Graph Effective Resistance and Distributed Control: Electrical Analogy and Scalability

Technical Report*

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Abstract

We introduce the concept of matrix-valued effective resistance for undirected matrix-weighted graphs. These effective resistances are defined to be the square blocks that appear in the diagonal of the inverse of the matrix-weighted Dirichlet graph Laplacian matrix. However, they can also be obtained from a "generalized" electrical network that is constructed from the graph, and for which currents, voltages and resistances take matrix values. In a recent paper [3], we showed that effective resistances have a direct physical interpretation in several distributed control problems and can be used to characterize their stability and speed of convergence.

In this paper, we establish several properties of effective resistances. In particular, (i) we show that the effective resistances are monotone with respect to an appropriately defined graph embedding relation; (ii) we determine laws that characterize how effective resistances scale with the distance between nodes for regular-degree large scale graphs; and (iii) we extend these laws for irregular large scale graphs that exhibit appropriate denseness properties.

1 Introduction

This paper considers undirected graphs with a weight associated with each one of its edges. The edge-weights are symmetric positive definite matrices. For such graphs we introduce the concept of "effective resistances." The effective resistance of a node is defined to be the square matrix blocks that appear in the diagonal of the inverse of the matrix-weighted Dirichlet graph Laplacian matrix (cf. Section 2). The terminology "effective resistance" is motivated by the fact that these matrices also define a linear map from currents to voltages in a generalized electrical network that can be constructed from the undirected matrix-weighted graph. However, the voltages, currents, and resistances in this generalized electrical network take matrix values (cf. Section 3).

Effective resistance in "regular" electrical networks, where currents voltages and resistances are scalar valued, have been known to have far reaching implications in a variety of problems. Recurrence and transience in random walks in infinite networks [6] and the coverage and commute times of random walks in graphs [4] are determined by this effective resistance. There is a strong connection between variance of the estimate of a scalar valued variable from relative measurements defined on a graph and effective resistance, which was discovered by Karp et al. [8]. It was later shown by Barooah and Hespanha [2] that this analogy can be extended to vector measurements with matrix-valued covariances with the introduction of generalized electrical networks, with matrix-valued currents, voltages, and resistances. Correspondingly, the effective

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resistance in a generalized electrical network is matrix-valued. Any graph with positive definite matrix weights can be thought of a generalized electrical network where the edge weights are the inverse-resistances.

The generalized effective resistances of matrix-weighted graphs play an important role in several problems related to distributed control and estimation. We justify this statement in a recent paper [3], where we showed that effective resistances have a direct physical interpretation. In [3], we also showed that effective resistances can be used to characterize the stability and speed of convergence of distributed control and estimation algorithms. Our main contribution in this paper is to determine how the effective resistance scales with distance in large graphs and what key structural properties of the graph determine this scaling.

In Section 3, we review the analogy between graph effective resistances (defined through the Dirichlet graph Laplacian matrix) and effective resistances in generalized electrical networks. This section also discusses several well known results for "regular" electrical networks that can be adapted to generalized electrical networks. These include Rayleigh's monotonicity law and bounds on the effective resistances for infinite regular lattices and their fuzzes. A fuzz of a graph is obtained by connecting all pairs of nodes that lie at graphical distances smaller than a given integer [6].

Section 4 contains the main results of this paper. We determine how the effective resistance between two nodes scales with the distance between them in large graphs. We start by introducing notions of denseness/sparseness of graphs that provide necessary and sufficient conditions for a graph to be embedded or embed a regular lattice. These notions are based on the concept of "graph drawing," which is a map that assigns nodes to positions in Euclidean space. We show that a graph can be embedded in the fuzz of a d-dimensional lattice if and only if it can be drawn in $\mathbb{R}^d$ "without too much clutter." Conversely, the fuzz of a graph can embed a d-dimensional lattice if and only if its nodes can cover $\mathbb{R}^d$ without leaving large holes between them and still having enough edges so that nodes drawn close to each other have a small graphical distance between them. Sufficient conditions for lattice embedding appeared previously in [2, 6]. We show that the sufficient condition for a graph to be embedded in the fuzz of a lattice that appeared in [6] is also necessary. However, the sufficient condition for the fuzz of a graph to embed a lattice that appeared in [2] turned out not to be necessary and had to be refined.

Combining the effective resistance scaling laws for lattices, Rayleigh's monotonicity law, and the necessary and sufficient conditions for graph embedding in lattices, we obtain effective resistance scaling laws for every graph that satisfies appropriate denseness/sparseness conditions. Examination of these conditions reveal that they cover almost all graphs commonly encountered in distributed control and estimation problems, and therefore give us a powerful and comprehensive graph taxonomy.

