Real-time Control under Clock Offsets between Sensors and Controllers*

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ABSTRACT

This paper studies the impact of clock mismatches in spatially distributed real-time control systems. We consider a configuration in which sensor measurements are collected by one processor that transmits the measurements to another control/actuation processor through a network, but the two processors do not have a common clock. Due to the clock mismatch, there will be an offset between the actual time at which a measurement is taken and the time reported by the sensor. Our goal is to discover fundamental limitations to the ability to stabilize the control loop arising from the clocks mismatch. We consider time-varying bounded offsets and derive limitations on the offset bound for the stability of the feedback system. For the case of a scalar linear process, there exists a critical limitation, which depends on the level of instability of the plant and the nominal sampling period. In contrast, for the vector linear processes, if the process dynamics has at least two distinct real eigenvalues, then there is no fundamental limitation on the offset bound.

Categories and Subject Descriptors

C.2.4 [**Computer-Communication Networks**]: Distributed Systems; I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*Control theory*

Keywords

Clock offsets; networked control systems; fundamental limitations.

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1. INTRODUCTION

It is widely recognized that control systems can place severe real-time constraints on software systems. However, in the context of distributed control applications, it generally does not suffice to make sure that the different processors involved perform computations in a timely fashion. In fact, a significant challenge in such systems is that the different processors may not share a common notion of time, because their local clocks are not properly synchronized. In this paper, we study the potential impact of clock mismatches in a spatially distributed real-time control system.

We consider control systems where the components are spatially distributed and connected via networks. To pursue real-time control of such systems, one will be required to synchronize the local clocks of the system components [12]. Clock synchronization in distributed systems [1, 6] and protocols for synchronization [10,18,21] have been actively studied. In addition to such protocols, information from Global Positioning System (GPS) is often used as a reference of the global clock in practical systems. However, attaining synchronization through communication channels is in general difficult: It has been shown that synchronization is possible only under certain conditions [7]. Moreover, recent works have pointed out that GPS signals can be spoofed and hence are vulnerable against malicious attacks [2, 11].

Real-time control has been studied from several view points including providing algorithms for task scheduling [4,17] and dealing with delays and jitters [3, 22]. However, the fundamental problem that analyzing the impact of clock mismatch on control systems has been an open issue.

In this paper, we consider feedback systems where the sensor measuring the plant state and the controller operate locally and may not be synchronized. Due to this asynchronism, a reported plant state from the sensor may be taken at an instant different from that the controller supposes. This mismatch results in uncertainty in measurements and causes fundamental limitations in control. We consider the situation in which the offset between the clock in the sensor and that in the controller is bounded and may be time varying. This offset is unknown to both the sensor and the controller sides. Our objective is to clarify limitations on the tolerable offsets for stability. For the scalar plants case, we derive a necessary and sufficient condition for stability, which gives a tight upper bound on the offset. In contrast, for the vector case, there exists no practical limitation on the offset if the

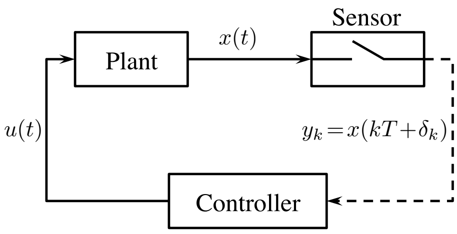


Figure 1: Feedback system.

plant has at least two different real modes.

Systems with variable sampling intervals have been actively studied from the perspective of robust control; see, e.g., [5,8,13,14]. In these papers, the controller is commonly given by a linear time-invariant system, where the inputs to the controller are updated at uncertain instants because of the variable sampling period. For the resulting feedback system, its stability against uncertain sampling instants has been analyzed, and sufficient conditions have been shown. Our recent work [20] in this line of research has dealt with constant clock offsets.

Unlike the above-mentioned works, we study the existence of a general causal control algorithm to stabilize the feedback loop in the presence of incorrect timing information. Inspired by the analysis of data rate limitations in control under quantization (see, e.g., [15,19]), we attempt to quantify how uncertainty in timing information translates into uncertainty in state estimation, under a causality constraint. Stabilization is impossible if the lowest level of uncertainty in state estimation results in an uncertainty set that grows unbounded. Analyzing fundamental limitations due to imperfect measurements in networked control systems has been a vital research area in the last decade. A wide variety of imperfections has been considered including quantization errors due to data rate limitations and packet losses caused by unreliability of channels; see [9, 16] and the references therein. In this research, the imperfection is caused by the clock offset between the sensor and the controller.

This paper is organized as follows: In Section 2, we formulate the problem. The definitions of the state estimation sets used to generate the control input are given in Section 3. In Section 4, we consider the case of scalar plants and provide the first main result. Section 5 is devoted to show the result for the vector systems case. In Section 6, a more practical setup is considered where the sensor output is quantized to discrete values. We consider the scalar plants case and generalize the result in Section 4. Finally concluding remarks are given in Section 7.

Notations: Throughout the paper, $\mathbb{Z}_+$ denotes the set of nonnegative integers; $\mathbf{0}$ (bold-faced zero) stands for the zero vector of an appropriate dimension; $o(\cdot)$ is the Landau symbol defined as $f = o(\phi)$ means that $f/\phi \to 0$; and $\|\cdot\|$ represents the Euclidean norm.

2. PROBLEM FORMULATION

Consider the feedback system depicted in Figure 1. The plant system to be controlled is the following continuous-time linear time-invariant process:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

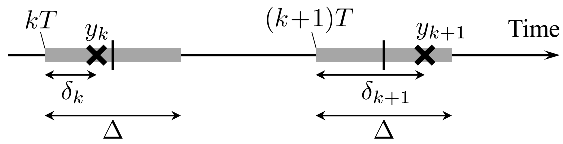


Figure 2: Time chart.

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. The unknown initial state $x(0)$ may be anywhere in $\mathbb{R}^n$. We introduce the following assumption on the plant.

Assumption 1. *The system* (1) *is unstable and controllable, and A is invertible.*

The plant state is periodically observed by a sensor and is sent to the controller. Let y_k, $k \in \mathbb{Z}_+$, be the kth observed state, and let $T > 0$ be the sampling period from the perspective of the controller clock. Figure 2 illustrates a time chart of sampling: The actual time instants at which the samples y_k are generated are represented by the crosses, whereas the gray boxes represent the uncertainty inherent to the timing information. due to the (variable) offset between the clocks of the sensor and the controller. We assume that the magnitude of the clock offset is bounded by $\Delta/2$. The offset bound Δ is smaller than the sampling interval T:

$$0 \leq \Delta < T. \tag{2}$$

There are various algorithms for the clock offset estimation, and the resulted estimation errors have been analyzed; see, e.g., [21]. These results are useful to determine the offset bound Δ. We denote by $kT+\Delta/2$ the kth nominal sampling instant (center of the timing uncertainty boxes in Figure 2). Then, y_k is given by

$$y_k = x(kT + \delta_k), \quad \delta_k \in [0, \Delta]. \tag{3}$$

Here, the actual sampling instant $kT + \delta_k$ according to the controller clock is unknown because of the unknown jitter δ_k. We note that the controller cannot determine δ_k from the time instant at which the sample y_k arrives since there may exist an unknown communication delay.

Based on the received observations $y_0, \ldots, y_k$, the controller determines the input $u(t)$, $t > kT + \Delta$. We make the following assumption.

Assumption 2. *The control input is constant during* $[kT, kT + \Delta]$ *for every* k*:*

$$u(t) = u_k, \quad \forall t \in [kT, kT + \Delta], \quad \forall k \in \mathbb{Z}_+.$$

This assumption implies that during the intervals in which the sensor may measure the plant state, the controller holds the input. For the inputs during $t \in (kT + \Delta, (k+1)T)$, we allow the controller to take any value.

Remark 1. Assumption 2 is required to prove the necessity results in the paper. We note that the sufficient conditions for stability hold even if we limit the class of controllers to piecewise constant ones.

