Effective Resistance of Gromov-Hyperbolic Graphs: Application to Asymptotic Sensor Network Problems

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***Abstract*— The technique of effective resistance has seen growing popularity in problems ranging from escape probability of random walks on graphs to asymptotic space localization in sensor networks. The results obtained thus far deal with such problems on Euclidean lattices, on which their asymptotic nature already reveals that the crucial issue is the large scale behavior of such lattices. Here we investigate how such results have to be amended on a class of graphs, referred to as Gromov hyperbolic, which behave in the large scale as negatively curved Riemannian manifolds. It is argued that Gromov hyperbolic graphs occur quite naturally in many situations. Among the results developed here, we will mention the nonvanishing probability of escape of a random walk to a Cantor set Gromov boundary and the facts that the space localization error of sensors networked in a Gromov hyperbolic fashion grows linearly with the distance to a sensor whose geographical position is known, but would become uniformly bounded in an idealized situation in which the geographical locations of the nodes at the Gromov boundary are known**

I. INTRODUCTION

The effective resistance between two nodes in a graph is the voltage drop that would be observed in a electrical network obtained by placing one resistor at each edge of the graph, when a unit current is injected in one of the nodes and removed from the other. In this paper, we extend the result of Doyle and Snell [12] on the effective resistance of a regular tree of degree bounded but greater than 2 and its ramifications in various asymptotic space localization and coordination problems for autonomous agents as initiated by Barooah and Hespanha [5], [3], [4].

A tree is regular if the degree of its nodes, except for the root, is constant. Under this degree condition, the effective resistance from the root of the tree to "infinity" is finite. The latter has the important consequence that a random walk on the tree has a finite probability of escaping to infinity. In the context of computer security, it follows that a worm is more likely to escape to infinity and therefore cause more damage than one that is recurrent (see Remark 1). In the context of localization based on relative measurements, this means that if only the absolute positions of the far-away leafs of a large tree are available, all nodes in the tree can still be accurately localized just based on noisy relative measurements between adjacent nodes in the graph [4]. The reader is referred to [5] for other connections between effective resistance and distributed control problems.

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The natural generalization of the concept of tree is that of Gromov hyperbolic graph [14]. Intuitively, a Gromov hyperbolic graph is a graph that has the property that it looks like a tree when viewed at a distance, where the concept of "viewing at a distance" is formalized in *large-scale geometry*, also referred to as *coarse geometry* or *asymptotic metric theory* [15]. The importance of Gromov hyperbolic graphs is that scale free, and hence Internet, graphs have that property [18], [23], [19]. Since Gromov hyperbolic graphs are a generalization of trees, the natural question is, "What is the effective resistance of a Gromov hyperbolic graph?" The importance of effective resistance of Gromov hyperbolic graphs stems from the above mentioned fact that effective resistances arise naturally in a variety of propagation and localization problems.

Before extending the result of Doyle and Snell, a technical question is whether such concept as "effective resistance between the root and infinity" can be defined for those hyperbolic graphs, in which the tree structure might not be completely obvious. A "root" of a hyperbolic graph is a so-called *quasi-pole* [10]. A *quasi-pole* Ω is a compact subset of an infinite graph G such that for every vertex v of G there exists a geodesic ray emanating from Ω and passing within a bounded distance of v. One of the premises of coarse geometry is that finite subsets, like Ω, can be coalesced to points. As such, Ω can be coalesced to $\omega \in \Omega$, which becomes the "root" of the graph. Once a root is chosen, the next issue is to define "infinity" so as to be able to define the effective resistance between the quasi-pole and infinity. In the context of the coarse geometry of Gromov hyperbolic graphs, "infinity" is the *Gromov boundary at infinity* of the graphs, $\partial_\infty G$. The latter is defined as the equivalence class of infinite geodesic rays emanating from the quasi-pole under the equivalence relation that two rays are equivalent if their Hausdorff distance remains finite [9, III.H.3].

While the Gromov boundary $\partial_\infty G$ of a regular tree can be easily visualized, this is not so when all that is known about G is that it is Gromov hyperbolic, since it could be anything ranging from $\mathbb{N}$ to the circle S^1, passing by the Cantor set, the Menger curve, etc. (See [21] for a survey.) The difficulty is that the topology of $\partial_\infty G$ dictates such application-relevant issues as the Cheeger isoperimetric constant and the exponential growth of balls [10]. As far as the present paper is concerned, in order to have finite effective resistance between Ω and $\partial_\infty G$, $\partial_\infty G$ must have the cardinality of the continuum.