2 Graph effective resistances

We now recall the basic definitions introduced in [3]. An *undirected matrix-weighted graph* is a triple $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$, where $\mathbf{V}$ is a set of n vertices; $\mathbf{E} \subset \mathbf{V} \times \mathbf{V}$ a set of m edges; and $\mathbf{W} := \{W_{u,v} \in \mathbb{R}^{k \times k} : (u, v) \in \mathbf{E}\}$ a set of symmetric positive definite matrix-valued weights for the edges of $\mathbf{G}$. Since we are dealing with undirected graphs, the pairs (u, v) and (v, u) denote the same edge. For simplicity we exclude the possibility of having multiple edges between the same pair of nodes and edges from a node to itself. However, all the results presented here are valid for graphs with parallel edges; we refer the interested reader to [1] for the details.

The *matrix-weighted Laplacian* of $\mathbf{G}$ is a $nk \times nk$ matrix L with k rows per vertex and k columns per edge such that the $k \times k$ block of L corresponding to the k rows associated with node $u \in \mathbf{V}$ and the k columns associated with node $v \in \mathbf{V}$ is equal to

$$\begin{cases} \sum_{v \in \mathcal{N}_u} W_{u,v} & u = v \\ -W_{u,v} & (u, v) \in \mathbf{E} \\ 0 & (u, v) \notin \mathbf{E}. \end{cases}$$

where $\mathcal{N}_u \subset \mathbf{V}$ denotes the set of neighbors of u, i.e., the set of nodes that have an edge in common with u. This matrix can be compactly defined in terms of the incidence matrix of $\mathbf{G}$ [3].

Given a subset $\mathbf{V}_o \subset \mathbf{V}$ consisting of $n_o \leq n$ nodes, the *matrix-weighted Dirichlet Laplacian for the boundary* $\mathbf{V}_o$ is a $(n - n_o)k \times (n - n_o)k$ matrix L_o obtained from the matrix-weighted Laplacian of $\mathbf{G}$ by removing all rows and columns corresponding to the nodes in $\mathbf{V}_o$.

We say that a graph $\mathbf{G}$ is *connected to* $\mathbf{V}_o$ if there is a path from every node in the graph to at least one of the boundary nodes in $\mathbf{V}_o$. It turns out that the L_o is positive definite if and only if $\mathbf{G}$ is connected to $\mathbf{V}_o$ [3]. For a connected graph $\mathbf{G}$, *node u's effective resistance to* $\mathbf{V}_o$, denoted by $R_u^{\text{eff}}(\mathbf{V}_o)$, is the $k \times k$ block in the main diagonal of L_o^{-1} corresponding to the k rows/columns associated with the node $u \in \mathbf{V}$. This terminology is justified by the results in Section 3.

3 Electrical analogy

We start by considering "regular" electrical networks with scalar valued currents and potentials. A *resistive electrical network* consists of an interconnection of purely resistive elements. Such interconnections are generally described by graphs whose nodes represent connection points between resistors and whose edges correspond to the resistors themselves. The effective resistance between two nodes in an electrical network is defined as the potential drop between the two nodes, when a current source with intensity equal to 1 Ampere is connected across the two nodes (cf. Figure 1). For a general network, the computation of effective

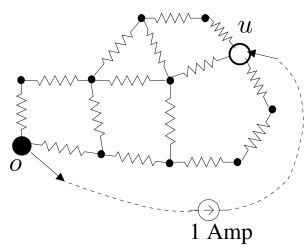


Figure 1: A resistive electrical network for which a 1 Ampere current was is injected at node u and extracted at the reference node o. The resulting potential difference $V_u - V_o$ is the effective resistance between u and o.

resistances relies on the usual Kirchoff's and Ohm's laws.

To establish the connection between graph effective resistances defined in the previous section and electrical networks we need to consider an abstract *generalized electrical network* in which currents, potentials, and resistors are $k \times k$ matrices. For such networks, Kirchoff's current law can be defined in the usual way, except that currents are added as matrices. Kirchoff's voltage law can also be defined in the usual way where potentials drops across edges are added as matrices. Kirchoff's voltage laws show the existence of a matrix valued node potential function. Ohm's law takes the following matrix form

$$V_e = R_e i_e,$$

where i_e is a generalized $k \times k$ matrix current flowing through the edge e of the electrical network, R_e is the generalized resistance of that edge, and V_e is a generalized $k \times k$ matrix potential drop across the edge e. Generalized resistances are always symmetric positive definite matrices.

The generalized electrical networks so defined share many of the properties of "regular" electrical networks. In particular, Kirchoff's and Ohm's laws uniquely define all edge currents and node voltages in a generalize electrical network when the potential at a particular *reference* node is fixed at some arbitrary value ("grounded"). The potential difference between pairs of nodes are the same irrespective of what the potential of the reference node is. This allow us to define the *generalized effective resistance* $R^{\text{eff}}_{u,v}$ between two nodes u and v as the potential difference between them when a generalized current equal to the $k \times k$ identity matrix is injected into one and extracted at the other.

The following result by Barooah and Hespanha [2] justifies the terminology of graph effective resistances in section 2 by showing that it is the same as the generalized effective resistance defined above.

