# Multiple model adaptive control, part 2: Switching*

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**Abstract**

This paper addresses the problem of controlling a continuous-time linear system with large modeling errors. We employ an adaptive control algorithm consisting of a family of linear candidate controllers supervised by a high-level switching logic. Methods for constructing such controller families have been discussed in the recent paper by the authors. The present paper concentrates on the switching task in a multiple model context. We describe and compare two different switching logics, and in each case study the behavior of the resulting closed-loop hybrid system.

*Keywords*: Linear system, modeling uncertainty, supervisory control, switching logic, hybrid system.

## 1 Introduction

In this paper we study the problem of controlling a continuous-time SISO linear system in the presence of large-scale modeling uncertainty, noise, and disturbances. The main idea behind the *supervisory control* approach to this problem is to employ logic-based switching among a family of candidate controllers. The need for switching stems from the fact that typically no single controller can guarantee a desired behavior when connected with the poorly modeled process. Such switching schemes provide an alternative to more traditional continuously tuned adaptive control algorithms. The switching control methodology considered here is "estimator-based", as it relies on an estimation procedure in carrying out on-line controller selection. For background on this subject, see [15, 16].

In addition to the given process and a family of candidate controllers, the overall supervisory control system contains three other subsystems: a *multi-estimator*, a *monitoring signal generator*, and a *switching logic*. The task of the switching logic is to generate a *switching signal* which determines, at each instant of time, the candidate controller that is to be placed in the feedback loop. Controller selection is based on the values of *monitoring signals*, which are obtained by taking integral norms of suitably defined *estimation errors* produced by the multi-estimator. The basic idea behind the switching strategy is to determine which of the monitoring signals is the smallest, and then choose the corresponding candidate controller. This amounts to applying a controller designed for the model that best fits the available data ("certainty equivalence").

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This is the second in the series of two papers examining the state-of-the-art of supervisory control methods for uncertain linear systems. The previous paper [1] has addressed various aspects of the problem of choosing the family of candidate controllers. The focus of this paper is on switching between such controllers. The closed-loop systems arising in the supervisory control framework are *hybrid systems*, as they combine discrete dynamics associated with the switching logic and continuous dynamics associated with the rest of the system. In what follows, we describe two different switching logics, and in each case analyze the behavior of the overall hybrid system. One of these switching logics is new, and enables us to overcome technical difficulties in those situations where the parameter space is a continuum.

The supervisory control design procedure outlined above is "modular", in the sense that the principles governing the design of the switching logic, of the multi-estimator, and of the candidate controllers are essentially independent of each other. The analysis of the overall system relies on basic properties of its different parts, but not on a particular way in which they are implemented. As a result, one gains the advantage of being able to use "off-the-shelf" control laws, rather than having to design control laws tailored to the specifics of the switching algorithm. Our goal therefore is to make maximal use of relevant concepts from non-adaptive control theory. While the class of systems treated in this paper can also be handled by the control algorithm described in [15, 16] which uses a different switching logic, the switching control design and analysis tools presented here are potentially applicable in a broader context; in fact, they have already been applied to certain classes of nonlinear systems in [6, 7, 10]. In addition, the proofs are considerably simplified compared to the ones given in earlier work.

## 2 Problem formulation

We assume that the uncertain process $\mathbb{P}$ to be controlled admits the model of a SISO finite-dimensional stabilizable and observable linear system with control input $u$ and measured output $y$, perturbed by a bounded disturbance input $d$ and a bounded output noise signal $n$. It is assumed known that the transfer function of $\mathbb{P}$ from $u$ to $y$ belongs to a family of admissible process model transfer functions

$$\bigcup_{p\in\mathcal{P}} \mathcal{F}(p)$$

where $p$ is a parameter taking values in some index set $\mathcal{P}$. Here, for each $p$, $\mathcal{F}(p)$ denotes a family of transfer functions "centered" around some known *nominal* process model transfer function $\nu_p$ (cf. below). Throughout the paper, we will take $\mathcal{P}$ to be a compact subset of a finite-dimensional normed linear vector space.

The problem of interest is to design a feedback controller that achieves output regulation, i.e., drives the output $y$ of $\mathbb{P}$ to zero, whenever the noise and disturbance signals are zero. Moreover, all system signals must remain (uniformly) bounded in response to arbitrary bounded noise and disturbance inputs. Everything that follows can be readily extended to the more general problem of set-point control (i.e., tracking an arbitrary constant reference $r$) with the help of adding an integrator in the feedback loop, as in [15, 16]. Such a modification would not introduce any significant changes as far as the principal developments of this paper are concerned. Control algorithms of the type described here can also be applied to the problem of disturbance suppression [4].

The set $\mathcal{P}$ can be thought of as representing the range of parametric uncertainty, while for each fixed $p \in \mathcal{P}$ the subfamily $\mathcal{F}(p)$ accounts for unmodeled dynamics. There are several ways of specifying allowable unmodeled dynamics around the nominal process model transfer functions $\nu_p$. For example, take two arbitrary numbers $\delta > 0$ and $\lambda_u \geq 0$. Then we can define

$$\mathcal{F}(p) := \{\nu_p(1+\delta_p^m) + \delta_p^a : \|\delta_p^m\|_{\infty,\lambda_u} \leq \delta, \|\delta_p^a\|_{\infty,\lambda_u} \leq \delta\}, \qquad p \in \mathcal{P} \tag{1}$$

where $\|\cdot\|_{\infty,\lambda_u}$ denotes the $e^{\lambda_u t}$-weighted $\mathcal{H}_\infty$ norm of a transfer function: $\|\nu\|_{\infty,\lambda_u} = \sup_{\omega\in\mathbb{R}} |\nu(j\omega - \lambda_u)|$. This yields the class of admissible process models treated in [15, 16]. Alternatively, one can define $\mathcal{F}(p)$ to be the ball of radius $\delta$ around $\nu_p$ with respect to the Vinnicombe metric [20]. Another possible definition is

$$\mathcal{F}(p) := \left\{ \frac{n_p + \delta_p^n}{m_p + \delta_p^m} : \|\delta_p^n\|_{\infty,\lambda_u} \leq \delta, \|\delta_p^m\|_{\infty,\lambda_u} \leq \delta \right\}, \qquad p \in \mathcal{P} \tag{2}$$

where $\nu_p = n_p/m_p$ is the normalized coprime factorization of $\nu_p$ (see, e.g., [22]). This is more general than (1) in that it allows for uncertainty about the pole locations of the nominal process model transfer functions. In the sequel, allowable unmodeled dynamics are assumed to be specified in either one of the aforementioned ways; we will refer to the positive parameter $\delta$ as the *unmodeled dynamics bound.*

Modeling uncertainty of the kind described above may be associated with unpredictable changes in operating environment, component failure, or various external influences. Typically, no single controller is capable of solving the regulation problem for the entire family of admissible process models. Therefore, one needs to develop a controller whose dynamics can change on the basis of available real-time data. Within the framework of supervisory control discussed here, this task is carried out by a "high-level" controller, called a *supervisor*, whose purpose is to orchestrate the switching among a parameterized family of *candidate controllers*

$$\{\mathbb{C}_q : q \in \mathcal{Q}\} \tag{3}$$

where $\mathcal{Q}$ is an index set. We require this controller family to be sufficiently rich so that every admissible process model can be stabilized by placing in the feedback loop the controller $\mathbb{C}_q$ for some index $q \in \mathcal{Q}$.