Since in this paper we use coarse geometric techniques in which we do not care about distortion as long as it is uniformly bounded, the *exact* value of the effective resistance $R(\omega, \partial_\infty G)$ is irrelevant. Is relevant, however, the question of whether the effective resistance is vanishing, finite ($0 < R(\omega, \partial_\infty G) < \infty$), or infinite. In fact, the latter are coarse geometry invariants.

Due to space limitations, most proofs have been omitted, but they are available in [20].

II. Mathematical Set-up

A. Graphs

A *graph* G is defined by its vertex set and its edge set, (V, E). The edge with end vertices u, v will be written uv. All graphs here have bounded local geometry, i.e., the degree of their nodes is uniformly bounded. As such, an infinite graph has infinitely many vertices. Edges are equipped with a *length function*, $\ell : E \to \mathbb{R}^+$, with the property that

$$\ell_{\min} := \inf_{e \in E} \ell(e) > 0, \qquad \ell_{\max} := \sup_{e \in E} \ell(e) < \infty \tag{1}$$

Upon identification of every edge with a homeomorph of the unit interval $[0,1]$, the length function is easily extended to Lebesgue-measurable subsets of the edges.

A *path* from x to y is a continuous map $p : [a,b] \to G$ such that $p(a) = x$, $p(b) = y$. The *length* of a path joining x to y is the sum of the lengths of the (subsets of) edges traversed by the path. The *distance* $d(x,y)$ between x, y is the infimum of the lengths of all paths joining x to y. The latter will sometimes be referred to as length distance. With this length distance, (G, d) becomes a *metric space*.

An *isometric embedding* $f : G \to H$ of the graph $G = (V_G, E_G)$ into the graph $H = (V_H, E_H)$ is a map (usually induced by a vertex transformation $V_G \to V_H$) that preserves the metric, viz., $d_H(f(x), f(y)) = d_G(x,y)$.

A *geodesic* $\gamma = [xy]$ from x to y in G is an isometric embedding $\gamma : [0, \ell(\gamma)] \to G$ such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = y$. Because of the bounded local geometry, every pair of points x, y can be joined by a geodesic $[xy]$ such that $\ell([xy]) = d(x,y)$. In such a situation, (G, d) is said to be a *geodesic metric space*.

B. Gromov hyperbolicity

A *geodesic triangle* $\triangle uvw$ on vertices u, v, w is defined as $[uv] \cup [vw] \cup [wu]$. A graph (G, d), or an arbitrary geodesic metric space (X, d) for that matter, is *Gromov hyperbolic* if there exists a $\delta_s < \infty$ such that, $\forall \triangle uvw$, we have $[vw] \subset N_{\delta_s}([uv]) \cup N_{\delta_s}([uw])$, where $N_{\delta_s}([uv])$ denotes the δ_s-neighborhood of $[uv]$ for the metric d. An equivalent characterization is the existence of a constant $\delta_i < \infty$ such that, $\forall \triangle uvw$, for points $i(v,w) \in [vw]$, $i(u,w) \in [uw]$, $i(u,v) \in [uv]$ such that $d(u, i(u,v)) = d(u, i(u,w))$, $d(v, i(v,w)) = d(v, i(v,u))$, $d(w, i(w,v)) = d(w, i(w,u))$, we have $\mathrm{diam}\{i(u,v), i(v,w), i(w,u)\} \leq \delta_i$. It can be shown that $\delta_s < \infty$ iff $\delta_i < \infty$ (see [9, Chap. III.H, Prop. 1.17]). Yet a third equivalent definition states that, in any quadrilateral, the largest sum of the lengths of pairs of opposite diagonals cannot exceed the medium sum by more than $2\delta_q < \infty$ (see [9, p. 411]).

The most extreme example of a Gromov hyperbolic graph is the tree, as $\delta_i = \delta_s = \delta_q = 0$.