Theorem 1 *Consider an undirected matrix-weighted graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ *and construct a generalized electric network with* $k \times k$ *edge-resistors* R_e, $e \in \mathbf{E}$ *that are numerically equal to the inverses of the corresponding edge-weights, i.e.,* $R_e = W_e^{-1}$, $e \in \mathbf{E}$. *For every single-node boundary* $\mathbf{V}_o := \{o\}$ *and every node* $u \in \mathbf{V} \setminus \mathbf{V}_o$, *node u's* effective resistance to $\mathbf{V}_o$ *is equal to the* generalized effective resistance between u and o. □

In the remaining of this section, we discuss several well known results for "regular" electrical networks that can be adapted to generalized electrical networks.

3.1 Rayleigh's Monotonicity Law

Rayleigh's Monotonicity Law states that if the edge-resistances in a (regular) electrical network are increased, then the effective resistance between any two nodes in the network can only increase. Conversely, a decrease in edge-resistances can only lead to a decrease in effective resistance. It turns out that Rayleigh's Monotonicity Law can be extended to generalized electrical networks.

For the problems considered here, it is convenient to consider not only increases in edge-resistances but also removing an edge altogether or introducing a new edge. The statement of a version of Rayleigh's Monotonicity Law that allow us to consider edge removal requires the introduction of a partial order for graphs. Given two undirected matrix-weighted graphs

$$\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}), \qquad \bar{\mathbf{G}} = (\bar{\mathbf{V}}, \bar{\mathbf{E}}, \bar{\mathbf{W}})$$

we say that $\mathbf{G}$ *can be embedded in* $\bar{\mathbf{G}}$, and write $\mathbf{G} \subset \bar{\mathbf{G}}$, if $\mathbf{V} \subset \bar{\mathbf{V}}$, $\mathbf{E} \subset \bar{\mathbf{E}}$, and

$$W_{u,v} \leq \bar{W}_{u,v}, \quad \forall (u, v) \in \mathbf{V}.$$

Here and below, given two symmetric matrices A and B, we write $A \geq B$ to mean that the matrix $A - B$ is positive semi-definite.

It was proved in [1] that Rayleigh's monotonicity law holds for generalized electrical networks. In view of Theorem 1, this leads to the following monotonicity result for graph effective resistances:

Theorem 2 (Rayleigh's Generalized Monotonicity Law) *Consider two undirected matrix-weighted graphs*

$$\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}), \qquad \bar{\mathbf{G}} = (\bar{\mathbf{V}}, \bar{\mathbf{E}}, \bar{\mathbf{W}})$$

such that $\mathbf{G} \subset \bar{\mathbf{G}}$. *For every single-node boundary* $\mathbf{V}_o := \{o\} \in \mathbf{V}$ *and every node* $u \in \mathbf{V} \setminus \mathbf{V}_o$, *we have that*

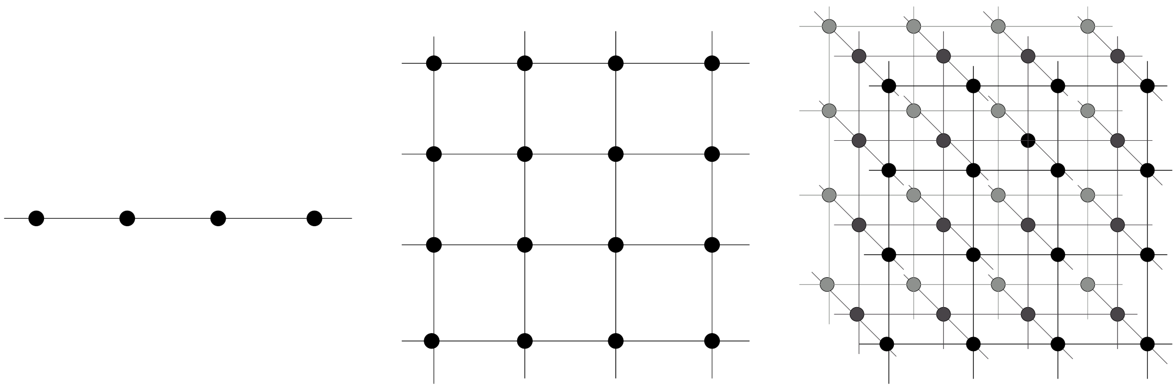
$$R^{\text{eff}}_u(\mathbf{V}_o) \geq \bar{R}^{\text{eff}}_{\bar{u}}(\mathbf{V}_o),$$

where $R^{\text{eff}}_u(\mathbf{V}_o)$ *denotes u's effective resistance to* $\mathbf{V}_o$ *with respect to the graph* $\mathbf{G}$ *and* $\bar{R}^{\text{eff}}_{\bar{u}}(\mathbf{V}_o)$ *denotes u's effective resistance to* $\mathbf{V}_o$ *with respect to the graph* $\bar{\mathbf{G}}$. □

3.2 Lattices, h-fuzzes, and their effective resistance

The effective resistances of several "regular" electrical networks have been studied in the literature on resistive electrical network. In this section we discuss a few graphs for which results can also be obtained for generalized electric networks.