In this paper, we will focus exclusively on the case when $\mathcal{Q} = \mathcal{P}$ (although in some situations it may be useful to choose $\mathcal{Q}$ different from $\mathcal{P}$; see [14]). For each $p \in \mathcal{P}$, we take for $\mathbb{C}_p$ a controller that stabilizes all the process models in $\mathcal{F}(p)$; such controllers should exist if unmodeled dynamics are sufficiently small. However, $\mathcal{P}$ is typically a continuum, and while it is possible to switch among a continuum of controllers [15, 16], it is often easier to work with a finite controller family. There are various reasons for considering this alternative, having to do with tractability of implementation and analysis. For example, parameters may enter in complicated ways, making estimation over a continuum intractable; the step of actually constructing a continuum of high-performance controllers for a continuum of process models is challenging and certainly constraining, especially if the controllers are to be designed using linear quadratic or $\mathcal{H}_\infty$ techniques. For linear MIMO or nonlinear systems these issues become even more severe.

Therefore, one may want to choose a different parameter space which is finite, and absorb the remaining parameter values into unmodeled dynamics. In carrying out such a program, one is faced with the problem of constructing a *finite* family of controllers having the property that every admissible process model is stabilized by at least one of these controllers. This problem has a solution if unmodeled dynamics are sufficiently small and if the topologies on the parameter space and on the space of transfer functions defining unmodeled dynamics are "compatible", in the sense that $\nu_p$

is contained in $\mathcal{F}(\bar{p})$ whenever $p$ and $\bar{p}$ are sufficiently close in $\mathcal{P}$ (see [1, Theorem 5.1]). The last assumption is always satisfied when unmodeled dynamics are defined via coprime factorizations or the Vinnicombe metric as outlined earlier, provided that $\nu_p$ depends continuously on $p$. The earlier work of Pait [17, 18] is also relevant in this regard.

The reader is thus referred to [1] for issues regarding the construction of controller families like (3), especially finite ones. In the sequel we will assume that a family of controllers is given, and discuss various ways of carrying out logic-based switching among these controllers to solve the output regulation problem for the unknown process $\mathbb{P}$.

## 3 Estimator-based supervisor

The supervisor consists of three subsystems (see Figure 1):

*multi-estimator* $\mathbb{E}$ – a dynamical system whose inputs are the output $y$ and the input $u$ of the process $\mathbb{P}$ and whose outputs are the signals $y_p$, $p \in \mathcal{P}$. Each $y_p$ would converge to $y$ asymptotically if the transfer function of $\mathbb{P}$ were equal to the nominal process model transfer function $\nu_p$ and there were no noise or disturbances.

*monitoring signal generator* $\mathbb{M}$ – a dynamical system whose inputs are the *estimation errors*

$$e_p := y_p - y, \qquad p \in \mathcal{P} \tag{4}$$

and whose outputs $\mu_p$, $p \in \mathcal{P}$ are suitably defined integral norms of the estimation errors, called *monitoring signals*.

*switching logic* $\mathbb{S}$ – a switched system whose inputs are the monitoring signals $\mu_p$, $p \in \mathcal{P}$ and whose output is a *switching signal* $\sigma$, taking values in $\mathcal{P}$, which is used to define the control law $u$.

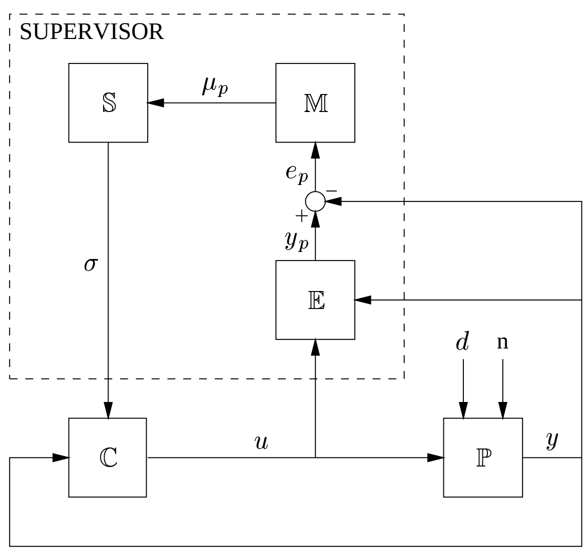


Figure 1: Supervisory control architecture

The auxiliary input $w$ to $\mathbb{P}$ accounts for the noise, disturbance, and possibly an external reference signal. We now briefly recall from [15] the key state-space equations for the different subsystems appearing in Figure 1. As $p$ ranges over $\mathcal{P}$, let

$$\begin{aligned} \dot{x}_{\mathbb{C}} &= A_p x_{\mathbb{C}} + b_p y \\ u &= k_p x_{\mathbb{C}} + r_p y \end{aligned}$$

be realizations of the transfer functions of the candidate controllers from (3), all sharing the same state $x_{\mathbb{C}}$. We then define the *multi-controller* $\mathbb{C}$ to be the system

$$\begin{aligned} \dot{x}_{\mathbb{C}} &= A_\sigma x_{\mathbb{C}} + b_\sigma y \\ u &= k_\sigma x_{\mathbb{C}} + r_\sigma y \end{aligned}$$

Such a multi-controller can be obtained using standard results from realization theory.

We assume that the multi-estimator is also realized in a state-shared fashion, as given by

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}} x_{\mathbb{E}} + b_{\mathbb{E}} y + d_{\mathbb{E}} u \\ y_p &= c_p x_{\mathbb{E}}\,, \qquad p \in \mathcal{P} \end{aligned}$$

with $A_{\mathbb{E}}$ a stable matrix. This type of structure is quite common in adaptive control. Note that even if $\mathcal{P}$ is an infinite set, the above dynamical system is finite-dimensional. In this case it formally has an infinite number of outputs; however, these outputs will only be needed in the minimization procedure used by the switching logic. Thus, if the associated minimization problem is tractable, then so is the implementation of the overall supervisory control algorithm.

Denoting by $x$ the composite state $(x'_{\mathbb{E}}, x'_{\mathbb{C}})'$ of the multi-estimator and the multi-controller, and using (4), we arrive at a system of the form

$$\begin{aligned} \dot{x} &= A_\sigma x + d_\sigma e_\sigma \\ y &= (c_{p^*}\ \ 0)\, x - e_{p^*} \\ u &= f_\sigma x + g_\sigma e_{p^*} \end{aligned} \tag{5}$$

Here $A_p$, $p \in \mathcal{P}$ are stable matrices all of whose eigenvalues have real parts smaller than $-\lambda_0$ for some $\lambda_0 > 0$, and $p^*$ is an (unknown) element of $\mathcal{P}$ such that the transfer function of $\mathbb{P}$ belongs to $\mathcal{F}(p^*)$, i.e., a "true" parameter value. In addition, the functions $p \mapsto A_p$, $p \mapsto c_p$, $p \mapsto d_p$, $p \mapsto f_p$, and $p \mapsto g_p$ are assumed to be continuous[1] on $\mathcal{P}$. See Sections IV and V of [15] for details on this construction.