The following theorem formulates an important property of Gromov hyperbolic spaces, a property that is sometimes even taken to be the definition of Gromov-hyperbolicity [10, Def. 1.1]:

Theorem 1: Let (X, d) be a δ_s-Gromov hyperbolic space. Let γ_1, γ_2 be two arc length parameterized geodesics such that $\gamma_1(0) = \gamma_2(0) = v^0$. Assume that, for some $R > 0$, $d(\gamma_1(R), \gamma_2(R)) > 3\delta_s$. Then any path p joining $\gamma_1(R+r)$, $\gamma_2(r+R)$ and outside the ball $B_{R+r}(v^0)$ is such that

$$\ell(p) \geq \delta_s 2^{\frac{r}{\delta_s} - 2}$$

This theorem has the (negative) consequence that, if an outage affects a ball $B_{R+r}(v^0)$, a message scheduled to transit along $[uv] \ni v^0$ will have to make a detour of *exponential* length $\delta_s 2^{\frac{r}{\delta_s} - 2}$ to circumvent the outage. As shown in [14], this bound can be refined to $\delta_s 2^{\frac{r+R}{\delta_s} - 1}$.

C. Quasi-isometry

A (λ, ϵ)-*quasi-isometric embedding* of the graph $G = (V_G, E_G)$ into the graph $H = (V_H, E_H)$ is a (not necessarily continuous) function $f : G \to H$ (usually induced by a vertex transformation $V_G \to V_H$) such that

$$\frac{1}{\lambda} d_G(x,y) - \epsilon \leq d_H(f(x), f(y)) \leq \lambda d_G(x,y) + \epsilon,$$

$\forall x, y \in G$. Such an f is *quasi-injective* in the sense that

$$f(x) = f(y) \quad \Rightarrow \quad d_G(x,y) \leq \epsilon\lambda, \quad \forall x, y \in G.$$

f is said to be a *quasi-isometry* if, in addition, there exists a constant c such that

$$d_H(f(G), z) \leq c, \quad \forall z \in H$$

which means that f is *quasi-surjective*. In this case, f has at least one *quasi-inverse* $f^{-q} : H \to G$ in the sense that there exists a constant d for which

$$d_G(f^{-q} \circ f(x), x) \leq d, \quad \forall x \in G, \tag{2}$$

$$d_H(f \circ f^{-q}(z), z) \leq d, \quad \forall z \in H, \tag{3}$$

and this quasi-inverse is both quasi-injective and quasi-surjective. Quasi-isometry is an equivalence relation.

D. Gromov boundary

A *geodesic ray* r emanating from $\omega \in V$ is an isometric embedding $r : [0, \infty) \to G$ such that $r(0) = \omega$. The same ray will sometimes be written as $r(0)/r(\infty)$. Two rays r, r' are said to be *equivalent* iff $\sup_{s \in [0,\infty)} d(r(s), r'(s)) < \infty$; equivalently, if $d_{\mathcal{H}}(r, r') < \infty$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance. The set of equivalence class of rays is the *Gromov boundary*, $\partial_\infty G$. The equivalence class of the ray r is written r_∞. Given two rays, $r^u_\infty, r^w_\infty \in \partial_\infty G$, parameterized by the arc length s, we define

$$\rho_\infty(r^u_\infty, r^w_\infty) := \lim \inf_{s \to \infty} e^{-\epsilon\left(s - \frac{1}{2} d_G(r^u(s), r^w(s))\right)}$$

The above is a *semimetric* [6, I.1.5] on $\partial_\infty G$. The idea behind this definition is that, in a tree-like graph, two rays

r^u, r^w emanating from the same quasi-pole ω will first follow the same path and then separate at some vertex $a(r^u, r^w)$, the *"common ancestor"* of the points $r^u(\infty), r^w(\infty)$. Then $\rho_\infty(r^u_\infty, r^w_\infty) = e^{-\epsilon d(\omega, a(r^u, r^w))}$. Another intuition behind ρ_∞ is as follows: Write the hyperbolic law of cosines in $\triangle \omega r^u(s) r^w(s)$ as embedded in a surface of small negative sectional curvature $\kappa = -\epsilon^2$:

$$\begin{aligned}\cosh \epsilon d_G\left(r^u(s), r^w(s)\right) = \\ &\cosh \epsilon d(\omega, r^u(s)) \cosh \epsilon d(\omega, r^w(s)) \\ &- \sinh \epsilon d(\omega, r^u(s)) \sinh \epsilon d(\omega, r^w(s)) \cos \hat{\omega}\end{aligned}$$