A *d-dimensional lattice*, denoted by $\mathbb{Z}_d$ is a graph that has one vertex for every point in $\mathbb{R}^d$ with integer coordinates and an edge between every two vertices corresponding to points with an Euclidean distance between them equal to one.



(a) 1-dimensional lattice $\mathbb{Z}_1$ (b) 2-dimensional lattice $\mathbb{Z}_2$ (c) 3-dimensional lattice $\mathbb{Z}_3$

Figure 2: Lattices

Figure 2 shows 1-, 2-, and 3-dimensional lattices. Lattices have infinitely many nodes and edges, and are therefore examples of *infinite graphs*. In practice, they serve as proxies for very large graphs.

We now need to define the concept of a fuzz of a graph, for which we introduce the notion of graphical distance. Given two nodes u and v of a graph $\mathbf{G}$, their *graphical distance*, denoted by $d_{\mathbf{G}}(u, v)$ is the minimum number of edges one has to traverse in going from one node to the other.

Given a graph $\mathbf{G}$ and an integer $h \geq 1$, *h-fuzz of* $\mathbf{G}$, denoted by $\mathbf{G}^{(h)}$, is a graph with the same set of nodes as $\mathbf{G}$ but with a larger set of edges. In particular, $\mathbf{G}^{(h)}$ has an edge between u and v whenever the graphical distance between u and v is less than or equal to h [6]. Figure 3 shows the 2-fuzz of the 2-dimensional lattice.

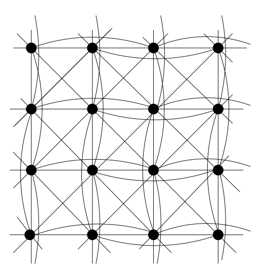
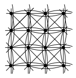
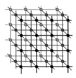


Figure 3: 2-fuzz of the 2-dimensional lattice.

It was shown by Doyle [5] that the scalar effective resistance in $\mathbf{G}^{(h)}$ (with all edges in both $\mathbf{G}^{(h)}$ and

Table 1: Effective resistance for lattices and their fuzzes

Graph		Effective resistance between u and v
	$\mathbb{Z}_1^{(h)}$	$\alpha_1 d_{\mathbb{Z}_1}(u,v) R_o \leq R^{\text{eff}}_{u,v} \leq \beta_1 d_{\mathbb{Z}_1}(u,v) R_o$
	$\mathbb{Z}_2^{(h)}$	$\alpha_2 \log\big(d_{\mathbb{Z}_2}(u,v)\big) R_o \leq R^{\text{eff}}_{u,v} \leq \beta_2 \log\big(d_{\mathbb{Z}_2}(u,v)\big) R_o$
	$\mathbb{Z}_3^{(h)}$	$\alpha_3 R_o \leq R^{\text{eff}}_{u,v} \leq \beta_3 R_o$

$\mathbf{G}$ having 1-Ohm resistors) will be lower than the corresponding effective resistance in $\mathbf{G}$ *only by a constant factor* as long as h is finite, and that constant will depend on h but not on the graphical distance between the two nodes. The arguments in [5] can be used to show that the same result holds for generalized electrical networks, which is stated in the next lemma (see [1] for details).

Lemma 1 *Let $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W}_o)$ and $\mathbf{G}^{(h)} = (\mathbf{V}, \mathbf{E}^{(h)}, \mathbf{W}_o)$ be two matrix-weighted graphs where the graphs $\mathbf{G}$ and $\mathbf{G}^{(h)}$ have bounded degree and all edges in both the graphs have equal positive definite weight W_o. Let $R^{\text{eff}}_{u,v}(\mathbf{G})$ be the effective resistance between two nodes u and v in $\mathbf{G}$ and $R^{\text{eff}}_{u,v}(\mathbf{G}^{(h)})$ be the effective resistance between u and v in the h-fuzz $\mathbf{G}^{(h)}$. The following relationship holds:*

$$\alpha R^{\text{eff}}_{u,v}(\mathbf{G}) \leq R^{\text{eff}}_{u,v}(\mathbf{G}^{(h)}) \leq R^{\text{eff}}_{u,v}(\mathbf{G}),$$

where $\alpha \in (0,1]$ is a constant independent of u and v.

The next lemma from [2] establishes the effective resistances in d-dimensional lattices and their fuzzes.