Our objective is to clarify how large Δ can be to ensure the stabilizability of the system in the following sense.

Definition 1. The feedback system is stabilizable if for any sequence of jitters $\{\delta_k\}_{k=0}^{\infty}$, $\delta_k \in [0, \Delta]$, and any initial state $x(0)$, there exists a controller such that the state converges to the origin, i.e., $\lim_{t\to\infty} x(t) = 0$.

3. COMPUTATION OF ESTIMATION SETS

In what follows, we present notations and definitions of estimation sets of the plant states, which will be refereed to in the following sections. Let I_k^- be the initial estimation set of $x(kT)$ based on the past observations $y_0, \dots, y_{k-1}$. From (3), y_k satisfies

$$y_k = \mathrm{e}^{A\delta_k} x(kT) + \int_{kT}^{kT+\delta_k} \mathrm{e}^{A(kT+\delta_k-\tau)} Bu(\tau)\mathrm{d}\tau \tag{4}$$

with the unknown parameter δ_k and the true state $x(kT)$. Thus, $x(kT)$ must lie in the following set J_k:

$$\begin{aligned} J_k &= \left\{ \mathrm{e}^{-A\delta_k} y_k - \int_{kT}^{kT+\delta_k} \mathrm{e}^{A(kT-\tau)} Bu(\tau)\mathrm{d}\tau : \delta_k \in [0, \Delta] \right\} \quad (5) \\ &= \left\{ \mathrm{e}^{-A\delta_k} \left(y_k + A^{-1}Bu_k \right) - A^{-1}Bu_k : \delta_k \in [0, \Delta] \right\}. \quad (6) \end{aligned}$$

Here, the second equality follows by Assumption 2. With this estimation set J_k, after receiving y_k, we obtain the updated estimation set I_k as

$$I_k := I_k^- \cap J_k. \tag{7}$$

The initial estimation set of $x((k+1)T)$ is computed using I_k by

$$I_{k+1}^- = \left\{ \mathrm{e}^{AT} \hat{x}_k + \int_{kT}^{(k+1)T} \mathrm{e}^{A((k+1)T-\tau)} Bu(\tau)\mathrm{d}\tau : \hat{x}_k \in I_k \right\}. \tag{8}$$

Since $I_0^- = \mathbb{R}^n$ from the setup, we have $I_0 = J_0$ for any J_0. By (8), the initial estimation set of $x(T)$ is

$$I_1^- = \left\{ \mathrm{e}^{-A\delta_0} a_0 + b_0 : \delta_0 \in [0, \Delta] \right\}, \tag{9}$$

where

$$\begin{aligned} a_0 &:= \mathrm{e}^{AT}(y_0 + A^{-1}Bu_0), \\ b_0 &:= -\mathrm{e}^{AT} A^{-1}Bu_0 + \int_0^T \mathrm{e}^{A(T-\tau)} Bu(\tau)\mathrm{d}\tau. \end{aligned}$$

After receiving y_1, we obtain the new estimation set

$$J_1 = \left\{ \mathrm{e}^{-A\delta_1} a_1 + b_1 : \delta_1 \in [0, \Delta] \right\}, \tag{10}$$

where $a_1 := y_1 + A^{-1}Bu_1$ and $b_1 := -A^{-1}Bu_1$.

We note that the set I_k given in (7) is a *tight* estimation set of $x(kT)$ in the following sense: For any $\hat{x}_k \in I_k$, there exists a path of the state $x(t), t \in [0, kT + \Delta]$, which results in $x(kT) = \hat{x}_k$ and corresponds to the observed outputs $y_0, y_1 \dots, y_k$ with possible delays $\{\delta_i\}_{i=0}^k$. We formally define tightness of an estimation set below.

Definition 2. For an estimation set $I_k \subset \mathbb{R}^n$ of $x(kT)$, which is constructed from I_0^-, $u(t)$, $t \in [0, kT + \Delta]$, and $y_0, y_1, \dots, y_k$, the estimation set is said to be tight if for every $\hat{x}_k \in I_k$, there exist $x_0 \in I_0^-$ and $\{\delta_i\}_{i=0}^k$, $\delta_i \in [0, \Delta]$, such that

$$\mathrm{e}^{AkT} x_0 + \int_0^{kT} \mathrm{e}^{A(kT-\tau)} Bu(\tau)\mathrm{d}\tau = \hat{x}_k, \tag{11}$$

$$\mathrm{e}^{A(iT+\delta_i)} x_0 + \int_0^{iT+\delta_i} \mathrm{e}^{A(iT+\delta_i-\tau)} Bu(\tau)\mathrm{d}\tau = y_i, \ \forall i. \tag{12}$$

Proposition 1. *The estimation set I_k defined in* (7) *is tight in the sense given in Definition 2.*

Proof. It is obvious from (5) that $I_0 = J_0$ is tight. Suppose that I_{k-1}, $k \geq 1$, is tight. That is, for any $\hat{x}_{k-1} \in I_{k-1}$, there exist $x_0 \in I_0^-$ and $\{\delta_i\}_{i=0}^{k-1}$, $\delta_i \in [0, \Delta]$, such that

$$\mathrm{e}^{A(k-1)T} x_0 + \int_0^{(k-1)T} \mathrm{e}^{A((k-1)T-\tau)} Bu(\tau)\mathrm{d}\tau = \hat{x}_{k-1}, \tag{13}$$
$$\mathrm{e}^{A(iT+\delta_i)} x_0 + \int_0^{iT+\delta_i} \mathrm{e}^{A(iT+\delta_i-\tau)} Bu(\tau)\mathrm{d}\tau = y_i,$$
$$i = 0, 1, \dots, k-1.$$

Now pick any $\hat{x}_k$ from I_k. Since $\hat{x}_k \in I_k^-$ and by (8), there exists $\hat{x}_{k-1} \in I_{k-1}$ such that

$$\mathrm{e}^{AT} \hat{x}_{k-1} + \int_{(k-1)T}^{kT} \mathrm{e}^{A(kT-\tau)} Bu(\tau)\mathrm{d}\tau = \hat{x}_k.$$

Substituting (13) into the above equation, we have that (11) holds with x_0 selected in (13). Furthermore, with the fact that $\hat{x}_k \in J_k$ and (5), there exists $\delta_k \in [0, \Delta]$ satisfying

$$\mathrm{e}^{A\delta_k} \hat{x}_k + \int_{kT}^{kT+\delta_k} \mathrm{e}^{A(kT+\delta_k-\tau)} Bu(\tau)\mathrm{d}\tau = y_k.$$

We substitute (11) into the above equation and then obtain (12) for the case that $i = k$, which concludes the proof. □

4. LIMITATION ON CLOCK OFFSET: SINGLE EIGENVALUE CASE

In this section, we consider the scalar plants case and derive a tight bound on the maximum offset Δ for stability. Consider the following system:

$$\dot{x}(t) = \lambda x(t) + u(t), \quad x(0) \in \mathbb{R}. \tag{14}$$

In the above system, from the controllability, we have assumed $B = 1$ without loss of generality. Moreover, from the instability it follows that $\lambda > 0$.

The following is the first main result.

Theorem 1. *The feedback system with the scalar plant* (14) *is stabilizable in the sense of Definition 1 if and only if the bound on the offset satisfies* $0 \leq \Delta < \bar{\Delta}$*, where*

$$\bar{\Delta} := \begin{cases} T & \text{if } 0 < T \leq \frac{1}{\lambda} \ln 3, \\ T - \frac{1}{\lambda} \ln(\mathrm{e}^{\lambda T} - 2) & \text{if } T > \frac{1}{\lambda} \ln 3. \end{cases} \tag{15}$$

We note that the two expressions in (15) are continuous. The limitation $\bar{\Delta} = T$ for the case $0 < T < (\ln 3)/\lambda$ is due to the assumption (2). The bound $T - \{\ln(\mathrm{e}^{\lambda T} - 2)\}/\lambda$ is decreasing with respect to λ and T when $T > (\ln 2)/\lambda$. This observation is consistent with the intuition that larger evolution rate $\mathrm{e}^{\lambda T}$ per sampling period will result in higher requirement in accuracy of the observations y_k. The following example shows how the bound $\bar{\Delta}$ varies with T.