We will demand that the following condition hold: there exist a positive number $\lambda$, positive constants $\delta_1, \delta_2$ that only depend on the unmodeled dynamics bound $\delta$ and go to zero as $\delta$ goes to zero, positive constants $B_1, B_2$ that only depend on the noise and disturbance bounds and go to zero as these bounds go to zero, and positive constants $C_1, C_2$ that only depend on the system's parameters and on initial conditions, such that along all solutions of the closed-loop system we have

$$\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau) d\tau \le B_1 e^{2\lambda t} + C_1 + \delta_1 \int_0^t e^{2\lambda\tau} u^2(\tau) d\tau \tag{6}$$

[1] The continuity assumption is made here for simplicity, and can be relaxed in most of what follows. It is of course vacuous if $\mathcal{P}$ is a finite set.

and

$$|e_{p^*}(t)| \leq B_2 + C_2 e^{-\lambda t} + \delta_2 e^{-\lambda t} \sqrt{\int_0^t e^{2\lambda\tau} u^2(\tau) d\tau} \tag{7}$$

The above inequalities represent the basic requirements being placed on the multi-controller and the multi-estimator, upon which the subsequent analysis depends. Intuitively, they mean that $y_{p^*}$ is a "good" approximation of $y$, and provide an explicit characterization of the error $e_{p^*} = y_{p^*} - y$ in the presence of noise, disturbances, and unmodeled dynamics. In the case when unmodeled dynamics are defined via the formula (1), it was shown in [15, 16] how one can construct a supervisory control system satisfying these requirements for each $\lambda \in (0, \min\{\lambda_u, \lambda_0\})$; the desired inequalities were immediate consequences of the equation (28) in [15]. A system with such properties can also be designed in a similar way for the case when unmodeled dynamics are defined via the formula (2) in terms of coprime factors (cf. [9]).

The constant $\lambda$ from (6) and (7) will play the role of a "weighting" design parameter in the definition of the monitoring signals; we assume, with no loss of generality, that $\lambda < \lambda_0$. Fix an arbitrary constant $\epsilon_\mu \geq 0$ (its role will become clear in Section 5.2). We generate the monitoring signals $\mu_p$, $p \in \mathcal{P}$ by the equations

$$\begin{aligned} \dot{W} &= -2\lambda W + \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix} \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix}', \qquad W(0) \geq 0 \\ \mu_p &:= \begin{pmatrix} c_p & -1 \end{pmatrix} W \begin{pmatrix} c_p & -1 \end{pmatrix}' + \epsilon_\mu, \qquad p \in \mathcal{P} \end{aligned} \tag{8}$$

where $W(t)$ is a symmetric nonnegative-definite $k \times k$ matrix, $k := \dim(x_{\mathbb{E}}) + 1$. Since $c_p x_{\mathbb{E}} - y = e_p \; \forall p \in \mathcal{P}$, this yields

$$\mu_p(t) = e^{-2\lambda t} \tilde{\mu}_p(0) + \int_0^t e^{-2\lambda(t-\tau)} e_p^2(\tau) d\tau + \epsilon_\mu, \qquad p \in \mathcal{P} \tag{9}$$

where $\tilde{\mu}_p(0) := \begin{pmatrix} c_p & -1 \end{pmatrix} W(0) \begin{pmatrix} c_p & -1 \end{pmatrix}'$. The underlying decision-making strategy employed by the supervisor basically consists in selecting for $\sigma$, from time to time, the candidate controller index $q$ whose corresponding monitoring signal $\mu_q$ is the smallest. The origin of this idea is the concept of certainty equivalence from parameter adaptive control. A precise description of a switching logic that implements this approach is given next.

# 4 Scale-independent hysteresis switching logic

So far we have closely followed the set-up of [15, 16]. However, the switching logic that we will consider here is different from the one employed there. Namely, we will work with the *scale-independent hysteresis switching logic* [5, 11], whose functioning is as follows (see Figure 2). Assumed given are continuous monitoring signals $\mu_p$, $p \in \mathcal{P}$ (to define the switching logic, we do not need to require that they be defined by the above formulas, although this is the case to which we will specialize later). Let us pick a number $h > 0$ called the *hysteresis constant*. First, we set $\sigma(0) = \arg\min_{p \in \mathcal{P}} \{\mu_p(0)\}$. Suppose that at a certain time $t_i$ the value of $\sigma$ has just switched to some $q \in \mathcal{P}$. We then keep $\sigma$ fixed until a time $t_{i+1} > t_i$ such that $(1+h) \min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\} \leq \mu_q(t_{i+1})$, at which point we set $\sigma(t_{i+1}) = \arg\min_{p \in \mathcal{P}} \{\mu_p(t_{i+1})\}$. When the indicated arg min is not unique, a particular value for $\sigma$ among those that achieve the minimum can be chosen arbitrarily. We refer the reader to [15]

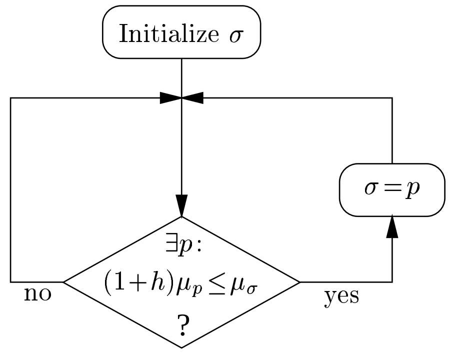


Figure 2: Scale-independent hysteresis switching logic

for a discussion of tractability issues regarding the minimization procedure (especially when $\mathcal{P}$ is a continuum).

Repeating the above steps, we generate a piecewise constant signal $\sigma$ which is continuous from the right everywhere. The overall supervisory control system is a hybrid system, $\sigma$ being its discrete state. If all $\mu_p$, $p \in \mathcal{P}$ are bounded below by some positive number, then infinitely fast switching (chattering) is avoided. In fact, there must be an interval $[0, T_{\max})$ of maximal length on which $\sigma$ exists, and there can only be a finite number of switches on each proper subinterval of $[0, T_{\max})$; see [5, 11] for details. In all the cases treated below, it will be clear that $T_{\max} = \infty$. Hybrid systems having the above property are known as *non-Zeno*.

One can easily see from the definition of the above switching logic that replacing the signals $\mu_p$, $p \in \mathcal{P}$ by their scaled versions

$$\bar{\mu}_p(t) := \Theta(t)\mu_p(t), \qquad p \in \mathcal{P}$$

where $\Theta$ is some positive function of time, would have no effect on $\sigma$; this is where the term "scale-independent" comes from. In the sequel, we assume that it is possible to choose $\Theta$ in such a way that the scaled signals $\bar{\mu}_p$, $p \in \mathcal{P}$ are strictly positive and monotone increasing (we will see later that the monitoring signals defined in the previous section satisfy this assumption). For analysis purposes we will always use the signals $\bar{\mu}_p$, $p \in \mathcal{P}$ that have this property, while the actual monitoring signals being implemented and used for the switching logic are the original ones $\mu_p$, $p \in \mathcal{P}$.

The analysis of Section 5 is restricted to the case when $\mathcal{P}$ is a finite set. In most situations of interest, the parameter space $\mathcal{P}$ is a continuum. However, it is often possible to work with a different parameter space which is finite, using the results of [1] (see Section 2). The implications of this approach, as well as some alternatives, will be discussed later. One reason for requiring $\mathcal{P}$ to be finite is to be able to use the result from [8] stated next. Given arbitrary numbers $t > t_0 \geq 0$, we will denote by $N_\sigma(t, t_0)$ the number of discontinuities of $\sigma$ on the interval $(t_0, t)$ or, more precisely, on the interval $(t_0, \min\{t, T_{\max}\})$ if the finiteness of $T_{\max}$ cannot be ruled out a priori.