Lemma 2 (Lattice Generalized Effective Resistances) *Consider a generalized electrical network obtained by placing generalized matrix resistances equal to R_o at the edges of the h-fuzz of the d-dimensional lattice, where h is a positive integer, $d \in \{1,2,3\}$, and R_o is a symmetric positive definite $k \times k$ matrix. There exist constants $\ell, \alpha_i, \beta_i > 0$ such that the formulas in Table 1 hold for every pair of nodes u, v at a graphical distance larger than ℓ.* □

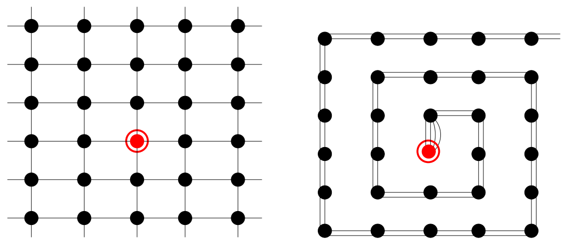
The fact that in a 1-dimensional lattice the effective resistance grows linearly with the distance between nodes can be trivially deduced from the well known formula for the effective resistance of a series of resistors (which generalizes to generalized electrical networks). In two-dimensional lattices the effective resistance only grows with the logarithm of the graphical distance and therefore the effective resistance grows very slowly with the distance between nodes. Far more surprising is the fact that in three-dimensional latices the effective resistance is actually bounded by a constant even when the distance is arbitrarily large.

4 Effective resistance scaling with distance: dense and sparse graphs

In this section we show how to combine the Electrical Analogy Theorem 1, Rayleigh's Generalized Monotonicity Law, and the Lattice Effective Resistance Lemma 2 to determine scaling laws of the effective resistance for more general classes of graphs.

A higher density of edges and nodes in a graph should lead to lower effective resistances. However, "naive" measures of node and edge density turn out to be misleading predictors for how the *effective resistance scales* *with distance*. We shall see that to predict these scalability laws one needs instead to determine deeper structural properties of the graph. First, however, we will discuss a few examples that will motivate our results.

4.1 Counterexamples to conventional wisdom



(a) $\mathbf{G}_1$ (b) $\mathbf{G}_2$

Figure 4: Two graphs drawn in 2-dimensional with the same number of nodes and edges per unit area and with the same node degree of four (except for a single node of $\mathbf{G}_2$ that has a higher degree of 6).

Consider the two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ in Figure 4 and for simplicity assume that all edges have the same edge-weights (and edge-resistances) equal to the identity matrix. For both graphs the number of nodes per unit area is exactly the same and the degree of every node in both the graphs is four (except for a single node of $\mathbf{G}_2$ that has a higher degree of 6). "Naive" notions of denseness would classify the graphs as similar.

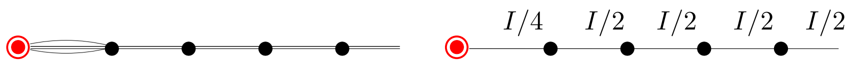
The graph $\mathbf{G}_1$ is a 2-dimensional lattice so from the Electrical Analogy Theorem 1 and the Lattice Effective Resistance Lemma 2, we conclude that the effective resistance from a node u to a single-node boundary grows asymptotically with the logarithm of the graphical distance from u to the boundary node. The exact same statement is true if we replace "graphical distance" by "Euclidean distance" because this drawing of the graph has the property that the graphical distance is always between the Euclidean distance and $\sqrt{2}$ times the Euclidean distance.

To study the scaling of effective resistance in $\mathbf{G}_2$ with distance, we redraw this graph as in Figure 5(a). It is now easy to recognize that $\mathbf{G}_2$ is also equivalent to an 1-dimensional lattice shown in Figure 5(b), with the edge resistances equal to $I/2$ (which corresponds to edge-weights equal to $2I$) at every edge, except the first one that for which it is $I/4$. This means that for $\mathbf{G}_2$ the effective resistance grows linearly with the graphical distance. Moreover, for the drawing of $\mathbf{G}_2$ in Figure 5(a), the exact same statement is true if we replace "graphical distance" by "Euclidean distance." However, for the drawing of $\mathbf{G}_2$ in Figure 4, the Euclidean distance provides little information about the effective resistance because nodes separated by small Euclidean distances can be separated by very large graphical distances (as one moves further away from the center node). Two conclusions can be drawn from these examples:

1. Two graphs with the same node degree can be drawn with the same node and edge densities, and yet they exhibit fundamentally different scaling laws of effective resistance with distance.

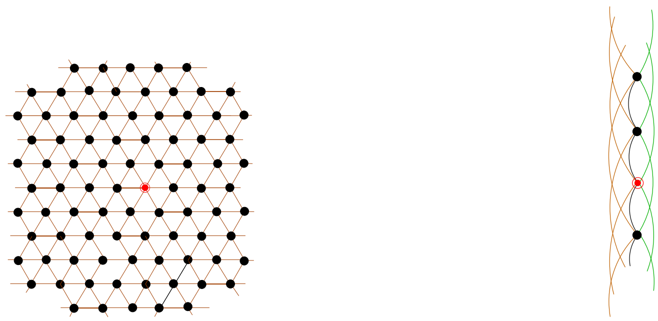
2. Some drawings of the graph appear to be more adequate than others to determine effective resistance scaling laws with respect to Euclidean distance.