Example 1. Consider the plant where $\lambda = 1.0$. In Figure 3, we plot the limitation $\bar{\Delta}$ versus the sampling period T. The solid line is the active bound $\bar{\Delta}$ and the vertical dotted lines indicate $T = (\ln 2)/\lambda$ (left) and $T = (\ln 3)/\lambda$ (right), respectively. For small T, the restriction due to (2) limits $\bar{\Delta}$ as T. At $T = (\ln 3)/\lambda$, $\bar{\Delta}$ takes its maximum and the two expressions in (15) coincide with each other.

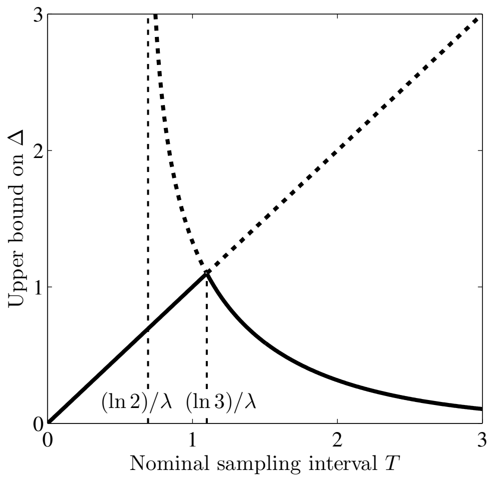


Figure 3: Upper bound on Δ versus sampling period T ($\lambda = 1.0$): The solid line is the active bound $\bar{\Delta}$ and the vertical dotted lines indicate $T = (\ln 2)/\lambda$ (left) and $T = (\ln 3)/\lambda$ (right), respectively.

4.1 Proof of Theorem 1

For scalar plants, by the definition (6), the estimation set J_k computed from y_k is written as

$$J_k = \left\{ \mathrm{e}^{-\lambda\delta_k} \left(y_k + \lambda^{-1} u_k \right) - \lambda^{-1} u_k : \delta_k \in [0, \Delta] \right\}. \quad (16)$$

Thus, J_k becomes a closed interval in $\mathbb{R}$. Therefore, with (7) and (8), I_k and I_{k+1}^-, $k \in \mathbb{Z}_+$, also result in intervals in $\mathbb{R}$. Notice that we have assumed $I_0^- = \mathbb{R}$.

Let l_k be the length of the interval I_k, which is the updated estimation set of $x(kT)$. It is clear that if $l_k \to 0$ as $k \to \infty$ then the feedback system is stabilizable. We also have that since I_k is a tight estimation set, if $\lim_{k\to\infty} l_k \neq 0$, then there exists a possible trajectory of $x(t)$ which does not converge to the origin no matter what we choose for $u(t)$. In the following, we consider a condition that $l_k \to 0$ to show the theorem. Suppose that I_{k-1} is given at $t = (k-1)T + \Delta$. Then one can compute I_k^- from (8). The length l_k is determined by I_k^- and J_k, that is, l_k depends on the true state $x_k \in I_k^-$, the jitter $\delta_k \in [0, \Delta]$, and the input $u(t)$, $t \in ((k-1)T + \Delta, kT)$. To guarantee stability for all possible x_k and δ_k, we have to care the worst case. Hence, we need a condition for

$$\max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \to 0 \text{ as } k \to \infty.$$

In addition, for the necessity part of the theorem, we pursue a condition guaranteeing

$$\min_{u(t), t\in((k-1)T+\Delta, kT)} \; \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \to 0 \text{ as } k \to \infty.$$

The following lemma gives the expansion rate of l_k.

LEMMA 1. *Given an interval I_{k-1}, $k \geq 1$. It follows that*

$$\min_{u(t), t\in((k-1)T+\Delta, kT)} \; \max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k = \frac{(1 - \mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}}{2} l_{k-1}. \quad (17)$$

PROOF. For simplicity of notation in the proof, we first introduce biased sets as follows: Define $\tilde{y} := y_k + \lambda^{-1} u_k$ and let $\tilde{J} := \{ z + \lambda^{-1} u_k : z \in J_k \}$. Then, from (16), this biased estimation set $\tilde{J}$ can be simply represented as

$$\tilde{J} = \begin{cases} \left[\mathrm{e}^{-\lambda\Delta} \tilde{y},\ \tilde{y} \right] & \text{if } \tilde{y} > 0, \\ \{0\} & \text{if } \tilde{y} = 0, \\ \left[\tilde{y},\ \mathrm{e}^{-\lambda\Delta} \tilde{y} \right] & \text{if } \tilde{y} < 0. \end{cases} \quad (18)$$

Similarly, define the biased sets

$$\tilde{I}^- := \left\{ z + \lambda^{-1} u_k : z \in I_k^- \right\},$$
$$\tilde{I} := \tilde{I}^- \cap \tilde{J} = \left\{ z + \lambda^{-1} u_k : z \in I_k \right\}.$$

Note that the length l_k of I_k is the same as that of $\tilde{I}$ and the analysis in this proof holds for any bias $\lambda^{-1} u_k$. In Figure 4, we illustrate the update procedure of the estimation set.

We now show that the input $u(t)$, $t \in ((k-1)T + \Delta, kT)$, achieving the minimum in (17) is the one which makes $\tilde{I}^-$ symmetric about the origin. In what follows, we will evaluate the worst case length $\max_{x_k, \delta_k} l_k$ for a given $\tilde{I}^-$. We start by describing the range of y_k under the knowledge that $x_k \in I_k^-$ and $\delta_k \in [0, \Delta]$. With (4), the set of all possible y_k is as follows:

$$y_k \in \left\{ \mathrm{e}^{\lambda\delta_k} x_k + \lambda^{-1}(\mathrm{e}^{\lambda\delta_k} - 1) u_k : x_k \in I_k^-,\ \delta_k \in [0, \Delta] \right\},$$

which results in

$$\tilde{y} \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0, \Delta],\ z \in \tilde{I}^- \right\}. \quad (19)$$

Thus, we can rewrite the maximization term in (17) as

$$\max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k = \max_{\tilde{y} \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0,\Delta], z \in \tilde{I}^- \right\}} l_k.$$

Next, we evaluate $\max_{\tilde{y}} l_k$ for each variation of $\tilde{I}^-$. In the rest of the proof, we assume $\tilde{y} \neq 0$ since by (18), it is clear that $\tilde{y} = 0$ does not maximize l_k. Let denote the lower bound and the upper bound of $\tilde{I}^-$ by p and q, respectively: $\tilde{I}^- = [p, q]$. Suppose that

$$|q| \geq |p|, \quad (20)$$

i.e., the midpoint of $\tilde{I}^-$ is nonnegative. For the case $|q| < |p|$, one can apply the same discussion below by flipping signs.

Consider the following two cases (i) $p \leq 0$, that is, $\tilde{I}^-$ contains the origin and (ii) $p > 0$, $\tilde{I}^-$ lies only in the positive area.

(i) $p \leq 0$: In this case, it follows that $q \geq 0$ from (20). By (18), we have

$$\tilde{I} = \begin{cases} \left[\mathrm{e}^{-\lambda\Delta} \tilde{y}, \min(q, \tilde{y}) \right] & \text{if } \tilde{y} > 0, \\ \left[\max(p, \tilde{y}), \mathrm{e}^{-\lambda\Delta} \tilde{y} \right] & \text{if } \tilde{y} < 0. \end{cases}$$

Taking maximums in both cases $\tilde{y} > 0$ and $\tilde{y} < 0$, and using (20), one can see that when $\tilde{y} = q$ the length l_k takes its maximum

$$\max_{\tilde{y} \in \left\{ \mathrm{e}^{\lambda\delta_k} z : \delta_k \in [0,\Delta], z \in \tilde{I}^- \right\}} l_k = (1 - \mathrm{e}^{-\lambda\Delta}) q. \quad (21)$$

The right-hand side takes its minimum under the conditions (20) and (i) for the case in which $|q| = |p|$ or $\tilde{I}^-$ is symmetry about the origin.