**Lemma 1** [8] *Let $\mathcal{P} = \{1, 2, \ldots, m\}$ with $m$ a positive integer. For arbitrary $t > t_0 \geq 0$ we have*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\mu}_{p^*}(t)}{\min_{p \in \mathcal{P}} \bar{\mu}_p(t_0)}\right) \tag{10}$$

*and*

$$\sum_{k=0}^{N_\sigma(t,t_0)} \big(\bar\mu_{\sigma(t_k)}(t_{k+1}) - \bar\mu_{\sigma(t_k)}(t_k)\big) \le m\big((1+h)\bar\mu_{p^*}(t) - \min_{p\in\mathcal{P}} \bar\mu_p(t_0)\big) \tag{11}$$

*where* $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ *are the discontinuities of* $\sigma$ *and* $t_{N_\sigma(t,t_0)+1} := t$.

This lemma expresses a general property of the switching logic, which does not depend on the structure of the system or on a particular definition of the monitoring signals. Let us now restrict our attention to the case when the monitoring signals are obtained from the equations (8). Setting $\Theta(t) := e^{2\lambda t}$, we see from the formula (9) that the scaled signals $\bar\mu_p$, $p \in \mathcal{P}$ are indeed monotone increasing, because they satisfy

$$\bar\mu_p(t) = \tilde\mu_p(0) + \int_0^t e^{2\lambda\tau} e_p^2(\tau)d\tau + \epsilon_\mu e^{2\lambda t}, \qquad p \in \mathcal{P} \tag{12}$$

Moreover, it is easy to ensure that $\mu_p(0) > 0\ \forall p \in \mathcal{P}$, either by setting $\epsilon_\mu > 0$ or by requiring $W(0)$ to be positive definite. Therefore, we can apply Lemma 1 and conclude that the inequalities (10) and (11) are valid. Since in this case the signals $\bar\mu_p$, $p \in \mathcal{P}$ are differentiable, the left-hand side of the inequality (11) equals the integral $\int_{t_0}^t \dot{\bar\mu}_{\sigma(\tau)}(\tau)d\tau$, which is to be interpreted as the sum of integrals over intervals on which $\sigma$ is constant. From (12) we have the following formula:

$$\dot{\bar\mu}_p(t) = e^{2\lambda t} e_p^2(t) + 2\lambda\epsilon_\mu e^{2\lambda t}, \qquad p \in \mathcal{P} \tag{13}$$

# 5 Analysis: finite $\mathcal{P}$

Take $\mathcal{P} = \{1, 2, \ldots, m\}$ as in the previous section. We now proceed to the analysis of the supervisory control system defined by (5), (8), and the switching logic described above. We will sometimes appeal to the state of the uncertain process $\mathbb{P}$, which we denote by $x_{\mathbb{P}}$.

## 5.1 No noise, disturbances, or unmodeled dynamics

We begin by considering the simple situation where there are no unmodeled dynamics ($\delta = 0$), i.e., the process $\mathbb{P}$ exactly matches one of the $m$ nominal process models, and where the noise and disturbance signals are zero ($n = d \equiv 0$). In this case, the constants $B_1, B_2, \delta_1, \delta_2$ in (6) and (7) are all zero. Let us take $\epsilon_\mu$ in the definition of the monitoring signals to be zero as well ($W(0)$ must then be taken positive definite; see Section 4). The inequality (6) gives $\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau \le C_1$, which together with (12) implies that $\bar\mu_{p^*}$ is bounded. It follows from (10) that $N_\sigma(t, t_0)$ is finite for arbitrary $t > t_0 \ge 0$. This means that the switching stops in finite time at some index $q^* \in \mathcal{P}$, i.e., there exists a time $T^*$ such that $\sigma(t) = q^*$ for $t \ge T^*$. Since $\bar\mu_{p^*}$ is bounded, we conclude from the definition of the switching logic that $\bar\mu_{q^*}$ is also bounded. In view of (12), we have in particular $e_{q^*} \in L^2$. Now it is not hard to see from (5) that, since $A_{q^*}$ is a stable matrix and $e_{p^*} \to 0$ by virtue of (7), we have $y \to 0$. Therefore, the output regulation problem is solved. In light of (5), (8), and detectability of $\mathbb{P}$, all the other signals remain bounded for all $t \ge 0$ (in fact, here we have $x_{\mathbb{P}} \to 0$, but this would no longer be true in the case of set-point control where $y \to r \neq 0$). We summarize this as follows.

**Proposition 2** *Suppose that the noise and disturbance signals are zero and there are no unmodeled dynamics, and set $\epsilon_\mu = 0$. Then all the signals in the supervisory control system remain bounded for every set of initial conditions such that $W(0) > 0$. Moreover, the switching stops in finite time, and we have $y(t) \to 0$ as $t \to \infty$.*

REMARK. Since the evolution of $x_{\mathbb{P}}$ and $x$ for $t \geq T^*$ is described by a linear time-invariant system, the rate of convergence in the above statement is actually exponential.

### 5.2 Noise and disturbances, no unmodeled dynamics

We now assume that bounded noise $n$ and disturbance $d$ are present, but there are no unmodeled dynamics. In this case the switching typically will not stop in finite time. In analyzing the supervisory control system under persistent switching, the following concept proves useful. Following [12], we say that a switching signal $\sigma$ has an *average dwell time* $\tau_{AD} > 0$ if there exists a nonnegative number $N_0$ such that the number of discontinuities of $\sigma$ on an arbitrary interval $(t_0, t)$ satisfies

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_{AD}}.$$

It was shown in [12] that if $\{B_p : p \in \mathcal{P}\}$ is a compact set of stable $n \times n$ matrices, then the switched linear system $\dot{z} = B_\sigma z$, $z \in \mathbb{R}^n$ is uniformly exponentially stable for all switching signals $\sigma$ with a sufficiently large average dwell time. Moreover, by choosing a large enough $\tau_{AD}$, the stability margin of this switched system can be made arbitrarily close to the smallest stability margin of the individual subsystems. This result will be crucial in the analysis given below.

Under the present assumptions, the inequalities (6) and (7) hold with some unknown but finite constants $B_1, B_2$. The parameters $\delta_1$ and $\delta_2$ are still zero, and $C_1$ and $C_2$ are positive constants as before. We take $\epsilon_\mu$ to be a positive number. From (6) and (12) we have

$$\bar{\mu}_{p^*}(t) \leq \tilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \epsilon_\mu e^{2\lambda t} \tag{14}$$

The formula (10) then yields

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\tilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \epsilon_\mu e^{2\lambda t}}{\epsilon_\mu e^{2\lambda t_0}}\right) \leq N_0 + \frac{t - t_0}{\tau_{AD}}$$

where

$$\tau_{AD} = \frac{\log(1+h)}{2\lambda m} \quad \text{and} \quad N_0 = 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\tilde{\mu}_{p^*}(0) + B_1 + C_1 + \epsilon_\mu}{\epsilon_\mu}\right)$$

The number $\tau_{AD}$ is the average dwell time of $\sigma$. It follows from the main result of [12] that if $\tau_{AD}$ is large enough, then the switched linear system $\dot{x} = A_\sigma x$ is uniformly exponentially stable, with stability margin $\lambda$. This implies that the switched system given by the first equation in (5) has a finite $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norm, which means that for some positive constants $g, g_0$ and for all $t \geq 0$ we have

$$e^{2\lambda t}|x(t)|^2 \leq g \int_0^t e^{2\lambda\tau} e^2_{\sigma(\tau)}(\tau) d\tau + g_0 |x(0)|^2 \tag{15}$$

We can guarantee that $\tau_{AD}$ is large enough by increasing the hysteresis constant $h$ and/or decreasing the weighting constant $\lambda$ if necessary. The desired lower bound for $\tau_{AD}$ can be calculated explicitly from the proof of the main result in [12]. In the sequel, we assume that $h$ and $\lambda$ have been chosen in such a way that (15) holds.