The two graphs in Figure 6 – a triangular lattice and a 3-fuzz of a 1-dimensional lattice – could have been used in place of the ones in Figure 4 to make these points. Actually, things are perhaps even more interesting for these graphs: The effective resistance in the 3-fuzz of the 1-dimensional lattice grows linearly



(a) (b)

Figure 5: (a)A 1-dimensional drawing of the graph G_2 in Figure 4, and (b)a drawing of a graph that is equivalent to the graph in (a) in terms of effective resistance.



(a) A triangular 2-dimensional lattice

(b) A 3-fuzz of a 1-dimensional lattice

Figure 6: Two different measurement graphs — a triangular 2-dimensional lattice and a 3-fuzz of a 1-dimensional lattice. Both graphs have the same node degree for every node but very different variance growth rates with distance.

with distance, whereas in the triangular lattice it grows only with the logarithm of distance, in spite of both graphs having the *same node degree of* 6. At this point the reader will have to take our word for the stated growth of the effective resistance in triangular lattices, but the tools needed to prove this fact are forthcoming.

4.2 Drawings of graphs

In graph theory, a graph is generally treated purely as a collection of nodes connected by edges, without any regard to the geometry determined by the nodes' locations. However, for the graphs that arise in distributed control problems there is an underlying geometry because nodes generally correspond to physical agents and their locations often determine the connectivity of the graph [3].

Graph drawings are used to capture the geometry of graphs in Euclidean space. The drawing of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ is simply a mapping of its nodes to points in some Euclidean space, which can formally be described by a function $f : \mathbf{V} \to \mathbb{R}^d$, $d \geq 1$. A drawing is also sometimes called a *representation* of a graph [7]. For a particular drawing f of a graph, we can define Euclidean distances between nodes, which are simply the distances in Euclidean space between the drawings of the nodes. In particular, given two nodes $u, v \in \mathbf{V}$ the *Euclidean distance between* u *and* v *induced by the drawing* $f : \mathbf{V} \to \mathbb{R}^d$ is defined by

$$d_f(u, v) := \|f(v) - f(u)\|,$$

where $\|\cdot\|$ denoted the usual Euclidean norm in d-space. Euclidean distances depend on the drawing and can be completely different from graphical distances. The two drawings of the same graph in Figure 4(b)

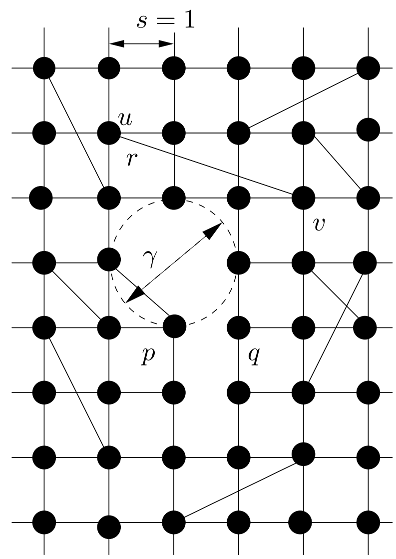


Figure 7: Drawing of a graph and the corresponding denseness/sparseness parameters: $s = 1$, $r = d_f(u, v) = \sqrt{10}$, $\gamma = 2$, and $\rho = d_f(p, q)/d_{\mathbf{G}}(p, q) = 1/5$.

and in Figure 5(a) should make this point quite clear. It is important to emphasize that the definition of drawing does not require edges to not intersect and therefore every graph has a drawing in any Euclidean space. In fact, every graph has infinitely many drawings.

For graphs that arise in distributed control problems in which nodes correspond to physical agents, there is a *natural drawing* that is obtained by associating each node to its position in 1-, 2- or 3-dimensional Euclidean space. In reality, all agents are situated in 3-dimensional space. However, sometimes it maybe more natural to draw them on a 2-dimensional Euclidean space if one dimension (e.g., height) does not vary much from node to node, or is somehow irrelevant. For *natural drawings, the Euclidean distance induced by the drawing is, in general, a much more meaningful notion of distance than the graphical distance.* In this paper we will see that the Euclidean distance induced by appropriate drawings provide the right measure of distance to determine scaling laws of effective resistance.

4.3 Measures of denseness/sparseness

For a particular drawing f and induced Euclidean distance d_f of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$, four parameters can be used to characterize denseness/sparseness. *Minimum node distance* denotes the minimum Euclidean distance between the drawing of any two nodes:

$$s := \inf_{\substack{u,v \in \mathbf{V} \\ v \neq u}} d_f(u, v).$$

Maximum connected range denotes the Euclidean length of the drawing of the longest edge:

$$r := \sup_{(u,v) \in \mathbf{E}} d_f(u, v).$$

Maximum uncovered diameter denotes the diameter of the largest open ball that can be placed in $\mathbb{R}^d$ with no drawing of a node inside it:

$$\gamma := \sup \Big\{ \delta : \exists B_\delta \text{ such that } f(u) \notin B_\delta, \ \forall u \in \mathbf{V} \Big\},$$

where the existential quantification spans over the balls B_δ in $\mathbb{R}^d$ with diameter δ. Finally, *asymptotic distance scaling* denotes the largest asymptotic ratio between the graphical and the Euclidean distance between two nodes:

$$\rho := \lim_{n\to\infty} \inf \left\{ \frac{d_f(u,v)}{d_{\mathbf{G}}(u,v)} : u, v \in \mathbf{V} \text{ and } d_{\mathbf{G}}(u,v) \geq n \right\}.$$

Essentially ρ provides a lower bound for the ratio between the Euclidean and the graphical distance for nodes that are very far apart. Figure 7 shows the drawing of a graph and the four corresponding parameters s, r, γ, and ρ.