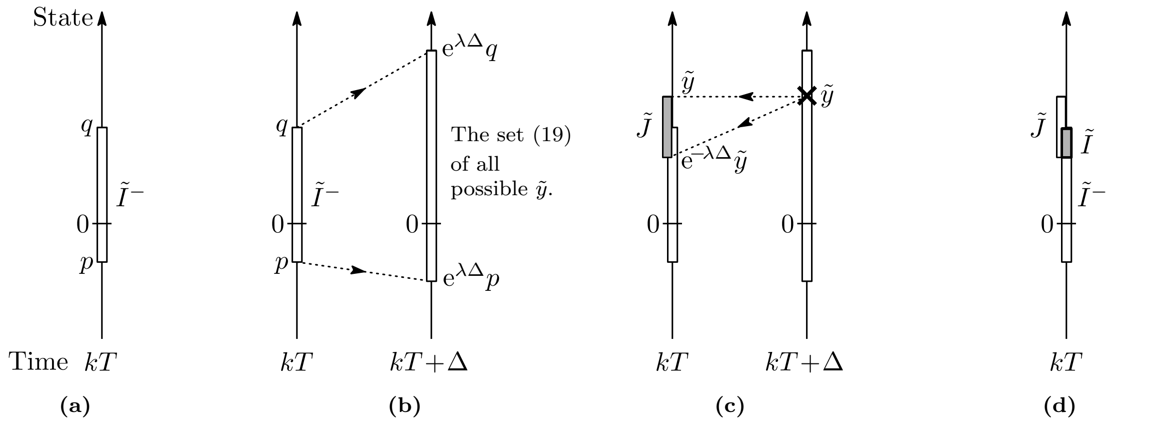


Figure 4: Update procedure of the estimation set: (a) The initial estimation set $\tilde{I}^-$ is depicted as the white box; (b) $\tilde{I}^-$ expands during the sensing interval $[kT, kT+\Delta]$; (c) After receiving $\tilde{y}$, the controller computes the set $\tilde{J}$ of the candidates of the state at kT which may result in $\tilde{y}$ somewhere in $[kT, kT+\Delta]$; (d) The updated estimation set $\tilde{I}$ is obtained as the intersection of $\tilde{I}^-$ and $\tilde{J}$.

(ii) $p > 0$: In this case, noticing that all possible $\tilde{y}$ are positive from (19), we obtain

$$\tilde{I} = \left[\max(p, \mathrm{e}^{-\lambda\Delta}\tilde{y}), \min(q, \tilde{y})\right].$$

Therefore,

$$\max_{\tilde{y}\in\left\{\mathrm{e}^{\lambda\delta_k}z:\delta_k\in[0,\Delta],z\in\tilde{I}^-\right\}} l_k = \begin{cases}(1-\mathrm{e}^{-\lambda\Delta})q & \text{if } q \geq \mathrm{e}^{\lambda\Delta}p,\\ q-p & \text{if } q < \mathrm{e}^{\lambda\Delta}p.\end{cases}$$

Note that $q-p$ in the right-hand side does not vary with the position of $\tilde{I}^-$. The other expression of $\max_{\tilde{y}} l_k$, $(1-\mathrm{e}^{-\lambda\Delta})q$, is the same formula for the case (i) as in (21). However, from the conditions (i) and (ii), it is obvious that q cannot be small as that for the case (i). Thus, $\tilde{I}^-$ satisfying (ii) cannot minimize $\max_{\tilde{y}} l_k$.

From the discussions (i) and (ii), we have that $\max_{\tilde{y}} l_k$ is minimized when $\tilde{I}^-$ is taken as satisfying $|q| = |p| = \mathrm{e}^{\lambda T} l_{k-1}/2$, and the minimum is

$$\min_{u(t),t\in((k-1)T+\Delta,kT)} \max_{x_k\in I_k^-,\delta_k\in[0,\Delta]} l_k = (1-\mathrm{e}^{-\lambda\Delta})\frac{\mathrm{e}^{\lambda T}}{2} l_{k-1}.$$

This concludes the proof. □

Proof of Theorem 1. From Lemma 1, the system (14) is stable if and only if

$$\frac{(1-\mathrm{e}^{-\lambda\Delta})\mathrm{e}^{\lambda T}}{2} < 1.$$

From this inequality and the assumption (2), we have the upper bound (15) on Δ. □

We now present a numerical example to illustrate behavior of the system.

Example 2. Consider the system where the plant parameter is $\lambda = 1.0$ and the nominal sampling period is $T = 2.0$. We employ the following control law: Divide the interval $((k-1)T + \Delta, kT)$ into two parts of the same length. In the first time period, the controller applies the constant input which makes I_k^- symmetric about $\lambda^{-1}u_k$. As we have seen in the proof of Lemma 1, this control input minimizes $\max_{x_k,\delta_k} l_k$, which is the length of the estimation set I_k in the worst case. The input in the later period and u_k are taken to be 0. We take the initial state as $x(0) = 1$ and chose δ_k uniformly randomly from $[0, \Delta]$. Note that these variables are unknown to the controller. We first consider the case that the offset bound is small and satisfies the stability condition in Theorem 1; suppose $\Delta = 0.30$, which is less than $\overline{\Delta} \approx 0.32$. In Figure 5a, we illustrate the state $x(t)$ and the input $u(t)$ by the solid line and the dashed line, respectively. Moreover, the observations y_k are depicted by the cross marks and the upper bounds and the lower bounds of the estimation sets I_k are represented by $\triangle$ and $\triangledown$, respectively. One can observe that as time progresses, the upper bounds and the lower bounds of I_k become closer, and the state successfully converges to 0. Next, consider the case $\Delta = 0.45$, which violates the stability limit $\overline{\Delta} \approx 0.32$. Figure 5b shows the behavior of the system in this case. We see that the state (solid line) oscillates and the estimation set I_k (triangle marks) does not shrink to a single point.

5. LIMITATION ON CLOCK OFFSET: MULTIPLE EIGENVALUES CASE

In this section, we consider plants with multi eigenvalues and study the maximum offset tolerable to achieve stability.

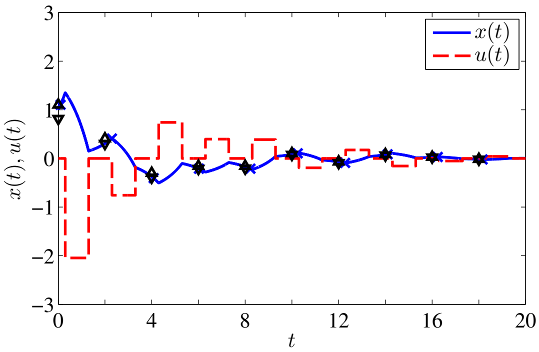
The following theorem is the main result in this section.

Theorem 2. *If A has at least two different real eigenvalues, then for any Δ satisfying* (2)*, the state of the feedback system with* (1) *can be reconstructed in finite steps.*

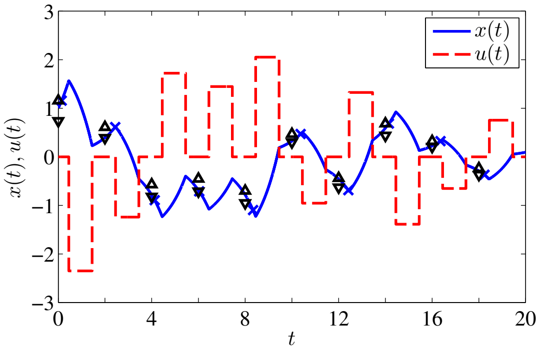
From this theorem and the controllability, the stabilizability result follows immediately.