Using (11), (13), and (14), we obtain

$$\int_0^t e^{2\lambda\tau} e^2_{\sigma(\tau)}(\tau) d\tau \leq m\big((1+h)(\tilde{\mu}_{p^*}(0) + B_1 e^{2\lambda t} + C_1 + \epsilon_\mu e^{2\lambda t}) - \epsilon_\mu e^{2\lambda t_0}\big)$$

Together with (15) this implies that

$$|x(t)|^2 \leq \big(gm(1+h)(\tilde{\mu}_{p^*}(0) + C_1) + g_0|x(0)|^2\big)e^{-2\lambda t} + gm(1+h)(B_1 + \epsilon_\mu)$$

Two conclusions can be drawn from the last formula. First, $x$ is bounded, and as in the previous subsection we can easily deduce from (5), (7), (8), and detectability of $\mathbb{P}$ that all system signals remain bounded. Note that the choice of the design parameters $\lambda$, $h$ and $\epsilon_\mu$ did not depend on the noise or disturbance bounds, in other words, explicit knowledge of these bounds is not necessary (we are merely requiring that such bounds exist). Second, if $n$ and $d$ equal or converge to zero, then $x$ will approach a neighborhood of the origin whose size is proportional to $g\epsilon_\mu$. A close examination of the last quantity (carried out in the appendix for the more general situation of nonzero unmodeled dynamics) reveals that it decreases to 0 as $\epsilon_\mu$ goes to 0, which means that we can make this neighborhood as small as desired by choosing $\epsilon_\mu$ sufficiently small. Moreover, $e_{p^*}$ will converge to zero because of (7), hence $y$ will also become arbitrarily small in view of (5). We arrive at the following result.

**Proposition 3** *Suppose that the noise and disturbance signals are bounded and there are no unmodeled dynamics. Then for an arbitrary $\epsilon_\mu > 0$ all the signals in the supervisory control system remain bounded for every set of initial conditions. Moreover, for every number $\epsilon_y > 0$ there is a value of $\epsilon_\mu$ leading to the property that if the noise and disturbance signals converge to zero, then for each solution there is a time $\bar{T}$ such that $|y(t)| \leq \epsilon_y$ for all $t \geq \bar{T}$.*

REMARK. We cannot simply let $\epsilon_\mu = 0$, as this would invalidate the above analysis even if $W(0) > 0$. However, by decreasing $\epsilon_\mu$ on-line (e.g., in a piecewise constant fashion), it is possible to recover asymptotic convergence of $y$ to zero under the (not very realistic) assumption that the noise and disturbance signals converge to zero.

### 5.3 Noise, disturbances, and unmodeled dynamics

If unmodeled dynamics are present, i.e., if the parameter $\delta$ is positive, then $\delta_1$ and $\delta_2$ in (6) and (7) are also positive. In this case, the analysis becomes more complicated, because we can no longer deduce from (6) that the switched system must possess an average dwell time. However, it is possible to prove that the above control algorithm, without any modification, is robust with respect to unmodeled dynamics in the following, "semi-global", sense.

**Theorem 4** *For arbitrary bounds on the noise and disturbance signals, the supervisory control system has the following properties:*

1. *For every positive value of $\epsilon_\mu$ and every number $E > 0$ there exists a number $\bar{\delta} > 0$ such that if the unmodeled dynamics bound $\delta$ is smaller than $\bar{\delta}$, then all signals remain bounded for every set of initial conditions such that $|x_{\mathbb{P}}(0)|, |x(0)| \leq E$.*

2. *For arbitrary positive numbers $E$ and $\epsilon_y$ there exist a value of $\epsilon_\mu$ and a number $\bar{\delta} > 0$ such that if the noise and disturbance signals converge to zero and the unmodeled dynamics bound $\delta$ is smaller than $\bar{\delta}$, then for each solution with $|x_{\mathbb{P}}(0)|, |x(0)| \leq E$ there is a time $\bar{T}$ such that $|y(t)| \leq \epsilon_y$ for all $t \geq \bar{T}$.*

Theorem 4 is proved in the appendix in a constructive manner, via a small-gain argument. The difficulty in obtaining a global result can be seen from the proof; it stems from the fact that the gains $g$ and $g_0$ in (15) depend on the initial conditions (through the chatter bound $N_0$). This is the price that one pays for allowing the switching to be slow only on the average.

There is an alternative approach which enables one to obtain the same result; it involves modifying the algorithm by introducing *normalized estimation errors*, allowing for a somewhat simpler analysis. Namely, one can define the monitoring signals by using, instead of (8), the equations

$$\begin{aligned}
\dot{W} &= -2\lambda W + \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix} \begin{pmatrix} x_{\mathbb{E}} \\ y \end{pmatrix}', \qquad W(0) \geq 0 \\
\dot{\zeta} &= -2\lambda\zeta + u^2, \qquad \zeta(0) \geq 0 \\
\mu_p &:= \frac{1}{1+\zeta} \begin{pmatrix} c_p & -1 \end{pmatrix} W \begin{pmatrix} c_p & -1 \end{pmatrix}' + \epsilon_\mu, \qquad p \in \mathcal{P}
\end{aligned}$$

This amounts to replacing the estimation errors $e_p$, $p \in \mathcal{P}$ by their normalized versions $\bar{e}_p := e_p/(1+\zeta)$, $p \in \mathcal{P}$. The formula (7) can be used to deduce that $\bar{e}_{p^*}$ is bounded, which immediately leads to the existence of an average dwell time as in the previous subsection. To finish the analysis, one still needs to apply a small-gain argument.

In [15, 16] a different switching logic was used, which ensures the existence of a dwell time (rather than just an average dwell time) and enables one to prove global stability in the presence of sufficiently small unmodeled dynamics. However, such an approach has its own disadvantages. First, the stability and robustness proofs become more complicated. Perhaps even more importantly, dwell time switching is not suitable for control of nonlinear systems because of the possibility of finite escape times, whereas the scale-independent hysteresis switching logic has already been successfully applied to various classes of nonlinear systems [6, 7, 10].

## 6 Local priority hysteresis switching logic

The analysis given in the previous section relied on the assumption that the controller family is finite. As discussed in [1], even if the range of parametric uncertainty is originally described by a continuum, in many cases of interest one can construct a finite family of stabilizing controllers that possess the desired robustness with respect to unmodeled dynamics. However, we cannot guarantee (at least with currently available analysis tools) that the switching algorithm proposed here preserves the robustness margins associated with the individual controllers. The reason for this can be explained as follows. To establish robustness with respect to unmodeled dynamics, one invariably uses a small-gain argument. However, in the inequality (11), which in the present case basically characterizes the $e^{\lambda t}$-weighted induced $\mathcal{L}_2$-gain from $e_{p^*}$ to $e_\sigma$, the number $m$ of controllers appears on the right-hand side. This implies that if the range of parametric uncertainty is large, which causes $m$ to be large, then the amount of unmodeled dynamics that the switching controller can tolerate becomes small, and might not be sufficient to cover the entire family of admissible process models. It is not hard to see that this problem cannot be overcome by simply increasing the number of controllers being switched.