4.3.1 Dense graphs

The drawing of a graph for which the maximum uncovered diameter is finite ($\gamma < \infty$) and the asymptotic distance scaling is positive ($\rho > 0$) is called a *dense drawing*. We say that a $\mathbf{G}$ *is dense in* $\mathbb{R}^d$ if there exists a dense drawing of the graph in $\mathbb{R}^d$. Intuitively, these drawing are "dense" in the sense that the nodes can cover $\mathbb{R}^d$ without leaving large holes between them and still having sufficiently many edges so that a small Euclidean distance between two nodes in the drawing guarantees a small graphical distance between them. In particular, for dense drawings there are always finite constants α, β for which

$$d_{\mathbf{G}}(u,v) \leq \alpha\, d_f(u,v) + \beta, \quad \forall u, v \in \mathbf{V}.$$

This fact is proved in [1]. Using the natural drawing of a d-dimensional lattice, one concludes that this graph is dense in $\mathbb{R}^d$. One can also show that a d-dimensional lattice can never be dense in $\mathbb{R}^{\bar{d}}$ with $\bar{d} > d$. This means, for example, that any drawing of a 2-dimensional lattice in the 3-dimensional Euclidean space will never be dense.

4.3.2 Sparse graphs

Graph drawings for which the minimum node distance is positive ($s > 0$) and the maximum connected range is finite ($r < \infty$) are called *civilized drawings*. This definition is essentially a refinement of the one given in [6], with the quantities r and s made to assume precise values. Intuitively, these drawings are "sparse" in the sense that one can keep the edges with finite lengths, without cramping all nodes on top of each other. We say that a graph $\mathbf{G}$ *is sparse in* $\mathbb{R}^d$ if *it can be drawn in a civilized manner in d-dimensional Euclidean space.*

Showing that a graph is sparse is in principle simple, all we have to do is to find a civilized drawing in the appropriate space—in sensor networks natural drawings generally provide a good starting point. For example, we can conclude from the natural drawing of a d-dimensional lattice that this graph is sparse in $\mathbb{R}^d$. In fact, any h-fuzz of a d-dimensional lattice is still sparse in $\mathbb{R}^d$. However, a d-dimensional lattice can never be drawn in a civilized way in $\mathbb{R}^{\bar{d}}$ with $\bar{d} < d$. This means, for example, that any drawing of a 3-dimensional lattice in the 2-dimensional Euclidean space will never be a civilized drawing. A 3-dimensional lattice is therefore not sparse in $\mathbb{R}^2$.

The notions of graph "sparseness" and "denseness" are mostly interesting for infinite graph, because every finite graph is sparse in all Euclidean spaces $\mathbb{R}^d$, $\forall d \geq 1$ and no finite graph can ever be dense in any Euclidean space $\mathbb{R}^d$, $\forall d \geq 1$. This is because any drawing of a finite graph that does not place nodes on top of each other will necessarily have a positive minimum node distance and a finite maximum connected range (from which sparse follows) and it is not possible to achieve a finite maximum uncovered diameter with a finite number of nodes (from which lack of denseness follows). However, in practice infinite graphs serve as proxies for very large graphs that, from the perspective of most nodes, "appear to extend in all directions as far as the eye can see." So conclusions drawn for sparse/dense infinite graphs hold for large graphs, at least far from the graph boundaries.

4.3.3 Sparseness, denseness, and embeddings

The notions of sparseness and denseness introduced above are useful because they provide a complete characterization for the classes of graphs that can embed or be embedded in lattices, for which the Lattice Effective Resistance Lemma 2 provides the precise scaling laws for the effective resistance.

Theorem 3 (Lattice Embedding) *Let* $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ *be a graph without multiple edges between the same pair of nodes.*

1. $\mathbf{G}$ *is sparse in* $\mathbb{R}^d$ if and only if $\mathbf{G}$ *can be embedded in an h-fuzz of a d-dimensional lattice. More precisely,*

$$\mathbf{G} \text{ is sparse in } \mathbb{R}^d \quad \Leftrightarrow \quad \exists h < \infty : \mathbf{G} \subset \mathbb{Z}_d^{(h)}$$