Corollary 1. *If A has at least two different real eigenvalues, then for any Δ satisfying* (2)*, the feedback system with* (1) *is stabilizable in the sense of Definition 1 in finite steps.*

Therefore, unlike the scalar case, there is no practical limitation on the offset if the plant dynamics has two different real eigenvalues. We may roughly explain this result as follows: At the time after receiving $y_k = x(kT + \delta_k)$, there are



(a) $\Delta = 0.30 < \bar{\Delta} \approx 0.32$



(b) $\Delta = 0.45 > \bar{\Delta} \approx 0.32$

Figure 5: Behavior of the system ($\lambda = 1.0$, $T = 2.0$, $\bar{\Delta} \approx 0.32$): The plant state $x(t)$ (solid), the input $u(t)$ (dashed), the observations y_k (cross), and the estimation sets I_k ($\triangle$: upper bounds, $\triangledown$: lower bounds).

$n+k+1$ unknown variables, i.e., $x(0)$ and $\delta_0, \dots, \delta_k$. On the other hand, if A has two different eigenvalues, we have $2(k+1)$ independent equations from evolution of the states corresponding to the eigenvalues. Hence, one can determine the exact state in finite steps. Moreover, since the plant is controllable, there exists an input which drives the state to the origin in a finite time interval.

The proof of Theorem 2 consists of two steps. We first consider a subsystem of (1). Since there exist eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ of A such that $\lambda_1 \neq \lambda_2$, via an appropriate coordinate transformation, we have

$$V^{-1}\dot{x}(t) = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & * \end{bmatrix} V^{-1}x(t) + V^{-1}Bu(t),$$

for some matrix $V \in \mathbb{R}^{n\times n}$. Extracting the first two rows of the above system, we obtain the subsystem

$$\dot{\xi}(t) = \Lambda\xi(t) + \nu(t), \tag{22}$$

where $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2)$, and ξ and ν are the corresponding states and the inputs, respectively.

As the first step of the proof, we consider computing estimation sets of $\xi(kT)$ from $y_0, \dots, y_k$. Instead of the estimation sets of x represented by J_k, I_k, and I_{k+1}^-, which are defined in (6)–(8), we denote the corresponding estimation sets of ξ by $\hat{J}_k$, $\hat{I}_k$, and $\hat{I}_{k+1}^-$. In light of (9), the estimation set $\hat{I}_1^-$ of $\xi(T)$ is expressed as

$$\hat{I}_1^- = \left\{ \mathrm{e}^{-\Lambda\delta_0}\hat{a}_0 + \hat{b}_0 : \delta_0 \in [0, \Delta] \right\}.$$

Here, $\hat{a}_0$ and $\hat{b}_0$ are defined as follows: Denote by ζ_k the first two elements of $V^{-1}y_k$, and let ν_k and $\nu(t)$ be the first two elements of $V^{-1}Bu_k$ and $V^{-1}Bu(t)$, respectively. Then, $\hat{a}_0$ and $\hat{b}_0$ are defined as

$$\begin{aligned} \hat{a}_0 &:= \mathrm{e}^{\Lambda T}(\zeta_0 + \Lambda^{-1}\nu_0), \\ \hat{b}_0 &:= -\mathrm{e}^{\Lambda T}\Lambda^{-1}\nu_0 + \int_0^T \mathrm{e}^{\Lambda(T-\tau)}\nu(\tau)\mathrm{d}\tau. \end{aligned} \tag{23}$$

Note that Λ is nonsingular since A is invertible. Moreover, by (10), it follows that

$$\hat{J}_1 = \left\{ \mathrm{e}^{-\Lambda\delta_1}\hat{a}_1 + \hat{b}_1 : \delta_1 \in [0, \Delta] \right\},$$
$$\text{where} \quad \hat{a}_1 := \zeta_1 + \Lambda^{-1}\nu_1, \quad \hat{b}_1 := -\Lambda^{-1}\nu_1. \tag{24}$$

We claim that the intersection of $\hat{I}_1^-$ and $\hat{J}_1$, which is the updated estimation set $\hat{I}_1$, can be a set containing only finite elements. This claim is formally stated in the following lemma.

Lemma 2. *There exists a control input $\nu(t)$, $t \in [0, T+\Delta]$ that makes the updated estimation set $\hat{I}_1 = \hat{I}_1^- \cap \hat{J}_1$ a set of finite points.*

To prove this lemma, we will need to consider the following problem: Let a, b, and c be nonzero vectors in $\mathbb{R}^2$; also fix $[T_1, T_2] \subset \mathbb{R}$ and $[T_3, T_4] \subset \mathbb{R}$. Define functions $f : [T_1, T_2] \to \mathbb{R}^2$ and $g : [T_3, T_4] \to \mathbb{R}^2$ as

$$f(t) := \mathrm{e}^{\Lambda t}a, \quad g(t) := \mathrm{e}^{\Lambda t}b + c.$$

Then, find the intersection of $\{f(t) : t \in [T_1, T_2]\}$ and $\{g(t) : t \in [T_3, T_4]\}$.

The following lemma states that the intersection can be a set of finite points if a, b, and c are chosen appropriately.

Lemma 3. *If at least one of the vectors a, b, and c is not an eigenvector of Λ, then the intersection of $\{f(t) : t \in [T_1, T_2]\}$ and $\{g(t) : t \in [T_3, T_4]\}$ becomes a set of finite points.*

Proof. See Section 5.1. $\square$

We now prove Lemma 2 using Lemma 3.

Proof of Lemma 2. Consider the statement of Lemma 3 with $a = \hat{a}_0$, $b = \hat{a}_1$, and $c = \hat{b}_0 - \hat{b}_1$. Then, $\hat{I}_1$ corresponds to the intersection of $\{f(t) : t \in [T_1, T_2]\}$ and $\{g(t) : t \in [T_3, T_4]\}$. Note that $\hat{a}_0$ and $\hat{a}_1$ can be assumed to be nonzero since if either $\hat{a}_0 = \mathbf{0}$ or $\hat{a}_1 = \mathbf{0}$, then $\hat{I}_1^-$ or $\hat{J}_1$ becomes a single point. Thus, such cases can be excluded.

Moreover, in what follows, we show that there exists an input to make $\hat{b}_0 - \hat{b}_1$ not to be $\mathbf{0}$ nor an eigenvector of Λ. Since $\nu(t) = \nu_0$, $t \in [0, \Delta]$, it follows from (24) and (24) that

$$\hat{b}_0 - \hat{b}_1 = -\Lambda^{-1}\mathrm{e}^{\Lambda(T-\Delta)}\nu_0 + \int_\Delta^T \mathrm{e}^{\Lambda(T-\tau)}\nu(\tau)\mathrm{d}\tau + \Lambda^{-1}\nu_1.$$

Note that the first and the third terms in the right-hand side are known to the controller. Furthermore, the integral term is the zero-state-response of (22). Because of the controllability of the original system (1), $x(kT)$ or $\xi(kT)$ can be set arbitrarily, and hence $\hat{b}_0 - \hat{b}_1$ can be an arbitrary vector with the appropriate input $\nu(t)$, $t \in (\Delta, T)$. Thus, with an input $\nu(t)$ such that $\hat{b}_0 - \hat{b}_1$ is not $\mathbf{0}$ nor an eigenvector of Λ, from Lemma 3, $\hat{I}_1$ becomes a set of finite points. $\square$

The second step, which completes the proof of Theorem 2, is to provide a procedure to determine the state using $\hat{I}_1$ consisting of finite points.

Proof of Theorem 2. Since each point of $\hat{I}_1$ corresponds to a candidate of the sequences of past offsets δ_0, δ_1, we can determine the estimations of $x(T)$ or I_1 as a set of finite points. From (8), the initial estimation set I_2^- of $x(2T)$ also results in finite point. Here, the number of the elements of I_2^- is the same as that of I_1. Thus, I_2 become also finite points and its number of the elements is less than or equal to that of I_2^-.