The above remarks suggest that when the unknown parameters take values in an infinite set, one may sometimes be forced to switch among an infinite family of controllers (as opposed to reducing the problem to that of switching among a finite number of controllers by means of a reparameterization). This is actually the case even in the absence of noise, disturbances, and unmodeled dynamics. If $\mathcal{P}$ is infinite, Lemma 1 is no longer valid, and the analysis given earlier breaks down (in fact, there exist examples in which the switching does not stop even though $\bar{\mu}_{p^*}$ is bounded). In what follows, we propose a new way of dealing with this problem, by means of modifying the switching logic. This enables us to demonstrate how the behavior of a hysteresis-based supervisory control system can be analyzed without relying on $\mathcal{P}$ being a finite set. There are actually several ways to achieve this goal, of which the one presented below is probably the simplest (see [13] for an alternative approach).

We now proceed to the description of the *local priority hysteresis switching logic*. The new switching logic is scale-independent (see Section 4). Its inputs are continuous monitoring signals $\mu_p$, $p \in \mathcal{P}$. One also needs to specify a positive hysteresis constant $h$ and another positive constant $\gamma$. A proper choice of $\gamma$ in the supervisory control context is discussed in the next section. For each $q \in \mathcal{P}$, we define the set

$$D_\gamma(q) := \{p \in \mathcal{P} : |q - p| \leq \gamma\}$$

where $|\cdot|$ stands for the norm in the ambient space of $\mathcal{P}$.

First, we set $\sigma(0) = \arg\min_{p\in\mathcal{P}}\{\mu_p(0)\}$. Suppose that at a certain time $t_i$ the value of $\sigma$ has just switched to some $q \in \mathcal{P}$. We then keep $\sigma$ fixed until a time $t_{i+1} > t_i$ such that the following inequality is satisfied:

$$(1+h)\min_{p\in\mathcal{P}}\{\mu_p(t_{i+1})\} \leq \min_{p\in D_\gamma(q)}\{\mu_p(t_{i+1})\}$$

at which point we set $\sigma(t_{i+1}) = \arg\min_{p\in\mathcal{P}}\{\mu_p(t_{i+1})\}$. As with the regular scale-independent hysteresis switching, if the arg min is not unique, a particular candidate for the value of $\sigma$ is chosen arbitrarily. The understanding here is that minimization over $D_\gamma(q)$ is computationally tractable if $\gamma$ is small.

Repeating the above steps, we generate a piecewise constant signal $\sigma$ which is continuous from the right everywhere. By the same argument as in [5, 11], one can show that chattering is avoided if all $\mu_p$, $p \in \mathcal{P}$ are bounded below by some positive number. In fact, there exists a maximal interval $[0, T_{\max})$ on which $\sigma$ is defined, and there can only be a finite number of switches on each proper subinterval of $[0, T_{\max})$. As before, in all the cases treated below we will have $T_{\max} = \infty$.

We begin to explore the properties of this new switching logic by proving the following result. We assume again that for some positive function of time $\Theta$ the scaled signals $\bar{\mu}_p(t) := \Theta(t)\mu_p(t)$, $p \in \mathcal{P}$ are strictly positive and monotone increasing. We also assume that $\mu_p(t)$ depends continuously on $p$ for every fixed $t \in [0, T_{\max})$.

**Lemma 5** *Suppose that $\bar{\mu}_{p^*}$ is bounded on $[0, T_{\max})$ for some $p^* \in \mathcal{P}$. Then the switching stops in finite time, i.e., there exists a time $T^* \in [0, T_{\max})$ such that $\sigma(t) = \sigma(T^*)$ for all $t \in [T^*, T_{\max})$. Moreover, $\bar{\mu}_{\bar{q}}$ is bounded on $[0, T_{\max})$ for some $\bar{q} \in D_\gamma(\sigma(T^*))$.*

PROOF. Let $K = \min_{p\in\mathcal{P}}\{\lim_{t\to T_{\max}} \bar{\mu}_p(t)\}$. The hypotheses of the lemma imply that $K$ is well defined and finite. Take an arbitrary fixed $\epsilon \in (0,h)$. We claim that there exists a time $T_\epsilon < T_{\max}$ such that $\bar{\mu}_p(T_\epsilon) \geq K/(1+\epsilon)$ for all $p \in \mathcal{P}$. Indeed, otherwise there would be a sequence $\{p_n\}$ in $\mathcal{P}$ and a sequence of times $\{t_n\}$ increasing to $T_{\max}$, with the property that $\bar{\mu}_{p_n}(t_n) < K/(1+\epsilon)$. Since $\mathcal{P}$ is compact, $\{p_n\}$ would contain an accumulation point $\bar{p}$ for which we would have $\bar{\mu}_{\bar{p}}(t) \leq K/(1+\epsilon)$ for all $t < T_{\max}$, contrary to the definition of $K$ (here we are using the fact that $\bar{\mu}_p$, $p \in \mathcal{P}$ are monotone increasing).

The sequence of values of $\sigma$ contains a subsequence $\{q_n\}$ which converges to some $q \in \mathcal{P}$. Pick a time $T^* \geq T_\epsilon$ such that $|\sigma(T^*) - q| < \gamma$. Then $q$ is an interior point of $D_\gamma(\sigma(T^*))$ relative to $\mathcal{P}$. Using the definition of the switching logic, it is straightforward to verify that if at least one more switch occurs at some $\bar{t} \geq T^*$, then we must have $\bar{\mu}_p(\bar{t}) \geq K(1+h)/(1+\epsilon) > K$ for all $p \in D_\gamma(\sigma(T^*))$. But this would imply that the values of $\sigma$ never re-enter $D_\gamma(\sigma(T^*))$ after the time $\bar{t}$, and we arrive at a contradiction. Therefore, $\sigma(t) = \sigma(T^*)$ for all $t \in [T^*, T_{\max})$.

Let us prove that $\bar{\mu}_{\bar{q}}(t) < K(1+h)$ for some $\bar{q} \in D_\gamma(\sigma(T^*))$ and for all $t < T_{\max}$. Suppose the contrary: that for every $p \in D_\gamma(\sigma(T^*))$ there is a time $T_p < T_{\max}$ such that $\bar{\mu}_p(T_p) \geq K(1+h)$. Since $D_\gamma(\sigma(T^*))$ is compact, the same argument as before shows that this would imply the existence of a time $\bar{T} < T_{\max}$ such that $\bar{\mu}_p(\bar{T}) \geq K(1+h)$ for all $p \in D_\gamma(\sigma(T^*))$. But this would force $\sigma$ to switch away from, and never re-enter, the set $D_\gamma(\sigma(T^*))$, which contradicts the assertion proved earlier. □

# 7 Analysis: infinite $\mathcal{P}$

Consider again the system (5). Here we assume the index set $\mathcal{P}$ to be infinite (typically a continuum), although all that follows is also valid when $\mathcal{P}$ is finite. Suppose that the monitoring signals are defined via the equations (8), and that the switching signal $\sigma$ is generated by the local priority hysteresis switching logic described in the previous section. We take the design parameter $\gamma$ for the switching logic to be small enough so that for every pair of indices $p_1, p_2 \in \mathcal{P}$ satisfying $|p_1 - p_2| \leq \gamma$ the matrix

$$A_{p_1,p_2} := A_{p_1} + d_{p_1}(c_{p_1} - c_{p_2}\ \ 0) \tag{16}$$

is stable. It is not hard to check that in view of stability of $A_p$ for each $p \in \mathcal{P}$, compactness of $\mathcal{P}$, and continuity of $c_p$ with respect to $p$, such a constant $\gamma > 0$ is guaranteed to exist.