where $\hat{\omega}$ is a short for $\angle r^u(s) \omega r^w(s)$. Since the arc length parameterization of geodesic rays is an isometry, $d(\omega, r^u(s)) = d(\omega, r^w(s)) = s$. Then, for a distance d such that $\epsilon d \gg 1$, $\cosh \epsilon d \approx e^{\epsilon d}$ and, for a small angle $\hat{\omega}$, $\cos \hat{\omega} \approx 1 - \frac{\hat{\omega}^2}{2}$. With these approximations, the hyperbolic law of cosines yields

$$e^{\epsilon d_G(r^u(s), r^w(s))} = 1 + e^{2\epsilon s} \frac{\hat{\omega}}{2}$$

In light of the above, ρ_∞ turns up to be the angle metric

$$\rho_\infty(r^u_\infty, r^w_\infty) = \lim \inf_{s \to \infty} \frac{\angle r^u(s) \hat{\omega} r^w(s)}{\sqrt{2}}$$

Now we make the semimetric ρ_∞ a metric by enforcing the triangle inequality

$$d_\infty(r^u_\infty, r^w_\infty) := \inf_{r^u = a^0, a^1, \ldots, a^n = r^w} \sum_{i=0}^{n-1} \rho_\infty(a^i, a^{i+1})$$

For $\epsilon < \log 2/\delta_G$, d_∞ is a *visual metric* in the sense that

$$\frac{1}{4} \rho_\infty(r^u_\infty, r^w_\infty) \leq d_\infty(r^u_\infty, r^w_\infty) \leq \rho_\infty(r^u_\infty, r^w_\infty)$$

(See [16, Sec. 2], [9, III.H.3.19].) This distance in turn defines a topology on $\partial_\infty G$.

The Gromov boundary might have varying cardinalities, and even Gromov boundaries with the same cardinality might be quite different. An example of card$(\partial_\infty G) = \aleph_0$ is provided by Figure 1. It should also be obvious that the cardinality of the Gromov boundary of a binary tree T_2 is $2^{\aleph_0}$, which is the cardinality of the continuum c. Another example of card$(\partial_\infty G) = c$ is provided by $G = \text{Cay}(\pi_1(M^n))$, the Cayley graph of the fundamental group of a compact Riemannian manifold M^n of curvature uniformly bounded from above by $\kappa < 0$ (see [11, V.58]). The *significant* difference between the continua $\partial_\infty T_2$ and $\partial_\infty \text{Cay}(\pi_1(M^n))$ is that the former is a Cantor set while the latter is S^{n-1}.

Other examples of Gromov boundaries that are circles include the Poincaré disk or any of its tessellation by the image of a fundamental domain [22, 4.10].

Theorem 2: Under a quasi-isometry $f : G \to H$, $G = (V_G, E_G)$ is Gromov hyperbolic iff $H = (V_H, E_H)$ is Gromov hyperbolic; furthermore, f induces a (quasi-Möbius) homeomorphism $f_\infty : \partial_\infty G \to \partial_\infty H$.

It transpires form the above that, for a Gromov hyperbolic graph to be quasi-isometric to a tree, it is necessary that its Gromov boundary be a Cantor set.

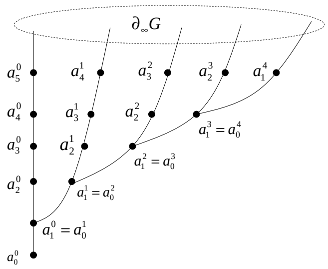

Fig. 1. A tree, the Gromov boundary of which has cardinality $\aleph_0$.

E. Isoperimetric behavior and exponential growth of balls

Given an infinite graph G, the *combinatorial Cheeger isoperimetric constant* is defined as

$$h(G) = \inf_{S \subseteq V_G} \frac{|\partial S|}{|S|}$$

where the infimum is over all finite subsets, $|S|$ denotes the cardinality, and

$$\partial S = \{v \in V \setminus S : v \in N(s) \text{ for some } s \in S\}.$$

Observe that ∂S can be interpreted as the boundary of S and therefore if $h(G) > 0$ the above means that the "volume" $|S|$ of a subset of vertices is bounded by a linear function of the "area" $|\partial S|$ of its boundary.