On the other hand, from the controllability, the controller can bring any point in I_2 to the origin by an appropriate input $u(t)$, $t \in (2T + \Delta, 3T)$. Hence, we can reduce the number of the elements of I_k at least by one per sampling interval. Repeating this procedure, we have I_k as a single point in finite steps. $\square$

We now show an example where I_1 becomes a single point. In this case, we can stabilize the system using only two observations y_0 and y_1.

Example 3. Consider the following system:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T = 1.5, \quad \Delta = 1.0.$$

Notice that Δ is greater than the bound (15) in the scalar case corresponding to each eigenvalue, 2 and 1, of A since $T - \left\{\ln(\mathrm{e}^{2T} - 2)\right\}/2 \approx 0.052$ and $T - \ln(\mathrm{e}^{T} - 2) \approx 0.59$. In Figure 6, we plot the estimation sets I_0, I_1^-, and J_1 computed from y_0 and y_1 in the state space $\mathbb{R}^2$. Since we assume $I_0^- = \mathbb{R}^2$, I_0 is equal to J_0. Note that the unknown state $x(0)$ and $x(T)$ are not used to compute the estimation sets. The figure shows that $I_1^- \cap J_1 = I_1$ becomes a single point, which equals to $x(T)$.

5.1 Proof of Lemma 3

In the following, we denote the images $\{f(t) : t \in [T_1, T_2]\}$ and $\{g(t) : t \in [T_3, T_4]\}$ simply by f and g, respectively. Since f and g are closed and bounded sets in $\mathbb{R}^2$, the intersection $f \cap g$ becomes a compact set. Thus, if the set $f \cap g$ has infinitely many elements, then there exists an accumulation point in $f \cap g$. For a point (t_0, s_0), $t_0 \in [T_1, T_2]$, $s_0 \in [T_3, T_4]$, in $f \cap g$, if the direction vectors $\frac{\mathrm{d}f}{\mathrm{d}t}(t_0)$ and $\frac{\mathrm{d}g}{\mathrm{d}t}(s_0)$ are parallel, i.e., there exists $\alpha \in \mathbb{R}$ such that $\frac{\mathrm{d}f}{\mathrm{d}t}(t_0) = \alpha \frac{\mathrm{d}g}{\mathrm{d}t}(s_0)$, we call (t_0, s_0) a *tangent point*.

We show Lemma 3 by proving that there is no accumulation point in $f \cap g$. We first establish that an accumulation point must be a tangent point as Lemma 4. Then, it will be proved that a tangent point also cannot to be an accumulation point, which is formally stated in Lemma 5. To state the first lemma, define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(t, s) := \mathrm{e}^{\Lambda t} a + \mathrm{e}^{\Lambda s} b + c.$$

The following lemma holds.

Lemma 4. *Suppose $h(t_0, s_0) = \mathbf{0}$ and (t_0, s_0) is not a tangent point of f and g. Then there exists a neighborhood of (t_0, s_0) in which $h(t, s) \neq \mathbf{0}$ holds except at (t_0, s_0).*

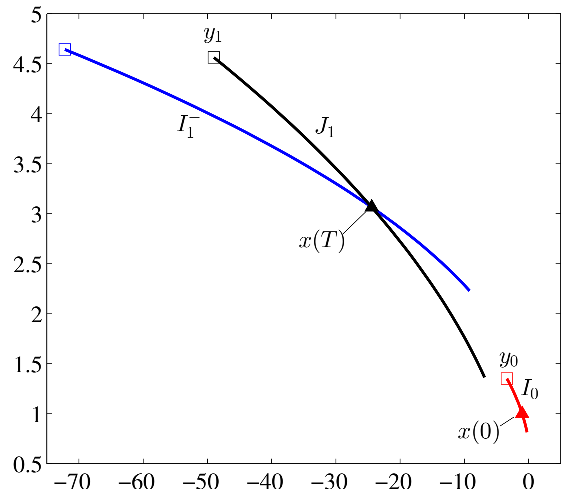


Figure 6: Estimation sets I_0, I_1^-, and J_1 in the state space $\mathbb{R}^2$ (horizontal axis: first element of x, vertical axis: second element of x).

Proof. Define h_1 and h_2 by

$$h_1(t) := \begin{cases} \frac{\mathrm{e}^{\Lambda(t-t_0)} - I}{t - t_0} \mathrm{e}^{\Lambda t_0} a & \text{if } t \neq t_0, \\ \Lambda \mathrm{e}^{\Lambda t_0} a & \text{if } t = t_0, \end{cases}$$

$$h_2(s) := \begin{cases} \frac{\mathrm{e}^{\Lambda(s-s_0)} - I}{s - s_0} \mathrm{e}^{\Lambda s_0} b & \text{if } s \neq s_0, \\ \Lambda \mathrm{e}^{\Lambda s_0} b & \text{if } s = s_0. \end{cases}$$

Then h_1 and h_2 are continuous and it follows that

$$h(t, s) = (t - t_0) h_1(t) + (s - s_0) h_2(s). \tag{25}$$

We will show that $h_1(t)$ and $h_2(s)$ are independent in a neighborhood of (t_0, s_0) by contradiction. Pick any $\epsilon > 0$ and consider the ball $B_\epsilon := \{(t, s) : \|(t, s) - (t_0, s_0)\| < \epsilon\}$. Assume that there exists $(t, s) \in B_\epsilon$ such that for some $\alpha_{t,s} \in \mathbb{R}$

$$h_1(t) = \alpha_{t,s} h_2(s). \tag{26}$$

From continuity of h_1 and h_2, there exist $l_1(t)$ and $l_2(s)$ such that $h_1(t) = h_1(t_0) + l_1(t)$, $h_2(s) = h_2(s_0) + l_2(s)$, and $\lim_{t \to t_0} l_1(t) = \lim_{s \to s_0} l_2(s) = \mathbf{0}$. For these vectors, it follows that

$$\|l_1(t) - \alpha_{t,s} l_2(s)\| \leq \|l_1(t)\| + |\alpha_{t,s}| \, \|l_2(s)\|.$$

Here, since $\|h_1(t)\|$ and $\|h_2(s)\|$ are bounded by positive constants respectively from above and below on B_ϵ, and by (26), $|\alpha_{t,s}|$ is bounded from above. Thus, for a sufficiently small ϵ, the right-hand side of the above inequality can be arbitrarily small. On the other hand, with (26), it follows that

$$\|l_1(t) - \alpha_{t,s} l_2(s)\| = \|h_1(t_0) - \alpha_{t,s} h_2(s_0)\|. \tag{27}$$

Since (t_0, s_0) is not a tangent point, for any $\alpha \in \mathbb{R}$, $h_1(t_0) \neq \alpha h_2(s_0)$. Therefore, the right-hand side of (27) is bounded from below as

$$\|h_1(t_0) - \alpha_{t,s} h_2(s_0)\| > \gamma,$$

where $\gamma := \inf_{\alpha\in\mathbb{R}} \|h_1(t_0) - \alpha h_2(s_0)\| > 0$ is the shortest distance from $h_1(t_0)$ to $\{\alpha h_2(s_0) : \alpha \in \mathbb{R}\}$. This contradiction shows that if we take a sufficiently small ϵ, then (26) does not hold for any point in B_ϵ. That is, $h_1(t)$ and $h_2(s)$ are linearly independent in B_ϵ. Recalling (25), we have that $h(t, s) = \mathbf{0}$ on B_ϵ only at $(t, s) = (t_0, s_0)$. $\square$

Now we show that a tangent point in $f \cap g$ also cannot to be an accumulation point if we choose a, b, or c not to be eigenvectors of Λ.

LEMMA 5. *Suppose that*

$$f(t_0) = g(s_0) \tag{28}$$

and (t_0, s_0) is a tangent point. If there exists a sequence $\{(t_k, s_k)\}_{k=1}^{\infty}$ converging to (t_0, s_0) and satisfying

$$f(t_k) = g(s_k), \quad \forall k \in \mathbb{Z}_+, \tag{29}$$

then a, b, and c are eigenvectors of Λ.