2. $\mathbf{G}$ *is dense in* $\mathbb{R}^d$ if and only if *(i) the d-dimensional lattice can be embedded in an h-fuzz of* $\mathbf{G}$ *for some positive integer h and (ii) every node of* $\mathbf{G}$ *is at an uniformly bounded graphical distance from another node of* $\mathbf{G}$ *that is also a node of* $\mathbb{Z}_d$*. More precisely,*

$$\mathbf{G} \text{ is dense in } \mathbb{R}^d \quad \Leftrightarrow \quad \exists h, c < \infty : \mathbf{G}^{(h)} \supset \mathbb{Z}_d \quad \& \quad \forall u \in \mathbf{V}\ \exists \bar{u} \in \mathbf{V}_{\mathrm{lat}}(\mathbf{G}) : d_{\mathbf{G}}(u, \bar{u}) \leq c,$$

where $\mathbf{V}_{lat}(\mathbf{G})$ *denotes the nodes of* $\mathbf{G}$ *that are mapped to nods in* $\mathbb{Z}_d$*.* □

The proof follows from simple geometric arguments, the interested reader is referred to [1] for the details.

4.4 Scaling laws for effective resistance

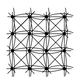
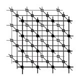
We are now finally ready to characterize scaling laws for graph effective resistances in terms of the denseness/sparseness properties of the graph. The following theorem does precisely this by combining Theorem 1, Rayleigh's Generalized Monotonicity Law, the Lattice Effective Resistance Lemma 2, and the Lattice Embedding Theorem 3.

Theorem 4 (Scaling of effective resistance) *Consider an undirected matrix-weighted graph* $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{W})$ *with matrix weights that satisfy* $R_{\min} \leq W_e^{-1} \leq R_{\max}$, $\forall e \in \mathbf{E}$ *for some symmetric positive definite matrices* $R_{\min}$, $R_{\max}$*. There exist constants* $\ell, \alpha_i, \beta_i > 0$ *such that the formulas in Table 2 hold for every single-node boundary* $\mathbf{V}_o := \{o\}$ *and every node u at an Euclidean to the boundary node o larger than* ℓ*.* □

At this point the reader may verify that the triangular lattice in Figure 6 is both dense and sparse in $\mathbb{R}^2$, which validates the statement made earlier that the effective resistance grows with the logarithm of distance in that graph.

One may ask whether it is common for the graphs that arise in distributed control/estimation problems to be sparse and/or dense in some Euclidean space $\mathbb{R}^d$. The answer happens to be "very much so" and this is often seen by considering the natural drawing of the graph. Recall that a natural drawing associates each node with its physical position in 1-, 2-, or 3-dimensional Euclidean space (cf. discussion in Section 4.2). All natural drawings are likely to be sparse in 3-dimensional space, since the only requirements for sparseness are that nodes not lie on top of each other and edges be of finite length. When agents lie in a 2-dimensional domain or when the third physical dimension is irrelevant, again the natural drawing is likely to be civilized in 2-dimensional space for the same reasons. It is slightly harder for a graph to satisfy the denseness requirements. Formally, a graph has to be infinite to be dense. However, what matters in practice are the properties of the graph "not too close to the boundary". Thus, a large graph satisfies the denseness requirements, as long as there are no big holes between nodes and sufficiently many interconnections between them. For

Table 2: Effective resistances for graphs that are sparse or dense. In the table, $d_f(u,o)$ denotes the Euclidean distance between node u and the reference node o, for any drawing f that establishes the graph's sparseness/denseness.

Euclidean space	Covariance matrix of the estimation error of x_u in a *sparse graph*	Covariance matrix of the estimation error of x_u in a *dense graph*
$\mathbb{R}$	$\alpha_1 d_f(u,o) R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_1 d_f(u,o) R_{\max}$
 $\mathbb{R}^2$	$\alpha_2 \log\big(d_f(u,o)\big) R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_2 \log\big(d_f(u,o)\big) R_{\max}$
 $\mathbb{R}^3$	$\alpha_3 R_{\min} \leq R^{\text{eff}}_{u,o}$	$R^{\text{eff}}_{u,o} \leq \beta_3 R_{\max}$

example, the commonly encountered model consisting of nodes that are Poisson distributed random points in 2-dimensional space with an edge between every pair of nodes that they are within a certain range are likely to be dense in 2-dimensional for a sufficiently large range (when compared to the intensity of the Poisson process). Underwater sensors filling a 3-dimensional volume using a similar model are likely to be dense in 3-dimensional space. In any case, almost all graphs that appear in distributed control/estimation problems are likely to fall into at least one of the classes - sparse or dense in some $\mathbb{R}^d$, $1 \leq d \leq 3$. The only graphs the authors are aware of that do not fall into any of these categories are regular degree infinite trees.

5 Conclusion

We introduced the concept of matrix-valued effective resistance for undirected matrix-weighted graphs, which were shown in a recent paper to have a direct physical interpretation in several problems related to distributed control and estimation and can be used to characterize their stability and speed of convergence [3].

We used an electrical analogy to establish scaling laws for the effective resistance both for regular-degree graphs and irregular graphs with appropriate denseness properties.

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