The exponential growth of balls is clearly related to the condition $h(G) > 0$. Indeed, if a subset S of vertices is "infected" at time $t = 0$, the number of vertices infected prior to or at time $t \geq 0$ is bounded from below by $|S|(1+h(G))^t$. It turns out that a Gromov hyperbolic graph need not have $h(G) > 0$. For the latter to hold, an extra condition on the boundary at infinity is needed:

Theorem 3: Let G be a infinite, bounded geometry, complete, Gromov hyperbolic graph with a quasi-pole. If, in addition, the infimum of the diameters of the connected components of $\partial_\infty G$ is > 0, then $h(G) > 0$, which implies that $\lambda_{\min}(L(G)) > 0$.

There is a slight refinement of the preceding result:

Theorem 4: $\lambda_{\min}(L(G)) \geq \frac{h^2(G)}{4}$.

III. INSTANCES OF COARSE GEOMETRY

Here we make the preceding concepts a little more pallatable by linking them with known engineering concepts.

A. Instances of hyperbolic graphs

Gromov hyperbolic graphs are an idealization of situations that readily occur in practice. Typically, we are given a set of agents $\{v^i\} =: V$, which attempt to form an ad hoc network, and which in doing so define a distance function $d : V \times V \to \mathbb{R}^+$. Typically, the distance $d(v^i, v^j)$ is the communication delay between v^i and v^j. As such, (V, d) becomes a metric space. The question is to determine the manifold M^n and its Riemannian metric g such that the agents can be thought of as operating on the manifold; precisely, the question is whether there exists an isometric embedding $(V,d) \hookrightarrow (M^n, g)$. Specifically, there exists an isometric embedding $(V,d) \hookrightarrow (\mathbb{H}^n, g)$, where $\mathbb{H}^n$ denotes the standard n-dimensional Riemannian manifold of constant sectional curvature $\kappa < 0$, iff $\det\{\cosh\left(d(v^i,v^j)\sqrt{-\kappa}\right)\}_{\{1,\ldots,k\}\times\{1,\ldots,k\}} = (-1)^{k+1}$. (See [6], [7].) Practically speaking, we cite a few specific situations that naturally lead to negatively curved networks.

a) Growth/preferential attachment graphs: They have been shown to be scaled Gromov hyperbolic [18], [19].

b) High power transceivers in a wireless sensor network: They have a tendency to network in a negatively curved graph, while the low power transceivers rather network in a positively curved graph [2].

c) Geometric optics propagation reflecting on convex obstacles: Assume that in the two-dimensional Euclidean space $\mathbb{R}^2$, we define a set of transceivers $\{v^i\}$ along with convex obstacles. Assume that, at least for some pairs of agents, communication can only be established by radio waves reflecting on convex obstacles. It can be shown that the embedding is in a negatively curved surface of a genus equal to the number of convex obstacles.

d) Delaunay triangulation of nonuniformly distributed vertices: Assume a set of agents $\{v^i\}$ are *nonuniformly* distributed in $\mathbb{R}^2$. The Delaunay triangulation (V,E) of a set of Gauss distributed agents is shown in Fig. 2 and is negatively curved in the following sense. Define the clustering coefficient as $\mathrm{cluster}(v^i) = \frac{\#\{(v^iv^j, v^iv^k) : v^jv^k \in E\}}{\binom{\deg(v^i)}{2}} \in [0,1]$. When $0 \le \mathrm{cluster}(v^i) \le \frac{1}{2}$, the curvature is locally negative [2], [13]. The distribution of $\mathrm{cluster}$ is shown in Fig. 3, indicating negative curvature.

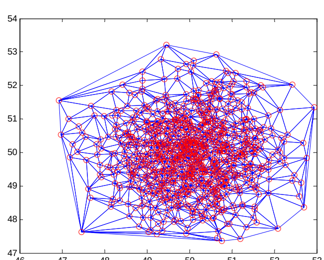

Fig. 2. Delaunay triangulation of nonuniformly distributed agents.

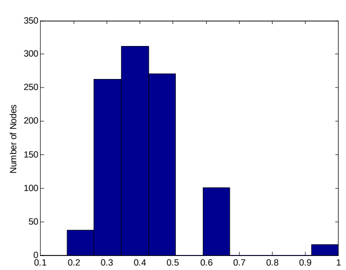

Fig. 3. Clustering coefficient distribution of Delaunay triangulation.