We restrict our discussion to the case when the noise and disturbance signals are zero ($n = d \equiv 0$) and there are no unmodeled dynamics ($\delta = 0$). Then the equations (6) and (7) hold with $B_1, B_2, \delta_1, \delta_2$ equal to zero. Let us take $\epsilon_\mu$ in the definition of the monitoring signals to be zero as well (the matrix $W(0)$ must then be positive definite; see Section 4). From (6), (12), and continuity of $c_p$ with respect to $p$, we see that the scaled monitoring signals given by $\bar{\mu}_p(t) = e^{2\lambda t}\mu_p(t)$, $p \in \mathcal{P}$ satisfy the hypotheses of Lemma 5. Therefore, the switching stops in finite time at some index $q^* \in \mathcal{P}$, i.e., there exists a time $T^*$ such that $\sigma(t) = q^*$ for $t \geq T^*$. Moreover, by the same lemma, $\bar{\mu}_{\bar{q}}$ is bounded for some $\bar{q} \in \mathcal{P}$ satisfying $|q^* - \bar{q}| \leq \gamma$. In view of (12), we see that $e_{\bar{q}} \in L^2$.

The first equation in (5) and the formula (16) imply that the evolution of $x$ for $t \geq T^*$ is given by

$$\dot{x} = A_{q^*}x + d_{q^*}e_{q^*} = A_{q^*,\bar{q}}x + d_{q^*}e_{\bar{q}}.$$

The matrix $A_{q^*,\bar{q}}$ is stable, hence $x$ converges to 0. Furthermore, since $e_{p^*} \to 0$ by virtue of (7), the output $y = (c_{p^*}\ \ 0)\,x - e_{p^*}$ of the process $\mathbb{P}$ also converges to 0. In light of (5), (8), and detectability of $\mathbb{P}$, all the other system signals remain bounded for all $t \geq 0$ (i.e., $T_{\max} = \infty$). We have thus proved the following counterpart of Proposition 2.

**Proposition 6** *Suppose that the noise and disturbance signals are zero and there are no unmodeled dynamics, and set $\epsilon_\mu = 0$. Then all the signals in the supervisory control system remain bounded for every set of initial conditions such that $W(0) > 0$. Moreover, the switching stops in finite time, and we have $y(t) \to 0$ as $t \to \infty$.*

REMARK. As in Proposition 2, since the system describing the evolution of $x_{\mathbb{P}}$ and $x$ for $t \geq T^*$ is linear time-invariant, the rate at which $y$ converges to zero must be exponential.

In the recent paper [13], we describe another switching logic for dealing with infinite parameter sets, which allows a successful treatment of noise, disturbances, and unmodeled dynamics.

# 8 Concluding remarks

Our goal has been to discuss recent progress in design and analysis of hysteresis-based supervisory control algorithms for uncertain linear systems. Although the results that we have obtained can also be achieved by using the methods described in [15, 16], the proofs given here are considerably simpler. Another important advantage of the present approach is that it is more suitable for control of nonlinear systems (as explained at the end of Section 5). In fact, we have already applied these methods to certain classes of nonlinear systems in the absence of noise, disturbances, and unmodeled dynamics (see [6, 7, 10]). In addressing the issue of robustness in the nonlinear case, the work on nonlinear extensions of the Vinnicombe metric [2, 21] and on stable factorizations of nonlinear systems [19, Chapter 5] may prove to be relevant. The Vinnicombe metric may also be useful for designing "safe" multiple model adaptive control algorithms, capable of guaranteeing that one never switches to a controller whose feedback connection with the process is unstable (cf. [3]). The new switching logic described in Section 6 is of interest in its own right, and its properties and implications are to be investigated further.

# A Proof of Theorem 4

From (5) and (6) one obtains

$$\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau \leq B_1 e^{2\lambda t} + D_1|\bar{x}(0)|^2 + \delta_1 F \int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau + \delta_1 G \int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau$$

where $\bar{x}$ denotes the composite state $(x'_{\mathbb{P}}, x')'$ of the process, the multi-estimator, and the multi-controller, $D_1$ is a positive constant, $F := 2\max_{p\in\mathcal{P}}|f_p|^2$, and $G := 2\max_{p\in\mathcal{P}}|g_p|^2$. Suppose that $\delta_1$ is small enough so that

$$\delta_1 G < 1 \tag{17}$$

Then we can write

$$\int_0^t e^{2\lambda\tau} e_{p^*}^2(\tau)d\tau \leq \bar{B}_1 e^{2\lambda t} + \bar{D}_1|\bar{x}(0)|^2 + \delta_1 \bar{F} \int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau \tag{18}$$

where $\bar{B}_1 := \dfrac{B_1}{1-\delta_1 G}$, $\bar{D}_1 := \dfrac{D_1}{1-\delta_1 G}$, and $\bar{F} := \dfrac{F}{1-\delta_1 G}$.

Let $W(0) = 0$ for simplicity. The formula (12) gives

$$\bar{\mu}_{p^*}(t) \leq \epsilon_\mu e^{2\lambda t} + \bar{B}_1 e^{2\lambda t} + \bar{D}_1|\bar{x}(0)|^2 + \delta_1 \bar{F} \int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau \tag{19}$$

For an arbitrary pair of positive numbers $C, C_0$ with $C \geq 1$, we can define the time $T = T(C, C_0) := \sup\{t \geq 0 : |x(t)|^2 \leq C|\bar{x}(0)|^2 + C_0\epsilon_\mu\} \leq \infty$. For all $t \leq T$ we have

$$\int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau \leq \frac{C|\bar{x}(0)|^2 + C_0\epsilon_\mu}{2\lambda} e^{2\lambda t} \tag{20}$$

hence

$$\bar{\mu}_{p^*}(t) \leq \epsilon_\mu \Omega e^{2\lambda t} \tag{21}$$

where

$$\Omega := 1 + \frac{\bar{B}_1}{\epsilon_\mu} + \frac{\bar{D}_1|\bar{x}(0)|^2}{\epsilon_\mu} + \delta_1 \bar{F} \frac{C|\bar{x}(0)|^2 + C_0\epsilon_\mu}{2\lambda\epsilon_\mu}$$

The inequality (10) now implies that for $T \geq t > t_0 \geq 0$ we have

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log \Omega + \frac{2\lambda m}{\log(1+h)}(t - t_0)$$

Thus on the interval $[0, T]$ we have the average dwell time $\tau_{AD} = \dfrac{\log(1+h)}{2\lambda m}$. We know from the results of [12] that if $h$ is large enough and $\lambda$ is small enough, then the switched system given by the first equation in (5) has a finite $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norm on this interval, as expressed by the inequality (15). Moreover, the gains $g$ and $g_0$ satisfy the bounds $g \leq D_3\Omega^{1-r}$ and $g_0 \leq D_4\Omega^{1-r}$ for some $r \in (0, 1)$ and some positive constants $D_3, D_4$ that are independent of $C$ and $C_0$.