PROOF. Since (t_0, s_0) is a tangent point, we have

$$\mathrm{e}^{\Lambda t_0} a = \alpha \mathrm{e}^{\Lambda s_0} b \tag{30}$$

for some constant $\alpha \neq 0$. Moreover, from (28) it follows that

$$\mathrm{e}^{\Lambda t_0} a = \mathrm{e}^{\Lambda s_0} b + c. \tag{31}$$

With (30) and (31), we have

$$c = (\alpha - 1)\mathrm{e}^{\Lambda s_0} b. \tag{32}$$

Since c is nonzero, $\alpha \neq 1$.

We show that there exists no sequence $\{(u_k, v_k)\}_{k=1}^{\infty}$ such that (29) holds with $t_k = t_0 + u_k$ and $s_k = s_0 + v_k$ and (u_k, v_k) goes to $(0, 0)$ by contradiction. Suppose b is not an eigenvector of Λ. Then since $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ and $\lambda_1 \neq \lambda_2$, all elements of b are nonzero. From the definitions of f and g and (30), we have

$$\begin{aligned} f(t_k) &= f(t_0 + u_k) = f(t_0) + \alpha(\mathrm{e}^{\Lambda u_k} - I)\mathrm{e}^{\Lambda s_0} b, \\ g(s_k) &= g(s_0 + v_k) = g(s_0) + (\mathrm{e}^{\Lambda v_k} - I)\mathrm{e}^{\Lambda s_0} b. \end{aligned}$$

Thus by (28) and (29), it follows that

$$\alpha(\mathrm{e}^{\Lambda u_k} - I)b = (\mathrm{e}^{\Lambda v_k} - I)b.$$

Since both elements of b are nonzero,

$$\alpha(\mathrm{e}^{\lambda_i u_k} - 1) = \mathrm{e}^{\lambda_i v_k} - 1 \tag{33}$$

holds for $i = 1, 2$. By (33), v_k satisfies $v_k = v_{\lambda_i}(u_k)$ where

$$v_\lambda(u) := \frac{1}{\lambda} \ln \left\{ \alpha(\mathrm{e}^{\lambda u} - 1) + 1 \right\}.$$

Taking derivative of $v_\lambda(u)$, we have

$$\frac{\mathrm{d}v_\lambda}{\mathrm{d}u} = \frac{\alpha \mathrm{e}^{\lambda u}}{\alpha(\mathrm{e}^{\lambda u} - 1) + 1}, \quad \frac{\mathrm{d}^2 v_\lambda}{\mathrm{d}u^2} = \frac{\alpha\lambda \mathrm{e}^{\lambda u}(1 - \alpha)}{\{\alpha(\mathrm{e}^{\lambda u} - 1) + 1\}^2}.$$

Thus $v_\lambda(u)$ can be written as

$$v_\lambda(u) = \alpha u + \frac{1}{2}\alpha(1 - \alpha)\lambda u^2 + o(u^2).$$

Define $d(u)$ by

$$\begin{aligned} d(u) &:= v_{\lambda_1}(u) - v_{\lambda_2}(u) \\ &= \frac{1}{2}\alpha(1 - \alpha)(\lambda_1 - \lambda_2)u^2 + o(u^2). \end{aligned}$$

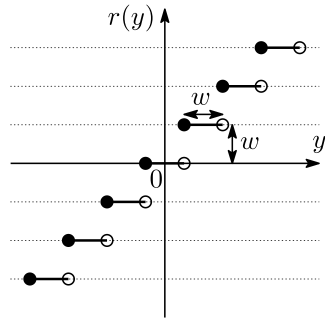


Figure 7: Static uniform quantizer.

We show that $d(u)$ has an isolated zero at $u = 0$. Define $\tilde{d}(u)$ as

$$\tilde{d}(u) := \begin{cases} u^{-2} d(u) & \text{if } u \neq 0, \\ \frac{1}{2}\alpha(1 - \alpha)(\lambda_1 - \lambda_2) & \text{if } u = 0. \end{cases}$$

Then $\tilde{d}(u)$ is continuous at $u = 0$. However,

$$\tilde{d}(0) = \frac{1}{2}\alpha(1 - \alpha)(\lambda_1 - \lambda_2) \neq 0.$$

Therefore, from continuity of $\tilde{d}(u)$, there is a neighborhood of 0 in which $\tilde{d}(u)$ or $d(u)$ has no zero except at $u = 0$. This contradicts the existence of the sequences $\{(u_k, v_k)\}_k$ or $\{(t_k, s_k)\}_k$. Thus, b must be an eigenvector of Λ.

Finally, from (30) and (32), if b is an eigenvector of Λ, then a and c are also eigenvectors. $\square$

6. STABILIZATION UNDER CLOCK OFFSET AND QUANTIZATION

So far, we have considered that the reported states y_k are real values. However, signals in networked control systems are often quantized to discrete values because of the limitation on sensor resolution and restrictions on dealing with signals in digital processors. In this section, we consider a more practical situation and extend the result for the scalar plants case in Section 4. Here, we introduce a quantizer at the sensor side and study that how the two types of imperfection in measurements, quantization errors and clock offsets, affect on the stabilizability of the system.

Consider the scalar system in (14). The plant state is periodically observed and then quantized by the following *static* and *uniform quantizer* $r(\cdot)$:

$$r(y) = iw, \quad y \in \left[\left(i - \frac{1}{2}\right) w, \left(i + \frac{1}{2}\right) w \right), \quad i \in \mathbb{Z}, \tag{34}$$

where $w > 0$ is the width of the quantization cells. In Figure 7, we show the relation of the input and the output of the quantizer $r(y)$. Then, by (3), the kth reported state y_k available at the controller side is represented as

$$y_k := r(x(kT + \delta_k)), \quad \delta_k \in [0, \Delta].$$

The following theorem gives an upper bound on the state achievable under the quantization and the clock offset.

THEOREM 3. *If $0 \leq \Delta < \bar{\Delta}$, that is, if the offset bound Δ satisfies the stability limitation given in Theorem 1, then*

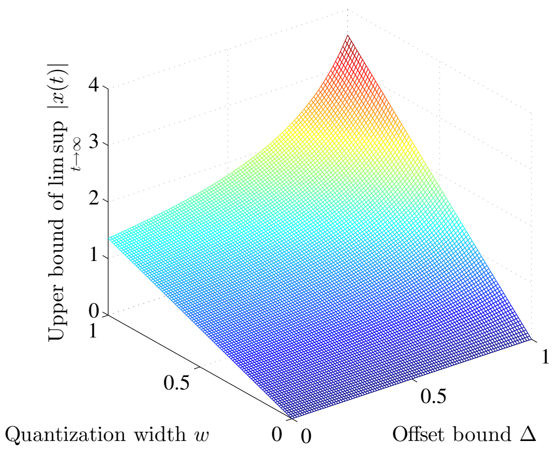


Figure 8: Upper bound of $\limsup_{t\to\infty} |x(t)|$: $\lambda = 1.0$, $T = 1.0$, which is greater than $(\ln 2)/\lambda \approx 0.69$.

there exists a control input $u(t)$, $t \geq 0$, such that

$$\limsup_{t\to\infty} |x(t)| \leq \begin{cases} \frac{e^{\lambda T}}{2} w & \text{if } 0 < T \leq \frac{1}{\lambda}\ln 2, \\ \frac{e^{\lambda T}}{2} \frac{w}{(e^{\lambda\Delta}-1)(1-e^{\lambda T}/2)+1} & \text{if } T > \frac{1}{\lambda}\ln 2. \end{cases} \tag{35}$$

We remark that under $0 \leq \Delta < \overline{\Delta}$ and $T > (\ln 2)/\lambda$, the two expressions in (35) satisfy

$$\frac{e^{\lambda T}}{2} w < \frac{e^{\lambda T}}{2} \frac{w}{(e^{\lambda\Delta}-1)(1-e^{\lambda T}/2)+1}.$$

This fact is consistent with one's intuition in which we have tighter upper bound on $\limsup_{t\to\infty} |x(t)|$ for smaller sampling interval T. The example below illustrates how the offset bound Δ and the width of quantization w affect on the upper bound (35).