B. Instances of quasi-isometries

The examples are plentiful, the most obvious being that a Euclidean lattice is quasi-isometric to the Euclidean space *of the same dimension*. Precisely, the natural embedding $\mathbb{Z}^n \to \mathbb{R}^n$ is a $(\lambda = 1, \epsilon = 1)$-quasi-isometry. More generally, we have the following:

Theorem 5: The drawing function $f : G \to \mathbb{E}^n$ of a graph G that is both sparse and dense in the n-dimensional Euclidean space $\mathbb{E}^n$ is a (λ,ϵ)-quasi-isometry for finite constants $\lambda, \epsilon > 0$.

IV. Effective Resistance: A Generalized Rayleigh Monotonicity Principle

The graph $G = (V,E)$ can be made into an electrical network by replacing every edge by a resistor of resistance $R_e = \ell(e)$. If $\mathbf{j}(v^i)$ denotes the current injected at node v^i, and $v(v^i)$ denotes the voltage at node v^i relative to some ground potential, then $j = Lv$, where L is the *Laplacian operator* defined as $(Lv)_i = \sum_{v^j \in N(v^i)} \frac{(v_i - v_j)}{R_{e(v^i,v^j)}}$, where $N(v^i) := \{v \in V : vv^i \in E\}$ and $R_{e(v^i,v^j)}$ is the resistance of the edge v^iv^j. If $\ell(e) = R_e = 1$, $\forall e \in E$, then $L = \mathrm{diag}(\deg(v^i)) - A$, where A is the adjacency matrix. It is easily verified that $L \ge 0$. To define the effective resistance at a driving point vertex v^i, a subset V_0 of short-circuited vertices, connected to the graph, is declared at ground potential, $v(V_0) = 0$. If V_0 is finite and at finite distance from some driving point vertex $v^i \notin V_0$, the *effective resistance* is defined as $R(v^i, V_0) = \left(L_0^{-1}\right)_{ii}$, where L_0 is the operator obtained after removing the rows and columns of L corresponding to V_0, easily seen to be invertible. If the graph is infinite, viewing $L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ as a bounded Hilbert space operator requires $v(\partial_\infty G) = 0$, so that the 0 reference potential is naturally set across $\partial_\infty G$. By the same token, boundedness of L requires $i(v^i) = 0, \forall v^i \in \partial_\infty G$. Assuming for a moment that $\lambda_1(L) > 0$, where $\lambda_1(L) = \inf \mathrm{spec}(L)$, then $\left(L^{-1}\right)_{ii}$ is the *effective resistance* $R(v^i, \partial_\infty G)$ between v^i and $\partial_\infty G$. The latter consideration makes $\lambda_1(L) > 0$ a crucial issue:

Theorem 6: $R(v^i, \partial_\infty G) < \bar{R} < \infty$, $\forall v^i \in V_G$ iff $\lambda_1(L(G)) > 0$.

One can also show that effective resistance satisfies the following triangle inequality

$$R_G(u,w) \le R_G(u,v) + R_G(v,w), \quad \forall u,v,w \in V_G.$$

Hence effective resistance is a distance, which is different from the length distance, unless G is a tree.

The following monotonicity result can be viewed as a generalization of Rayleigh's Monotonicity Law [12] to "quasi-embeddings":

Theorem 7 (Rayleigh Monotonicity Principle): Consider two uniformly-bounded degree graphs $G = (V_G, E_G)$, $H = (V_H, E_H)$ for which there exists a quasi-injective function $f : G \to H$ and constants $\lambda, \epsilon < \infty$ such that

$$d_H(f(x), f(y)) \le \lambda\, d_G(x,y) + \epsilon, \quad \forall x,y \in G. \tag{4}$$

Then there exists a finite constant $\alpha > 0$ such that

$$R_H(f(x), f(y)) \leq \alpha R_G(x, y), \quad \forall x, y \in V_G. \quad (5)$$

If, in addition, f is *quasi-surjective* then we also have that

$$R_H(z, w) \leq \alpha R_G(f^{-q}(z), f^{-q}(w)) + \beta, \ \forall z, w \in V_H, \ (6)$$

where $f^{-q}(\cdot)$ is a quasi-inverse of f and β a finite constant. When f is actually surjective, then (6) holds with $\beta = 0$. In the expressions above, R_G and R_H denote effective resistances in electrical networks constructed from the graphs G and H, respectively.