From (11) and (21) it also follows that

$$\int_0^t e^{2\lambda\tau} e^2_{\sigma(\tau)}(\tau) d\tau \leq m(1+h)\epsilon_\mu \Omega e^{2\lambda t}$$

Combined with (15), this implies that for all $t \leq T$ we have

$$\begin{aligned}
|x(t)|^2 &\leq gm(1+h)\epsilon_\mu\Omega + g_0|x(0)|^2 e^{-2\lambda t} \leq m(1+h)D_3\epsilon_\mu\Omega^2 + D_4|\bar{x}(0)|^2\Omega \\
&= D_5 D_6^2 \epsilon_\mu + \frac{D_5\delta_1^2\bar{F}^2C_0^2\epsilon_\mu}{4\lambda^2} + \frac{D_5 D_6 \delta_1 \bar{F} C_0 \epsilon_\mu}{\lambda} + \frac{D_5\bar{D}_1^2 + \bar{D}_1 D_4}{\epsilon_\mu}|\bar{x}(0)|^4 \\
&+ \frac{D_5\delta_1^2\bar{F}^2C^2}{4\lambda^2\epsilon_\mu}|\bar{x}(0)|^4 + (2D_5\bar{D}_1 + D_4)D_6|\bar{x}(0)|^2 + \frac{D_5 D_6\delta_1\bar{F}C}{\lambda}|\bar{x}(0)|^2 \\
&+ \frac{(2D_5\bar{D}_1 + D_4)\delta_1\bar{F}C}{2\lambda\epsilon_\mu}|\bar{x}(0)|^4 + \frac{(2D_5\bar{D}_1 + D_4)\delta_1\bar{F}C_0}{2\lambda}|\bar{x}(0)|^2 + \frac{D_5\delta_1^2\bar{F}^2CC_0}{2\lambda^2}|\bar{x}(0)|^2
\end{aligned}$$

where $D_5 := m(1+h)D_3$ and $D_6 := 1 + \bar{B}_1/\epsilon_\mu$.

Suppose that $|\bar{x}(0)| \leq E$, where $E$ is given. We can choose $C_0$ large enough so that the inequality

$$D_5 D_6^2 < \frac{C_0}{3}$$

holds, then choose $C$ large enough so that the inequalities

$$\frac{D_5\bar{D}_1^2 + \bar{D}_1 D_4}{\epsilon_\mu}E^2 < \frac{C}{7} \quad \text{and} \quad (2D_5\bar{D}_1 + D_4)D_6 < \frac{C}{7}$$

hold, and finally choose $\delta_1$ small enough so that the inequalities

$$\begin{gathered}
\frac{D_5\delta_1^2\bar{F}^2C}{4\lambda^2\epsilon_\mu}E^2 < \frac{1}{7}, \quad \frac{D_5 D_6\delta_1\bar{F}}{\lambda} < \frac{1}{7}, \quad \frac{(2D_5\bar{D}_1 + D_4)\delta_1\bar{F}}{2\lambda\epsilon_\mu}E^2 < \frac{1}{7}, \\
\frac{(2D_5\bar{D}_1 + D_4)\delta_1\bar{F}C_0}{2\lambda} < \frac{C}{7}, \quad \frac{D_5\delta_1^2\bar{F}^2C_0}{2\lambda^2} < \frac{1}{7}
\end{gathered} \tag{22}$$

hold. Using the above calculations, it is straightforward to check that this implies $|x(t)|^2 < C|\bar{x}(0)|^2 + C_0\epsilon_\mu$ for all $t \le T(C, C_0)$, thus for these values of $C$ and $C_0$ we actually have $T = \infty$. It now follows from (5), (7), (8), (18), (20), and detectability of $\mathbb{P}$ that the solutions remain bounded, which proves the first statement of the theorem.

Let us use (11) and (19) again, combined with the fact that the switched system has a finite $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_2$ induced norm:

$$\int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau \le \bar{g}\int_0^t e^{2\lambda\tau}e^2_{\sigma(\tau)}(\tau)d\tau + \bar{g}_0|x(0)|^2$$

where the gains $\bar{g}$ and $\bar{g}_0$ are of the same type as $g$ and $g_0$ above. We obtain

$$\begin{aligned}\int_0^t e^{2\lambda\tau}e^2_{\sigma(\tau)}(\tau)d\tau &\le m(1+h)\Big[(\epsilon_\mu + \bar{B}_1)e^{2\lambda t} + \bar{D}_1|\bar{x}(0)|^2 + \delta_1\bar{F}\int_0^t e^{2\lambda\tau}|x(\tau)|^2 d\tau\Big] \\ &\le m(1+h)\Big[(\epsilon_\mu + \bar{B}_1)e^{2\lambda t} + \Big(\bar{D}_1 + \delta_1\bar{F}\bar{g}_0\Big)|\bar{x}(0)|^2 + \delta_1\bar{F}\bar{g}\int_0^t e^{2\lambda\tau}e^2_{\sigma(\tau)}(\tau)d\tau\Big]\end{aligned}$$

Choose $\delta_1$ to satisfy the inequalities (17) and (22), and then decrease it if necessary so as to have $m(1+h)\delta_1\bar{F}\bar{g} < 1$. This implies

$$\int_0^t e^{2\lambda\tau}e^2_{\sigma(\tau)}(\tau)d\tau \le D_7 + D_8(\epsilon_\mu + \bar{B}_1)e^{2\lambda t}$$

for suitable constants $D_7$ and $D_8$. It follows from (15) that

$$|x(t)|^2 \le (gD_7 + g_0|x(0)|^2)e^{-2\lambda t} + gD_8(\epsilon_\mu + \bar{B}_1)$$

If $n$ and $d$ converge to zero, then $\bar{B}_1$ converges to zero. We also have

$$g\epsilon_\mu \le D_3\Big(\epsilon_\mu + \bar{B}_1 + \bar{D}_1|\bar{x}(0)|^2 + \delta_1\bar{F}\frac{C|\bar{x}(0)|^2 + C_0\epsilon_\mu}{2\lambda}\Big)^{1-r}\epsilon_\mu^r$$

Since $0 < r < 1$, for every given bound $E$ on $|\bar{x}(0)|$ and every $\epsilon > 0$ the quantity on the right-hand side can be made smaller than $\epsilon$ by choosing $\epsilon_\mu$ sufficiently small, selecting $C_0, C, \delta_1$ in accordance with the above inequalities, and decreasing $\delta_1$ further if needed. Moreover, since $B_2$ in (7) converges to zero, it is not difficult to verify that $|e_{p^*}(t)|$ is bounded by a quantity that converges to

$$\delta_2\sqrt{\bar{F}\frac{C|\bar{x}(0)|^2 + C_0\epsilon_\mu}{2\lambda}}$$

as $t \to \infty$. This can be made arbitrarily small by taking $\delta_2$ sufficiently small. Thus $\lim_{t\to\infty}|y(t)| = \lim_{t\to\infty}|(c_{p^*}\ \ 0)\,x - e_{p^*}|$ will not exceed a prescribed $\epsilon_y > 0$ if the design parameter $\epsilon_\mu$ and the unmodeled dynamics bound $\delta$ are small enough, which proves the second statement of the theorem. $\square$

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