Example 4. Consider the scalar system where $\lambda = 1.0$ and $T = 1.0$. In Figure 8, we have plotted the upper bound on $\limsup_{t\to\infty} |x(t)|$ against the offset bound Δ and the width of quantization w. Note that $T = 1.0 > (\ln 2)/\lambda \approx 0.69$. Hence, the figure represents the second expression in (35). We observe that the bound (35) is monotonically increasing linearly with w and exponentially with Δ.

To show Theorem 3, we follow the approach in the proofs of Theorem 1 and Lemma 1. The main difference for this case is that, due to the imperfection by quantization, the estimation set J_k computed from the observation y_k now has two uncertain parameters; the offset and the quantization error.

Proof of Theorem 3. With (16) and the definition of the quantizer (34), the estimation set J_k based on the quantized observation y_k results in

$$J_k = \Big\{ e^{-\lambda\delta_k} \left(y_k + r_k + \lambda^{-1} u_k \right) - \lambda^{-1} u_k \\ : \delta_k \in [0, \Delta], r_k \in \left[-\frac{w}{2}, \frac{w}{2} \right] \Big\}. \tag{36}$$

Here, r_k is the quantization error in y_k. By following the proof of Lemma 1, we introduce the notations:

$$\tilde{y} := y_k + \lambda^{-1} u_k, \quad \tilde{J} := \{ z + \lambda^{-1} u_k : z \in J_k \}.$$

From (36), $\tilde{J}$ is expressed as

$$\tilde{J} = \begin{cases} \left[e^{-\lambda\Delta} \left(\tilde{y} - \frac{w}{2} \right), \tilde{y} + \frac{w}{2} \right] & \text{if } \tilde{y} - \frac{w}{2} > 0, \\ \left[-\frac{w}{2}, \frac{w}{2} \right] & \text{if } -\frac{w}{2} \leq \tilde{y} \leq \frac{w}{2}, \\ \left[\tilde{y} - \frac{w}{2}, e^{-\lambda\Delta} \left(\tilde{y} + \frac{w}{2} \right) \right] & \text{if } \tilde{y} + \frac{w}{2} < 0. \end{cases} \tag{37}$$

Now fix I_k^- and find an upper bound of the length l_k of the updated estimation set $I_k = I_k^- \cap J_k$. Let $\tilde{I}^- := \{ z + \lambda^{-1} u_k : z \in I_k^- \} = [p, q]$ and $\tilde{I} := \tilde{I}^- \cap \tilde{J}$. In view of the proof of Lemma 1, we take the control input $u(t)$, $t \in ((k-1)T + \Delta, kT)$ to make $\tilde{I}^-$ symmetry about the origin. Under such this input, we have

$$|q| = |p| = \frac{e^{\lambda T} l_{k-1}}{2}. \tag{38}$$

We show that under this control input, the length l_k of I_k or $\tilde{I}$ is bounded as follows:

$$\max_{x_k \in I_k^-, \delta_k \in [0,\Delta]} l_k \leq \begin{cases} w & \text{if } l_{k-1} \leq 2e^{-\lambda T} w, \\ (1 - e^{-\lambda\Delta}) \frac{e^{\lambda T}}{2} l_{k-1} + w e^{-\lambda\Delta} & \text{else.} \end{cases} \tag{39}$$

From (38), $p < 0$ and $q > 0$, and hence by (37), we have

$$\tilde{I} = \begin{cases} \left[e^{-\lambda\Delta} \left(\tilde{y} - \frac{w}{2} \right), \min\left(q, \tilde{y} + \frac{w}{2} \right) \right] & \text{if } \tilde{y} \geq \frac{w}{2}, \\ \left[\max\left(p, -\frac{w}{2} \right), \min\left(q, \frac{w}{2} \right) \right] & \text{if } -\frac{w}{2} \leq \tilde{y} < \frac{w}{2}, \\ \left[\max\left(p, \tilde{y} - \frac{w}{2} \right), e^{-\lambda\Delta} \left(\tilde{y} + \frac{w}{2} \right) \right] & \text{if } \tilde{y} < -\frac{w}{2}. \end{cases}$$

Therefore,

$$l_k \leq \begin{cases} (1 - e^{-\lambda\Delta}) q + w e^{-\lambda\Delta} & \text{if } \tilde{y} \geq \frac{w}{2}, \\ w & \text{if } -\frac{w}{2} \leq \tilde{y} < \frac{w}{2}, \\ -(1 - e^{-\lambda\Delta}) p + w e^{-\lambda\Delta} & \text{if } \tilde{y} < -\frac{w}{2}. \end{cases}$$

Substituting (38) into the above upper bounds yields the following inequality between l_k and l_{k-1}:

$$l_k \leq \begin{cases} w & \text{if } -\frac{w}{2} \leq \tilde{y} < \frac{w}{2}, \\ \left(1 - e^{-\lambda\Delta} \right) \frac{e^{\lambda T}}{2} l_{k-1} + w e^{-\lambda\Delta} & \text{else.} \end{cases}$$

Comparing these two upper bounds, we see that

$$w \geq \left(1 - e^{-\lambda\Delta} \right) e^{\lambda T} \frac{l_{k-1}}{2} + w e^{-\lambda\Delta}$$

is equivalent to $l_{k-1} \leq 2e^{-\lambda T} w$. Thus, the inequality (39) holds.

We next consider an upper bound on $\limsup_{k\to\infty} l_k$ from (39) and derive the bound (35) on $\limsup_{t\to\infty} |x(t)|$. Define the sequence $\{l'_k\}_{k=0}^{\infty}$ by

$$l'_{k+1} := \left(1 - e^{-\lambda\Delta} \right) \frac{e^{\lambda T}}{2} l'_k + w e^{-\lambda\Delta}, \quad l'_0 := l_0.$$

Under the condition $0 \leq \Delta < \overline{\Delta}$, l'_k converges to

$$\frac{w}{(e^{\lambda\Delta} - 1)(1 - e^{\lambda T}/2) + 1},$$

and this limit satisfies

$$w \geq \frac{w}{(e^{\lambda\Delta} - 1)(1 - e^{\lambda T}/2) + 1}$$

for the case $0 < T \le (\ln 2)/\lambda$. Hence, by (39), it follows that

$$\limsup_{k\to\infty} l_k \le \begin{cases} w & \text{if } 0 < T \le \frac{1}{\lambda}\ln 2, \\ \frac{w}{(e^{\lambda\Delta}-1)(1-e^{\lambda T}/2)+1} & \text{if } T > \frac{1}{\lambda}\ln 2, \end{cases} \tag{40}$$

for any $x(0)$ and $\delta_k \in [0, \Delta]$, $k \in \mathbb{Z}_+$. Finally, for all $t \in [kT, (k+1)T]$, $k \in \mathbb{Z}_+$, the plant state is bounded as

$$|x(t)| \le e^{\lambda T}|x(kT)| \le e^{\lambda T}\frac{l_k}{2}.$$

This fact and (40) conclude the theorem. $\square$

7. CONCLUSION

This paper has studied stabilization of a linear system using an asynchronous pair of a sensor and a controller. We have derived conditions on the clock offset for stabilization of the system with a single controller. The condition in the scalar plants case gives a tight limitation on the offset, which depends on the level of instability of the plant and the sampling period. For the vector plants case, we have shown that if the plant has at least two different real poles, then the system is stabilizable in finite steps for any large offsets. Although a stabilizing algorithm has been given in the proof, it is based on controllability gramians and is not very suitable for practical controllers. It would be an interesting topic left for future work to study practical control algorithms. Another topic is the investigation of quantization effects for the vector plants case. In this paper, quantization has been considered in the scalar plants case. The main challenge in the vector case is to evaluate the volume of the intersection of I_k^- and J_k, which result in bands of exponential curves in $\mathbb{R}^n$. We also would like to consider more practical situations including control under noises and uncertainty in the plant dynamics.

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