Comput. Science, p. 387–401. Springer-Verlag, Berlin

6. Hespanha J. P, Ortega A, Vasudevan L (2002) Towards the control of linear systems with minimum bit-rate In: Proc. of the Int. Symp. on the Math. Theory of Networks and Syst.
7. Khalil H. K (1996) Nonlinear systems. Prentice Hall, Upper Saddle River, NJ
8. Kurtaran B (1979) Corrections and extensions to "decentralized stochastic control with delayed sharing information pattern". IEEE Trans. on Automat. Contr. AC-24(4):656–657
9. Kushner H (2001) Heavy traffic analysis of controlled queueing and communication networks, vol. 47 of Applications of Mathematics. Springer, Berlin
10. Lian F.-L (2001) Analysis, Design, Modeling , and Control of Networked Control Systems PhD thesis University of Michigan Ann Arbor, MI
11. Liberzon D (2002) A note on stabilization of linear systems using coding and limited communication p. 836–841. In: Proc. of the 41st Conf. on Decision and Contr.
12. Liberzon D, Hespanha J. P (2004) Stabilization of nonlinear systems with limited information feedback To appear in the IEEE Trans. on Automatic Control
13. Matveev A. S, Savkin A. V (2003) The problem of state estimation via asynchronous communication channels with irregular transmission times. IEEE Trans. on Automat. Contr., 48(4):670–676
14. Nair G. N, Evans R. J (2000) Communication-limited stabilization of linear systems p. 1005–1010., In: Proc. of the 39th Conf. on Decision and Contr.
15. Oksendal B (2000) Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin
16. Revuz D, Yor M (1999) Continuous Martingales and Brownian Motion. Springer, Berlin
17. Schuss Z (1980) Theory and Applications of Stochastic Differential Equations. Wiley Series in Probability and mathematical statistics. John Wiley and Sons
18. Tatikonda S (2000) Control Under Communication Constrains PhD thesis MIT Cambridge, MA
19. Witsenhausen H. S (1968) A counterexample in stochastic optimum control. SIAM J. Contr. 6(1):131–147
20. Witsenhausen H. S (1971) Separation of estimation and control for discrete time systems. Proceedings of the IEEE 59(11):1557–1566
21. Wong W. S, Brockett R. W (1997) Systems with finite communication bandwidth constraints–part I: State estimation problems. IEEE Trans. on Automat. Contr. 42(9)
22. Wong W. S, Brockett R. W (1999) Systems with finite communication bandwidth constraints–II: Stabilization with limited information feedback. IEEE Trans. on Automat. Contr. 44(5)
23. Xu Y, Hespanha J. P (2004) Optimal communication logics for networked control systems In: Proc. of the 43rd Conf. on Decision and Contr. to appear
24. Yook J. K, Tilbury D. M, , Soparkar N. R (2002) Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators. IEEE Trans. Contr. Syst. Technol, 10(4):503–518