The following result can be obtained by applying Theorem 7 to a quasi-isometry f and to its quasi-inverse f^{-q}.

Corollary 1: Suppose that f is a (λ, ϵ)-quasi-isometry between two uniformly-bounded degree graphs $G = (V_G, E_G)$, $H = (V_H, E_H)$. Then there exist finite constants $\alpha, \beta > 0$ such that

$$\frac{1}{\alpha} R_G(x, y) - \beta \leq R_H(f(x), f(y)) \leq \alpha R_G(x, y), \quad (7)$$

$\forall x, y \in V_G$, where R_G and R_H denote effective resistances in electrical networks constructed from the graphs G and H, respectively. When f is actually injective, then (7) holds with $\beta = 0$.

V. TWO ASYMPTOTIC PROBLEMS

Here we formulate two asymptotic problems amenable to effective resistance methods:

1) The escape probability of a random walk on a Gromov-hyperbolic graph, closely related to the effective resistance between the root and the Gromov boundary of the graph.
2) The asymptotically space localization error at large distance from the reference sensor, closely related to the asymptotic effective resistance between two points in the limit of large distance.

A. Escape probability of random walks

The exponential growth of balls is relevant to a propagation scheme in which every node infects *every* neighboring node. Other processes include random walks on a graph—that is, processes through which an infected node infects *only one* of its neighbors chosen at random. More specifically, if the walker is at v^i, the probability of jumping to $v^j \in N(v^i)$ is

$$p(v^i, v^j) = \frac{\frac{1}{R_{e(v^i, v^j)}}}{\sum_{v \in N(v^i)} \frac{1}{R_{e(v, v^j)}}}$$

The problem is to determine whether there is a nonvanishing probability of the walk reaching the Gromov boundary at infinity. The latter is called *escape probability*. A slight generalization of [12, Sec. 1.3.4] yields

$$p_{\text{esc}}(\omega, \partial_\infty G) = \frac{\frac{1}{R(\omega, \partial_\infty G)}}{\sum_{v^i \in N(\omega)} \frac{1}{R_{e(\omega, v^i)}}}$$

where ω is a vertex of a quasi-pole. In other words, the probability of escape from a vertex of a quasi-pole to the boundary at infinity is proportional to the reciprocal of the effective resistance between ω and the Gromov boundary. As already said, the mere fact that the graph is Gromov hyperbolic with a quasi-pole does not provide enough data to determine an exact value for $R(\omega, \partial_\infty G)$, but it is enough to determine whether or not $R(\omega, \partial_\infty G)$ is finite.

Remark 1. The difference between a randomly walking worm escaping to infinity and an exponentially growing percolating worm is a matter of the gap between $\lambda_1(L(G))$ and $h^2(G)/4$, as specified by Theorem 4.

B. Space localization

Assume that autonomous agents distributed in the Euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$ communicate via the graph G. Assume that a subset of vertices V_0 has known geographical position and that the position of the other agents has to be computed from noisy measurement of the distances $d_G(v^i, v^j)$, $v^i v^j \in E$, and azimuth data. The following is proved in [4]:

Theorem 8: If $V_0 = \{v^0\}$, then the unbiased least square estimation error of the geographical position $x(v^i)$ of v^i is given by

$$E(||x(v^i) - \hat{x}(v^i)||^2) = R_G(v^i, v^0).$$

If G is Gromov hyperbolic, and under the idealized situation where $V_0 = \partial_\infty G$, we have

$$E(||x(v^i) - \hat{x}(v^i)||^2) = R(v^i, \partial_\infty G).$$

VI. RESULTS

The solution to the two problems defined in the previous section relies on a generic theorem, stating that a Gromov hyperbolic graph is quasi-isometric to a tree. Several such results are already known (see, e.g., [8]), but here we need to set-up the quasi-isometry in a specific way that easily translates to effective resistance and geographic estimation errors.

Theorem 9: A planar Gromov hyperbolic graph G with a quasi-pole Ω and a Cantor Gromov boundary is (quasi-surjectively) quasi-isometric to a tree T. Furthermore, if $\ell(e) = 1$, $\forall e \in E$, the quasi-isometry can be taken such that

$$d_G(u, v) - \delta_i \leq d_T(u, v) \leq \lambda d_G(u, v), \quad \forall u, v \in V$$

with $\lambda = 7\delta_i + 2$.

We note that, in [8], it is shown that there exists a quasi-isometric embedding from a binary tree to a Gromov hyperbolic graph with Cantor Gromov boundary. The problem is that this embedding is not in general quasi-surjective and hence cannot be used to construct a quasi-isometry between the graph and the binary tree.

A. Escape probability

Theorem 10: Let G be a bounded geometry, Gromov hyperbolic graph with a quasi-pole Ω and a Gromov boundary that is a Cantor set. Then $0 < R(\omega, \partial_\infty G) < \infty$, i.e., there is a nonvanishing probability of escape from the quasi-pole to the boundary at infinity $\partial_\infty G$.

We observe that in [8] a quasi-isometric embedding $f : T \to G$ is set up from a binary tree T to a Gromov hyperbolic graph G. This immediately leads to $R_G(u, v) \leq R_T(u, v)$ and the upper bound $R_G(\omega, \partial_\infty G) < \infty$. As the preceding shows, the difficulty is the lower bound.

As an illustration, consider a regular tree with node degree 3. In this case, the Gromov boundary is the collection of all paths from the root to infinity. Since on the way from the root to $\partial_\infty G$, each edge gives birth to two other edges, the number of paths from the root to ∂_∞, hence the cardinality of $\partial_\infty G$, is $2^{\aleph_0}$, the cardinality of the continuum. It also follows that the paths can be binary encoded, the same way as the Cantor set is.

To provide a counterexample that $\aleph_0$, the cardinality of $\mathbb{N}$, is not enough to guarantee $R(\omega, \partial_\infty G) < \infty$, consider the following tree construction: Let $A^0 = a_0^0 a_1^0 a_2^0 ...$ be a semi-infinite string of vertices such that $d(a_i^0, a_{j+1}^0) = 1$. We attach, at $a_1^0 \in A^0$, a replica A^1 of A^0. Specifically, $a_1^0 = a_0^1$. Then we attach at $a_1^1 \in A^1$ another replica A^2. Specifically, $a_1^1 = a_0^2$. From here on, we iterate by attaching further replica A^k, $k \geq 3$, such that $a_1^k = a_0^{k+1}$. Clearly, the cardinality of $\partial_\infty G$ is $\aleph_0$. As far as the resistance is concerned, $R(a_0^0, \partial_\infty G) = 1 + R(a_0^1, \partial_\infty G)$. But by symmetry, $R(a_0^0, \partial_\infty G) = R(a_0^1, \partial_\infty G)$. And the second to last inequality implies $R(a_0^0, \partial_\infty G) = \infty$.

Corollary 2: In a Gromov hyperbolic graph with a quasi-pole, the mutually exclusive properties

- $R(\omega, \partial_\infty G) = 0$
- $0 < R(\omega, \partial_\infty G) < \infty$
- $R(\omega, \partial_\infty G) = \infty$

are coarse geometry invariants.

B. Space localization

Theorem 11: Given a Gromov hyperbolic graph G with a quasi-pole ω and a Cantor set Gromov boundary, the effective resistance $R_G(\omega, v^i)$, for reference nodes $V_0 = \{\omega\}$ at the quasi-pole, grows linearly as $d_G(\omega, v^i)$,

$$E(||x(v^i) - \hat{x}(v^i)||^2) = O(d(\omega, v^i)),$$

For reference nodes $V_0 = \partial_\infty G$ at the Gromov boundary,

$$E(||x(v^i) - \hat{x}(v^i)||^2) = O(1)$$

It may be said that the graph G in Theorem 11 has dimension 2, because the integer dimension of its Cantor boundary is 1. For a 2-dimensional Euclidean lattice, it was shown in [4] that the variance of the error is $O(\log d(\omega, v^i))$. The latter and our major result are consistent, as the Euclidean lattice is more wired up than a tree-like graph, so that it is not surprising that the error is smaller in the former.

VII. Conclusion

We have shown that a Gromov hyperbolic graphs with a quasi-pole and a Cantor Gromov boundary is quasi-isometric to a tree and therefore exhibits finite effective resistance to infinity. We have shown that this property has important implications to problems related to propagation and localization problems. However, some coordination problem calls for new challenges, as some standard flocking algorithms [17], [1] enforce a positively curved formation graph, rather than a negatively curved graph such as a Gromov hyperbolic one. This issues remains open for future research.

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